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PROXIMITY MAPS FOR CERTAIN SPACES

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1. Introduction

Let K be a nonempty subset of a normed linear space X and let $x \in X$. An element k_0 in K satisfying

$$\|x - k_0\| = d(x, K) := \inf_{k \in K} \|x - k\|$$

is called a best approximation to x from K. For any $x \in X$, the set of all best approximations to x from K is denoted by

$$P_K(x) = \{k \in K : \|x - k\| = d(x, K)\}.$$

The set K is called proximinal (resp., Chebyshev) if for every $x \in X$, $P_K(x)$ is nonempty (resp., a singleton).

Let K be a proximinal subset of X. The set-valued map $P_K : X \longrightarrow 2^K$ thus defined is called the metric projection onto K and the kernel of the metric projection P_K is the set

$$\begin{aligned} \ker P_K : &= \{x \in X : 0 \in P_K(x)\} \\ &= \{x \in X : \|x\| = d(x, K)\}. \end{aligned}$$

A map $p : X \longrightarrow K$ which associates with each element of X one of its best approximation in K is called a proximity map.

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In this paper, we are interested in proximity maps which are continuous, linear or Lipschitz continuous. In section 2, we extend a result of continuous proximity maps in $L_p(S,X)$ in [4] where X is a Banach space and (S,Ω,μ) is a σ-finite measure space and the existence of linear proximity maps and Lipschitz continuous maps are discussed. In section 3, we consider the space $C(S,Y)$ of all continuous maps f from a compact Hausdorff space S into a Banach space Y and prove that if $C(S,H)$ has a continuous proximity map then H has a continuous proximity map. In section 4, we discuss some results on the proximinality in $L(X,Y)$.

2. Proximity maps for $L_p(S,G)$

Let X be a Banach space, G a closed subspace of X and (S,Ω,μ) be a σ-finite measure space.

DEFINITION 2.1. Let (M,d) be a metric space. A Borel measurable function from S to M is called *strongly measurable* if it is the pointwise limit of a sequence of simple Borel measurable functions from S to M.

For $1\le p<\infty$, $L_p(S,X)$ is the Banach space consisting of (equivalence classes of) strongly measurable functions $f:S\longrightarrow X$ such that $\int\|f(s)\|^p d\mu(s)$ is finite. For $p=\infty$, $L_\infty(S,X)$ is the Banach space of essentially bounded strongly measurable functions $f:S\longrightarrow X$. For $F\in L_p(S,X)$,

$$\|f\|_p=\Big(\int\|f(s)\|^p d\mu(s)\Big)^{\frac{1}{p}}\quad 1\le p<\infty,$$

and

$$\|f\|_\infty=\text{ess sup}_{s\in S}\|f(s)\|.$$

For $A\in\Omega$ and a strongly measurable function $f:S\to X$, we write I_A for the characteristic function of A and $I_A\otimes f$ denoted the function $F(s)=I_A(s)f(s)$. In particular, for $A\in\Omega$ and $x\in X$, $(I_A\otimes x)(s)=I_A(s)x$.

THEOREM 2.2. [11] *Let $1<p<\infty$. Then the following are equivalent:*

(i) *$L_p(S,G)$ is proximinal in $L_p(S,X)$;*

(ii) *$L_1(S,G)$ is proximinal in $L_1(S,X)$.*

THEROEM 2.3. *Let G be a closed subspace of X and $1 \le p < \infty$. Then*

(i) *If G has a continuous proximity map, then $L_p(S,G)$ is proximinal.*

(ii) *If $L_p(S,G)$ is proximinal in $L_p(S,X)$, then G is proximinal in X. Moreover, if $L_p(S,G)$ has a continuous proximity map, then G has a continuous proximity map.*

