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SOME GEOMETRIC SOLVABILITY THEOREMS IN TOPOLOGICAL VECTOR SPACES

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1. Introduction

The aim of this paper is to present theorems on the existence of zeros for mappings defined on convex subsets of topological vector spaces with values in a vector space. In addition to natural assumptions of continuity, convexity, and compactness, the mappings are subject to some geometric conditions. In the first theorem, the mapping satisfies a "Darboux-type" property expressed in terms of an auxiliary numerical function. Typically, this function is, in this case, related to an order structure on the target space. We derive an existence theorem for "obtuse" quasiconvex mappings with values in an ordered vector space. In the second theorem, we prove the existence of a "common zero" for an arbitrary (not necessarily countable) family of mappings satisfying a general "inwardness" condition again expressed in terms of numerical functions (these numerical functions could be duality pairings (more generally, bilinear forms)). Our inwardness condition encompasses classical inwardness conditions of Leray-Schauder, Altman, or Bergman-Halpern types.

All topological spaces are assumed to be Hausdorff and vector spaces are real.

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2. Obtuse Mappings

To start, let D be a set and let F be a topological vector space ordered by a closed cone K (assume that $K \cap (-K) = \{0\}$) with dual cone $K^{'}$. Assume that $K' \neq \emptyset$; this happens e.g. when K is proper and has nonempty interior (this follows from a theorem of Krein and Rutman, see for instance Corollary 1.6.3 in Jameson [4]). Let $\kappa \in K^{'}$ be a bounded positive functional and define a real function $B : F \times F \longrightarrow \mathbf{R}$ by putting $B(v,u) = \kappa(v)\kappa(u), (u,v) \in F \times F$. Let $\phi : D \longrightarrow K \cup (-K)$ be a mapping and assume that the disjoint sets $D_+ = \{x \in D; \phi(x) \in K \backslash \{0\}\}$ and $D_- = \{x \in D; \phi(x) \in -K \backslash \{0\}\}$ are both non-empty. Then, any given point $x \in D$ is either a zero for ϕ, or it satisfies the inequality:

(1) $\quad$ there exists $y \in D$ with $B(\phi(y), \phi(x)) < 0$

Furthermore, assuming that D has a topological structure and that $\forall v \in \phi(D)$, the real function $x \longmapsto B(v, \phi(x))$ is upper semicontinuous on D, then both sets D_+ and D_- are open in D. Under these conditions, it is obvious that ϕ has a zero provided D is connected.

What happens for a general mapping B that is not necessarily related to an order structure?

In the framework of topological vector spaces, an answer is based on convexity and on the next definition motivated by the above discussion.

Definition 2.1. Let D be a set, F a vector space, and B a real function on $F \times F$. A mapping $\phi : D \longrightarrow F$ is said to be $B-$obtuse at a given point $x \in D$ if and only if the inequality (1) is satisfied whenever $\phi(x) \neq 0$. The map ϕ is said to be $B-$obtuse on D if it is so at every point of D.

Example 2.2. Let $D = [-1, 1]$ and $F = \mathbf{R}$. Consider the mapping ψ : $\mathbf{R})|x|, x \in \mathbf{R}, \theta \in (0, \frac{\pi}{4})$, whose graph consists of the half-lines forming the angle $\alpha = \pi - 2\theta \in (\frac{\pi}{2}, \pi)$.

Let

$$B(x,y) := \cos(\widehat{(x, \psi(x)), (y, \psi(y))}), x, y \in \mathbf{R}.$$

Note that $B(x,x) > 0$ for any $x \in \mathbf{R}$. Let $\phi : D \longrightarrow \mathbf{R}$ be any mapping such that $\phi(x_*) < 0$ and $\phi(x^*) > 0$ for some $x_*, x^* \in D$. It is readily seen that ϕ is $B-$obtuse on D.