Proof. (i) Let $\pi : X \longrightarrow G$ be a continuous proximity map. Let $\mathcal{S}(X), \mathcal{S}(G)$ and $\mathcal{S}(\ker P_G)$ be the class of simple integrable functions with values in X, G and $\ker P_G$, respectively. For $u = \sum_{i=1}^{n} I_{E_i} \otimes x_i \in \mathcal{S}(X)$ ($E_i \cap E_j = \emptyset$ if $i \neq j$), let

$$v = \sum_{i=1}^{n} I_{E_i} \otimes y_i \qquad \text{and} \qquad w = \sum_{i=1}^{n} I_{E_i} \otimes w_i$$

where $y_i = \pi(x_i) \in P_G(x_i)$ and $w_i = x_i - y_i$. Then $v \in \mathcal{S}(G)$ and $w \in \mathcal{S}(\ker P_G)$. Then one can easily obtain that

$$(*) \qquad v \in P_{L_p(S,G)}(u), \qquad w \in L_p(S, \ker P_G) \quad \text{and} \quad u = v + w.$$

Let $f \in L_p(S,X)$. Then there exists a sequence $\{f_n\}$ of simple integrable functions in $L_p(S,X)$ such that $\|f_n(s) - f(s)\| \to 0$. Then, by $(*)$, $f_n = g_n + h_n$ for some $g_n \in P_{L_p(S,G)}(f_n)$ and $h_n \in L_p(S, \ker P_G)$ (g_n and h_n simple integrable). Define $g : X \to G$ by $g(s) = \pi(f(s))$. Since π is continuous and $\pi(f_n(s)) = g_n(s)$, $g_n(s) = \pi(f_n(s)) \to \pi(f(s)) = g(s)$. Hence g is strongly measurable. Since for any $s \in S$, $g(s)$ is a best approximation of $f(s)$, it follows that $g \in P_{L_p(S,G)}(f)$.

(ii) Since (S, Ω, μ) is σ-finite, we can assume $S = \cup_{n \in N} A_n$, $A_n \in \Omega$ such that $A_n \subset A_{n+1}$ and $\mu(A_n) < \infty$ for each $n \in N$. Then there must be $k_0 \in N$ such that $0 < \mu(A_{k_0}) < \infty$. Let $x \in X$. Define $f_x : S \to X$ by

$$f_x(s) = \mu(A_{k_0})^{\frac{1}{p}-1} (I_{A_{k_0}} \otimes x)(s)$$

for all $s \in S$.

Then $f_x \in L_p(S,X)$. By the assumption, there exists $f_0 \in L_p(S,G)$ such that $\|f_x - f_0\|_p = d(f_x, L_p(S,G))$. So

$$\begin{aligned}\|f_x - f_0\|_p &\le \|f_x - \mu(A_{k_0})^{\frac{1}{p}-1} I_{A_{k_0}} \otimes g\|_p \\ &= \mu(A_{k_0})^{\frac{1}{p}-1} \|I_{A_{k_0}} \otimes x - I_{A_{k_0}} \otimes g\|_p \\ &= \mu(A_{k_0})^{\frac{1}{p}-1} \Big(\int_{A_{k_0}} \|x-g\|^p d\mu(s)\Big)^{\frac{1}{p}} \\ &= \mu(A_{k_0})^{\frac{2}{p}-1} \|x-g\|\end{aligned}$$

for all $g \in G$. Since $\|f_x(s) - f_0(s)\| \le \|f_x(s) - h(s)\|$ a.e. for any strongly measurable function $h : S \to G$, $f_0 = I_{A_{k_0}} \otimes f_0$ [11]. Put $x_0 = \int f_0(s) d\mu(s)$. Then

$$\begin{aligned}\|x - \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} x_0\| &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \|\mu(A_{k_0})^{\frac{1}{p}} x - x_0\| \\ &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \|\int (f_x(s) - f_0(s)) d\mu(s)\| \\ &\le \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int \|f_x(s) - f_0(s)\| d\mu(s) \\ &\le \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \Big(\int \|f_x(s) - f_0(s)\|^p d\mu(s)\Big)^{\frac{1}{p}} \mu(A_{k_0})^{1-\frac{1}{p}} \\ &\le \|x-g\|\end{aligned}$$

for all $g \in G$. Hence $\frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} x_0$ is a best approximation of x in G.

Let $P : L_p(S,X) \longrightarrow L_p(S,G)$ be a continuous proximity map. Define $Q : X \longrightarrow G$ by

$$Q(x) = \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (Pf_x)(s) d\mu(s).$$

Then Q is a proximity map for G. Now, let $x_n \to x$ in X. Then

$$\begin{aligned}\|f_{x_n} - f_x\|_p &= \|\mu(A_{k_0})^{\frac{1}{p}-1} I_{A_{k_0}} \otimes x_n - \mu(A_{k_0})^{\frac{1}{p}-1} I_{A_{k_0}} \otimes x\|_p \\ &= \mu(A_{k_0})^{\frac{1}{p}-1} \Big(\int_{A_{k_0}} \|x_n - x\|^p d\mu(s) \Big)^{\frac{1}{p}} \\ &= \mu(A_{k_0})^{\frac{2}{p}-1} \|x_n - x\| \to 0.\end{aligned}$$

Since P is continuous, we have $Pf_{x_n} \to Pf_x$ in $L_p(S,X)$. Since

$$\begin{aligned}\|Qx_n - Qx\| &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \| \int ((Pf_{x_n})(s) - (Pf_x)(s)) d\mu(s) \| \\ &\le \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \Big(\int \|(Pf_{x_n})(s) - (Pf_x)(s)\|^p d\mu(s) \Big)^{\frac{1}{p}} \mu(A_{k_0})^{1-\frac{1}{p}} \\ &= \frac{1}{\mu(A_{k_0})^{\frac{2}{p}-1}} \|Pf_{x_n} - Pf_x\|_p \to 0,\end{aligned}$$

Q is continuous. □

REMARK.

(i) When (S, Ω, μ) is a finite measure space, the above theorem is just Theorem 2.1 of [4].

(ii) In [11], You and Guo proved that if $L_1(S,G)$ is proximinal in $L_1(S,X)$, then G is proximinal in X. Thus the first part of Theorem 2.3 (ii) can be proved by Theorem 2.2 but we proved it directly.

THEOREM 2.4. *Let G be a closed subspace of X and $1 \le p < \infty$. Then the following are equivalent:*

(i) *G has a linear proximity map;*

(ii) *$L_p(S,G)$ has a linear proximity map.*

Proof. (i)⇒(ii) Let π be a linear proximity map of X onto G. Define $\Phi_\pi : L_p(S,X) \to L_p(S,G)$ by $\Phi_\pi(f) = \pi \circ f$. Then Φ_π is a proximity map for $L_p(S,G)$. Take any f_1, $f_2 \in L_p(S,X)$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(\Phi_\pi(\alpha f_1))(s) &= (\pi \circ \alpha f_1)(s) = \pi(\alpha f_1(s)) \\ &= \alpha \pi(f_1(s)) = \alpha(\Phi_\pi(f_1))(s)\end{aligned}$$

for all $s \in S$ and

$$\begin{aligned}(\Phi_\pi(f_1 + f_2))(s) &= (\pi \circ (f_1 + f_2))(s) = \pi(f_1(s) + f_2(s)) \\ &= \pi(f_1(s)) + \pi(f_2(s)) = (\Phi_\pi(f_1))(s) + (\Phi_\pi(f_2))(s)\end{aligned}$$

for all $s \in S$. Hence $L_p(S, G)$ has a linear proximity map.