THEOREM 2.3. *Let X be a convex subset in a topological vector space E, D be a line segment in X, F be a vector space, B a real function on $F \times F$ with $B(u,u) \geq 0$ for all $u \in F$, and let $\phi : E \longrightarrow F$ be a mapping satisfying the following hypotheses:*

(i) *for any $v \in \phi(D)$, $x \longmapsto B(v, \phi(x))$ is upper semicontinuous on D;*

(ii) *Given any three points $x, y_1, y_2 \in D$ with $B(\phi(y_i), \phi(x)) < 0, i = 1, 2$, we have:*

$$B(\phi(ty_1 + (1-t)y_2), \phi(x)) < 0, \forall t \in (0,1).$$

Then ϕ has a zero in D, i.e. a point $x^* \in D$ with $\phi(x^*) = 0$, if and only if the restriction of ϕ to D is $B-$obtuse.

Proof. The necessity readily follows from Definition 2.1. We only prove the sufficiency. Note first that D is compact and define a multifunction $\Phi : D \longrightarrow 2^D$ by putting:

$$\Phi(x) := \{y \in D : B(\phi(y), \phi(x)) < 0\}, \ x \in D.$$

Since ϕ is $B-$obtuse on D, it suffices to show the existence of $x^* \in D$ with $\Phi(x^*) = \emptyset$ to conclude that $\phi(x^*) = 0$.

Assume for a contradiction that $\Phi(x) \neq \emptyset$ for any $x \in D$, and decompose the set $M := \Phi(D)$ as $M = M_1 \cup M_2$ where

$$M_1 := \cup_{x \in D}\{\Phi(x) : \Phi(x) \text{ is convex}\} \quad \text{and}$$
$$M_2 := \cup_{x \in D}\{\Phi(x) : \Phi(x) \text{ is not convex}\}.$$

We claim that the set M_2 is nonempty. Indeed, if M_2 were empty, then $M = M_1$ i.e. Φ is convex-valued on D.

Moreover, the continuity assumption (i) implies that $\Phi^{-1}(y)$ is an open subset of D for any $y \in D$. By the Browder-Fan fixed point theorem (cf for instance [1]), it would have a fixed point $x_* \in D$, i.e., $B(\phi(x_*), \phi(x_*)) < 0$ which contradicts the positivity of B. Hence, there exists $\hat{x} \in D$ such that $\Phi(\hat{x})$ is nonconvex, i.e., there exist $y_1, y_2 \in \Phi(\hat{x})$ and $t \in (0,1)$ with $ty_1 + (1-t)y_2 \notin \Phi(\hat{x})$. This contradicts assumption (ii) and ends the proof. □

REMARK. (1) Condition (i) is obviously verified when ϕ is continuous and B is separately continuous. Condition (ii) is automatically satisfied when for any $u \in \phi(D)$, the real function $y \longmapsto B(\phi(y), u)$ is quasiconvex.

(2) Clearly, one could very well replace the segment D by the whole set X provided X is either compact or the additional following weaker compactness condition holds:

there exist a compact subset K of X and a compact convex subset C of X such that:

$$\text{for any } \ x \in X \setminus K, \ \text{ there exists } \ y \in C \ \text{ with } \\ B(\phi(y), \phi(x)) < 0 \ \text{ provided } \ \phi(x) \neq 0.$$

(3) This theorem extends to multivalued ϕ in an obvious manner.

As an immediate consequence of Theorem 2.3, we obtain a solvability theorem for a quasiconvex mapping with values in an ordered space. Let us recall (see Jeyakumar *et al.* [5]) that given a convex subset X of a vector space and a vector space F partially ordered by a cone K, a mapping $\phi : X \longrightarrow F$ is said to be *quasiconvex* on X if and only if:

$$\text{for any } \ y_1, y_2 \in X \text{ and } \ v \in F,$$
$$\phi(y_i) - v \in -K, i = 1, 2, \Rightarrow \phi(ty_1 + (1-t)y_2) - v \in -K, \ \text{ for } \ t \in [0, 1].$$

Recall (see Martelloti and Salvadori [6]) that given a topological space X and an order complete topological Riesz space F with proper cone K having nonempty interior $int(K)$, a mapping $\phi : X \longrightarrow F$ is said to be *lower semicontinuous* at $x_0 \in X$ if

$$\text{for any } \ \epsilon \in int(K), \ \text{ there exists a neighborhood } \ U_{x_0} \text{ of } x_0 \\ \text{with } \ (\phi(x) - \phi(x_0) + \epsilon) \in int(K) \ \text{ for all } \ x \in U_{x_0}.$$

The mapping ϕ is said to be lower semicontinuous in X if it is so at every point of X.

COROLLARY 2.4. *Let X be a convex subset of a topological vector space, F be an order complete topological Riesz space F with proper cone K having nonempty interior,and let $\phi : X \longrightarrow F$ be a lower semicontinuous quasiconvex mapping satisfying:*

(i) *for each $x \in X$ with $\phi(x) \neq 0$, there exists $y \in X$ such that $\phi(x) - \phi(y) \in K \setminus \{0\}$.*

Assume that any one of the following conditions holds:

(a) *X is a line segment;*

(b) *X is compact;*

(c) *there exist a compact subset D of X and a compact convex C of X such that:*

$$\text{for each } x \in X\backslash D, \text{ there exists } y \in C \text{ with } \phi(x) - \phi(y) \in K \setminus \{0\}.$$

Then there exists $x^* \in X$ such that $\phi(x^*) = 0$.

Proof. Apply Theorem 2.3 (with Remark 2) with the function $B(v,u) = \kappa(v-u), (v,u) \in F \times F$, where $\kappa \in K'$ is a bounded positive linear functional, and note that the quasiconvexity of ϕ implies that of the real function $\kappa\phi$ and that the lower semicontinuity of ϕ implies that of the real function $\kappa\phi$ ((Proposition 2.7 in [6]). □

Another interesting consequence of Theorem 2.3 is the

COROLLARY 2.5. *Let E be a topological vector space and $B : E \times E \longrightarrow \mathbf{R}$ be a mapping that is odd with respect to one argument and satisfies $B(u,u) > 0$ for $u \in E \setminus \{0\}$. Let X be a compact convex subset of E and let $f : X \longrightarrow X$ be a continuous mapping. Assume that f is periodic on X (i.e. $f^2(x) = x$) and that for any $u \in E$, the function $x \longmapsto B(f(x) - x, u)$ is quasiconvex on X. Then f has a fixed point.*

Proof. Note that the field $\phi(x) = f(x) - x$ is $B-$obtuse on X and apply Theorem 2.3. □

3. Inward mappings

A modification in Definition 2.1 leads to the following concept of "inwardness".

DEFINITION 3.1. Let X be a subset of a vector space E, F a vector space, B a real function on $F \times E$. A mapping $\phi : X \longrightarrow F$ is said to be $B-$inward at a point $x \in X$ if and only if:

(2) there exists $y \in X$ with $B(\phi(x), x - y) < 0$ whenever $\phi(x) \neq 0$.

The mapping ϕ is said to be $B-$ inward on X if it is so at every point of X.

This concept is more general than the classical concept of "weak inwardness" of Bergman and Halpern [2]. The reader is refereed to Park [7] for fixed point theorems for a large class of multifunctions satisfying classical inwardness conditions.

PROPOSITION 3.2. *Assume that $E = F$ is a normed space (with norm $|.|$), and that $B(.,.)$ is a symmetric bilinear form which is continuous and coercive. If ϕ is weakly inward at a point $x \in X$, (i.e. $\phi(x)$ belongs to the contingent cone of Bouligand $T_X(x) := \{y \in E$: there exist $t_n \to 0^+$ and $v_n \to y$ such that for all n, $x + t_n v_n \in X\}$ to X at x), then ϕ is $B-$inward at x.*
The converse is false.

Proof. Indeed, assume that $\phi(x) \in T_X(x)$, $\phi(x) \neq 0$, and that $0 < \epsilon\sqrt{c/\alpha} < |\phi(x)|$, (where $C \geq 0$ is a constant with $|B(u,v)| \leq C|u||v|$ for all $u, v \in E$, and $\alpha > 0$ is a constant with $B(u,u) \geq \alpha|u|^2$ for all $u \in E$). Let N be a positive integer such that:

$$|\phi(x) - v_N| < \min\{\epsilon, \epsilon\sqrt{c/\alpha}\}, v_N \in E, x_N = x + t_N v_N \in X.$$