(ii)$\Rightarrow$(i) For $x \in X$, define $f_x : S \to X$ by

$$f_x(s) = \mu(A_{k_0})^{\frac{1}{p}-1}(I_{A_{k_0}} \otimes x)(s)$$

for all $s \in S$. Then $f_x \in L_p(S, X)$. Let $P : L_p(S, X) \longrightarrow L_p(S, G)$ be a linear proximity map. Define $Q : X \longrightarrow G$ by

$$Q(x) = \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (Pf_x)(s) d\mu(s).$$

Then Q is a proximity map for G. Note that $f_{x+y} = f_x + f_y$ and $f_{\alpha x} = \alpha f_x$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Thus for every $x, y \in X$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}Q(x+y) &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (Pf_{x+y})(s) d\mu(s) \\ &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (P(f_x + f_y))(s) d\mu(s) \\ &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (Pf_x)(s) d\mu(s) + \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (Pf_y)(s) d\mu(s) \\ &= Q(x) + Q(y)\end{aligned}$$

and

$$\begin{aligned}Q(\alpha x) &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (Pf_{\alpha x})(s) d\mu(s) \\ &= \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int \alpha (Pf_x)(s) d\mu(s) \\ &= \alpha Q(x).\end{aligned}$$

Hence Q is linear. □

THEOREM 2.5. [3] *Let G be a proximinal subspace of X. Then the following are equivalent:*

(i) *G has a linear proximity map;*
(ii) *$\ker P_G$ contains a closed subspace W such that $X = G \oplus W$.*

Moreover, if (ii) *holds, then a linear proximity map for G can be defined by*

$$p(g+w) = g, \quad g+w \in G \oplus W.$$

DEFINITION 2.6. [2] A subspace G of a Banach space X is called *1-complemented* in X if there is a closed subspace W of X such that $X = G \oplus W$ and the projection $P : X \to W$ is a contractive projection.

In [2], Deeb and Khalil proved that if G is 1-complemented in X, then G is proximinal in X.

THEOREM 2.7. *Let G be a subspace of a Banach space X. Then following are equivalent:*

(i) *G is 1-complemented in X;*
(ii) *G is proximinal in X and has a linear proximity map.*

Proof. (i) $\Rightarrow$ (ii) Let $X = G \oplus W$ and $P : X \to W$ be a contractive projection. Let $w \in W$. Then for any $g \in G$, $\|P(w-g)\| = \|w\| \leq \|w-g\|$. Thus $W \subset \ker P_G$. Hence G has a linear proximity map by Theorem 2.5.
(ii) $\Rightarrow$ (i) If G has a linear proximity map, then $\ker P_G$ contains a closed subspace W such that $X = G \oplus W$. Moreover, $p(g+w) = g$, $g+w \in G \oplus W$ is a linear proximity map for G and so p is the projection of X onto G along W by Theorem 2.5. Thus $I - p : X \to W$ is a projection. Since $\|(I-p)(x)\| = \|x - p(x)\| \leq \|x\|$ for all $x \in X$, $I-p$ is contractive. Hence G is 1-complemented. □

In [2], Deeb and Khalil proved that if G is 1-complemented in X and (S, Ω, μ) is a finite measure space, then $L_\infty(S, G)$ is 1-complemented in $L_\infty(S, X)$. Hence if G has a linear proximity map and (S, Ω, μ) is a finite measure space, then $L_\infty(S, G)$ has a linear proximity map.

THEOREM 2.8. *Let G be a closed subspace of X and $1 \leq p < \infty$. Then the following are equivalent:*

(i) *G has a Lipschitz continuous proximity map;*
(ii) *$L_p(S, G)$ has a Lipschitz continuous proximity map.*

Proof. (i)⇒(ii) Let $\pi : X \longrightarrow G$ be a Lipschitz continuous proximity map. Define $\Phi_\pi : L_p(S, X) \to L_p(S, G)$ by $\Phi_\pi(f) = \pi \circ f$. Then Φ_π is a proximity map and

$$\begin{aligned}\|\Phi_\pi(f) - \Phi_\pi(g)\|_p &= \Big(\int \|\pi(f(s)) - \pi(g(s))\|^p d\mu(s)\Big)^{\frac{1}{p}} \\ &\leq \lambda\Big(\int \|f(s) - g(s)\|^p d\mu(s)\Big)^{\frac{1}{p}} \\ &= \lambda \|f - g\|_p\end{aligned}$$

for some $\lambda > 0$.

(ii)⇒(i) For $x \in X$, define $f_x : S \to X$ by

$$f_x(s) = \mu(A_{k_0})^{\frac{1}{p}-1}(I_{A_{k_0}} \otimes x)(s)$$

for all $s \in S$. Then $f_x \in L_p(S, X)$. Let $P : L_p(S, X) \longrightarrow L_p(S, G)$ be a Lipschitz continuous proximity map. Define $Q : X \longrightarrow G$ by

$$Q(x) = \frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} \int (Pf_x)(s) d\mu(s).$$

Then Q is a proximity map for G and

$$\begin{aligned}\|Q(x) - Q(y)\| &\leq \frac{1}{\mu(A_{k_0})^{\frac{2}{p}-1}} \|Pf_x - Pf_y\|_p \\ &\leq \frac{1}{\mu(A_{k_0})^{\frac{2}{p}-1}} \lambda \|f_x - f_y\|_p \\ &= \lambda \|x - y\|\end{aligned}$$

for some $\lambda > 0$. □

LEMMA 2.9. [9] *For every finite dimensional Chebyshev subspace G of a normed linear space E, the metric projection P_G is continuous.*

LEMMA 2.10. [9] *Every closed linear subspaces G of a uniformly convex Banach space E is a Chebyshev subspace where the metric projection P_G is continuous.*

LEMMA 2.11. [10] *If G is a Chebyshev set and approximatively compact in a metric space E, then P_G is continuous.*

For $1 \leq p \leq \infty$, You and Guo [11] proved that if $f_0 \in L_p(S, G)$ is a best approximation of $f \in L_p(S, X)$ in $L_p(S, G)$, then there exists a null set N such that $\|f(s) - f_0(s)\| \leq \|f(s) - q(s)\|$ for all $s \in S \setminus N$ and for all strongly measurable function $q : S \to G$. Thus $\|f(s) - f_0(s)\| \leq \|f(s) - g\|$ for all $s \in S \setminus N$ and $g \in G$. Thus if G is Chebyshev in X and f_1 is another best approximation of f in $L_p(S, G)$, then $f_0(s) = f_1(s)$ for all $s \in S \setminus N$. Hence $f_0 = f_1$.

THEOREM 2.12. *Let G be a closed subspace of X. Then for $1 \leq p < \infty$, $L_p(S, G)$ is Chebyshev in $L_p(S, X)$, if one of the following assumptions holds:*

(i) *G is finite dimensional and Chebyshev.*
(ii) *X is uniformly convex.*
(iii) *G is Chebyshev and approximately compact.*

Proof. This follows from Lemma 2.9, Lemma 2.10, Lemma 2.11, Theorem 2.3 and the above remark. □

LEMMA 2.13. [9] *If G is a proximinal hyperplane in a normed linear space E, then G has a linear proximity map.*

THEOREM 2.14. *Let G be a proximinal subspace of X. Then for $1 \leq p < \infty$, $L_p(S, G)$ is proximinal in $L_p(S, X)$ and $L_p(S, G)$ has a linear proximity map, if G is of codimension 1.*

Proof. This follows from Lemma 2.13 and Theorem 2.4. □

THEOREM 2.15. [7] *Let E be a normed linear space and $G \subset E$ a proximinal subspace. Compare the following two statements:*

(i) *G has a continuous proximity map $s : E \to G$ such that $s(x) = 0$ for each $x \in E$ with $0 \in P_G(x)$.*

(ii) *P_G is lower semi-continuous.*

We have (i) $\Rightarrow$ (ii) *and if G is complete, also* (ii) $\Rightarrow$ (i).