Clearly, $2B(\phi(x), x - x_N) = -2t_N B(\phi(x), v_N)$. By the choice of ϵ, we have:

$$\sqrt{\alpha}|\phi(x)|(\sqrt{\alpha}|\phi(x)| - \epsilon\sqrt{c}) > 0,$$

hence,

$$C\epsilon^2 < C\epsilon^2 + \alpha|\phi(x)|^2 - \sqrt{c\alpha}\epsilon|\phi(x)| < (\sqrt{\alpha}|\phi(x)| - \sqrt{c}\epsilon)^2 + \alpha|\phi(x)|^2.$$

On the other hand, $(\sqrt{\alpha}|\phi(x)| - \sqrt{c}\epsilon)^2 < \alpha|v_N|^2$, thus:

$$C\epsilon^2 < \alpha[|\phi(x)|^2 + |v_N|^2],$$

which implies that:

$$t_N[C\epsilon^2 - \alpha(|\phi(x)|^2 + |v_N|^2)] < 0.$$

Moreover,

$$\begin{aligned}\alpha[|\phi(x)|^2 + |u_N|^2] - 2B(\phi(x), u_N) &\leq B(\phi(x), \phi(x)) + B(u_N, u_N) \\ &\quad - 2B(\phi(x), u_N) \\ &= B(\phi(x) - u_N, \phi(x) - u_N) \\ &\leq C|\phi(x) - u_N|^2 \\ &< C\epsilon^2\end{aligned}$$

which implies that:

$$-2t_N B(\phi(x), v_N) < t_N[C\epsilon^2 - \alpha(|\phi(x)|^2 + |v_N|^2)] < 0.$$

Therefore, $B(\phi(x), x - x_N) < 0$ and (2) is thus satisfied with $y = x_N$.

To see that the converse is not true, let X be the closed unit disk in $\mathbf{R}^2$ with boundary ∂X and let A be the $x - axis$ in $\mathbf{R}^2$. Let $\phi : X \to A \subset \mathbf{R}^2$ be a continuous function which values on ∂X are given by:

$$\phi(\cos\theta, \sin\theta) := \begin{cases} -1 + \cos\theta, & \text{if } 0 \leq \theta < 3\pi/4; \\ -4 + 4\theta/\pi - \sin\theta, & \text{if } 3\pi/4 \leq \theta < \pi; \\ 4 - 4\theta/\pi + \sin\theta, & \text{if } \pi \leq \theta < 5\pi/4; \\ -1 + \cos\theta, & \text{if } 5\pi/4 \leq \theta < 2\pi. \end{cases}$$

We claim that ϕ is $B-$inward but not weakly inward on X (here $B(x, y) = \langle x, y\rangle$ is the inner product in $\mathbf{R}^2$). Indeed, since X is convex, for any $x \in \partial X$, the Bouligand cone coincides with the tangent cone $\overline{\bigcup_{t>0} \frac{1}{t}(X - x)}$. Observe now that the tangent line to ∂X at the point $(-\sqrt{2}/2, \sqrt{2}/2) = (\cos(3\pi/4), \sin(3\pi/4))$ intersects A at the point $(-\sqrt{2}, 0)$ and that $\phi(\cos(3\pi/4), \sin(3\pi/4)) = -1 - \sqrt{2}/2$ lies to the left of that tangent line. Thus $\phi(3\pi/4) \notin T_X((-\sqrt{2}/2, \sqrt{2}/2))$ (note that ϕ fails to be weakly inward on two symmetric arcs of ∂X). We claim however that ϕ is $B-$inward on ∂X. To see this, first note that for $x_0 = (-1, 0)$, $\phi(x_0) = \phi(\pi) = 0$. Now observe that $\phi(\partial X) = ([-1-\sqrt{2}/2, 0]) \times \{0\}$. Hence, for any $x \in \partial X$ with $x \neq x_0$ and $\phi(x) \neq 0$, the angle between $\phi(x)$ and $(x - x_0)$ lies in the interval $(\pi/2, 3\pi/2)$ thus $\langle\phi(x), x - x_0\rangle < 0$ and (2) is satisfied. □

REMARK. In view of this proposition, when $E = F$ and since $T_X(x) = E$ for every interior point x of X, it suffices to assume the condition of $B-$inwardness only at boundary points of X.