COROLLARY 2.16. *Suppose that G is a proximinal subspace of X. If P_G is lower semi-continuous, then $L_p(S, G)$ is a proximinal subspace of $L_p(S, X)$ $(1 \leq p < \infty)$.*

3. Proximity maps for $C(S, H)$

If S is a compact Hausdorff space and Y is a Banach space, $C(S, Y)$ denotes the Banach space of all continuous maps f from S into Y with norm defined by

$$\|f\| = \sup_{s \in S} \|f(s)\|.$$

THEOREM 3.1. [8] *Let H be a closed subspace of the Banach space Y. Let S be a compact Hausdorff space. For each $f \in C(S, Y)$,*

$$d(f, C(S, H)) = \sup_{s \in S} d(f(s), H).$$

THEOREM 3.2. [8] *If there is a continuous proximity map of Y onto H, then $C(S, Y)$ is proximinal in $C(S, Y)$ and in fact it has a continuous proximity map.*

THEOREM 3.3. *If $C(S, H)$ is proximinal in $C(S, Y)$ then H is proximinal in Y. Moreover, if $C(S, H)$ has a continuous proximity map, then H has a continuous proximity map.*

Proof. For $y \in Y$, define $f_y : S \longrightarrow Y$ by $f_y(s) = y$ for all $s \in S$. Then $f_y \in C(S, Y)$. By the assumption, there exists $g \in C(S, H)$ such that $\|f_y - g\| = d(f_y, C(S, H))$. By Theorem 3.1,

$$\begin{aligned} d(f_y, C(S, H)) &= \sup_{s \in S} d(f_y(s), H) \\ &= d(y, H) \end{aligned}$$

and hence

$$\begin{aligned}\|y - g(s)\| &= \|f_y(s) - g(s)\| \\ &\leq \|f_y - g\| \\ &= d(y, H)\end{aligned}$$

for all $s \in S$. Thus H is proximinal in Y.

Let $A : C(S,Y) \longrightarrow C(S,H)$ be a continuous proximity map. Fix any $s_0 \in S$. Define $Q : Y \longrightarrow H$ by $Qy = (Af_y)(s_0)$. Then Q is a proximity map. Suppose that $y_n \to y$ in Y. Then $Af_{y_n} \to Af_y$ and

$$\begin{aligned}\|Q(y_n) - Q(y)\| &= \|(Af_{y_n})(s_0) - (Af_y)(s_0)\| \\ &\leq \|Af_{y_n} - Af_y\| \to 0.\end{aligned}$$

Hence Q is continuous. □

THEOREM 3.4. *If P_H is lower semi-continuous, then $C(S,H)$ is proximinal in $C(S,Y)$.*

Proof. Let $f \in C(S,Y)$. Define $\Phi : S \longrightarrow 2^H$ by

$$\Phi(s) = (P_H \circ f)(s),$$

i.e., $\Phi(s) = \{h \in H : \|f(s) - h\| = d(f(s), H)\}$. Take any $s_0 \in S$. Since P_H is lower semi-continuous, for every open set O in H such that $P_H(f(s_0)) \cap O \neq \emptyset$, there exists an open neighborhood V of $f(s_0)$ such that $P_H(y) \cap O \neq \emptyset$ for all $y \in V$. Since f is continuous at s_0, there exists an open neighborhood U of s_0 such that $f(U) \subset V$. Thus $P_H(f(s)) \cap O \neq \emptyset$ for all $s \in U$. Hence Φ is lower semi-continuous.

Note that each $\Phi(s)$ is a nonvoid, closed and convex subset of H. By Michael Selection Theorem, Φ has a continuous selection, say g. Thus $g \in C(S,H)$. Moreover,

$$\|f - g\| = \sup_{s\in S} \|f(s) - g(s)\| = \sup_{s\in S} d(f(s), H) = d(f, C(S,H)).$$

Hence $C(S,H)$ is proximinal in $C(S,Y)$. □

REMARK. Theorem 3.4 follows from Theorem 2.15 and Theorem 3.2. Moreover, $C(S, H)$ has a continuous proximity map. But we proved it directly.

COROLLARY 3.5. *Let H be a proximinal subspace of a Banach space Y. If one of the following holds:*

(i) *H is Chebyshev and approximately compact,*
(ii) *H is of codimension 1,*

then $C(S, H)$ is proximinal in $C(S, Y)$.

Proof. This follows from Lemma 2.11, Lemma 2.13 and Theorem 3.2. □

4. Proximity maps for $L(X, Y)$

Let $L(X, Y)$ be the space of all bounded linear operators from a Banach space X into a Banach space Y.

THEOREM 4.1. *Let G be a proximinal subspace of Y. If G has a linear proximity map, say π, then $L(X, G)$ has a linear proximity map.*

Proof. Define $P : L(X, Y) \to L(X, G)$ by

$$P(A) = \pi \circ A.$$

Let $x \in X$. Then $\pi(A(x))$ is a best approximation to $A(x)$ in G. Hence

$$\begin{aligned} \|A(x) - (P(A))(x)\| &= \|A(x) - \pi(A(x))\| \\ &\leq \|A(x) - \theta\| \end{aligned}$$

for all $\theta \in G$. So

$$\|A(x) - (P(A))(x)\| \leq \|A(x) - B(x)\|$$

for all $B \in L(X, G)$. Since x was arbitrary in X, $\|A - P(A)\| \leq \|A - B\|$ for all $B \in L(X, G)$. Thus $P(A)$ is a best approximation of A in $L(X, G)$. Since $P(\alpha A + \beta B) = \alpha P(A) + \beta P(B)$ for all $A,\ B \in L(X, Y)$ and $\alpha,\ \beta \in \mathbb{R}$, P is linear. □

Deeb and Khalil [2] proved that if G is 1-complemented in Y(or equivalently G has a linear proximity map), then $L(X, G)$ is proximinal in $L(X, Y)$.

COROLLARY 4.2. *Let G be a proximinal subspace of Y and of codimension 1. Then $L(X,G)$ is proximinal in $L(X,Y)$ and $L(X,G)$ has a linear proximity map.*

Proof. This follows from Lemma 2.13 and Theorem 4.1. □

References

[1] D. L. Cohn, *Measure theory*, Birkhaüser (1980).
[2] W. Deeb and R. Khalil, *Best approximation in L(X, Y)*, Math. Proc. Camb. Phil. Soc. **104** (1988), 527-531.
[3] F. Deutsch, *Linear selections for the metric projection*, J. Funct. Anal. **49** (1982), 269-292.
[4] D. Hussein and R. Khalil, *Best approximation in tensor product spaces*, Soochow J. Math. **18** (1992), 397-407.
[5] R. Khalil, *Best approximation in $L^p(I, X)$*, Math. Proc. Camb. Phil. Soc. **94** (1983), 277-279.
[6] R. Khalil and W. Deeb, *Best approximation in $L^p(\mu, X)$ II*, J. Approx. Theory **59** (1989), 296-299.
[7] H. Krüger, *A remark on the lower semi-continuity of the set-valued metric projection*, J. Approx. Theory **28** (1980), 83-86.
[8] W. A. Light and E. W. Cheney, *Approximation theory in tensor product spaces*, Lect. Notes in Math. Vol. 1169, Springer-Verlag, New York/Berlin, 1985.
[9] I. Singer, *The Theory of best approximation and functional analysis*, Society for Industrial and Appl. Math., Philadelphia, 1974.
[10] ———, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, New York, 1970.
[11] You Zhao-Yong and Guo Tie-Xin, *Point-wise best approximation in the space of strongly measurable functions with applications to best approximation in $L^p(\mu, X)$*, J. Approx. Theory **78** (1994), 314-320.

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