This remark motivates the next proposition.

PROPOSITION 3.3. *Assume that $E = F$, that $B(.,.) = \langle .,. \rangle$ is an inner product on E, and that X is a closed disk with center at the origin and radius $R > 0$ in E. If a mapping $f : X \to E$ satisifies any one of the following boundary conditions:*

(b_1) *(Browder) for any $x \in \partial X$ with $|\ f(x)| > R$, there exists $z \in I_X(x) := x + \bigcup_{a \geq 0} a(X - x)$ such that $|z - f(x)| < |x - f(x)|$.*

(b_2) *(Leray-Schauder) for any $x \in \partial X$ with $|f(x)| > R$ and any $\lambda \in (0,1)$, $x \neq \lambda f(x)$.*

(b_3) *$|x - f(x)| \neq |f(x)| - R$ for each $x \in \partial X$ with $|f(x)| > R$.*

(b_4) *(Altman) for any $x \in \partial X$ with $|\ f(x)| > R$, there exists $\alpha \in (1, \infty)$ such that $|f(x)|^\alpha - R^\alpha \leq |x - f(x)|^\alpha$.*

Then the field $\phi = f - i$ (where i is the inclusion $X \hookrightarrow E$) is $B-$inward on X.

Proof. To see this, let $x \in \partial X$ with $|f(x)| > R$ be arbitrarily fixed and assume that (b_1) is satisfied with $z = x + a(y - x)$ for some $a \geq 0$ and some $y \in X$. Note that since z cannot coincide with x, then $a \neq 0$ and $y \neq x$. It follows that:

$$\begin{array}{rrcl} & |z - f(x)| & < & |x - f(x)| \\ \Leftrightarrow & |a(y - x) - \phi(x)| & < & |\phi(x)| \\ \Rightarrow & |a(y - x) - \phi(x)|^2 & < & |\phi(x)|^2 \\ \Leftrightarrow & a|y - x|^2 & < & 2\langle \phi(x), y - x \rangle \\ \Rightarrow & \langle \phi(x), x - y \rangle & < & 0. \end{array}$$

Assume now that (b_2) holds.

If $\langle \phi(x), x \rangle = 0$, then $\langle \phi(x), x - R\phi(x)/|\phi(x)| \rangle = -R|\phi(x)| < 0$. If $\langle \phi(x), x \rangle < 0$ then $y = 0$ satisfies (2). If $\langle \phi(x), x \rangle > 0$, it follows from (b$_2$) that $\phi(x) = f(x) - x \neq \alpha x$ for any $\alpha \in \mathbf{R}$. Hence,

$$\begin{aligned}
& \langle \phi(x), x - R(x+\phi(x))/|x+\phi(x)| \rangle \\
= \; & [1/|x+\phi(x)|]\langle \phi(x), (|x+\phi(x)| - R)x - R\phi(x) \rangle \\
= \; & [1/|x+\phi(x)|][-R|\phi(x)|^2 + (|x+\phi(x)| - R)\langle \phi(x), x \rangle] \\
< \; & [1/|x+\phi(x)|][-R|\phi(x)|^2 + (|x| + |\phi(x)| - R)|\phi(x)||x|] \\
= \; & [R/|x+\phi(x)|][-|\phi(x)|^2 + |\phi(x)|^2] \\
= \; & 0,
\end{aligned}$$

where the strict inequality above follows from the triangle strict inequality $|x+\phi(x)| < |x| + |\phi(x)|$ because x and $\phi(x)$ are not colinear. Condition (2) is thus satisfied with $y = R(x+\phi(x))/|x+\phi(x)|$.

If (b$_3$) is satisfied then it holds:

(b$_3$)$'$ $\quad |f(x) - x| > |f(x)| - R$ for any $x \in \partial X$ with $|f(x)| > R$.

A quick calculation shows that (2) holds with $y = R(x+\phi(x))/|x+\phi(x)|$.

If (b$_4$) holds, since $R/|f(x)| \in (0,1)$, then:

$$\begin{aligned}
\frac{|f(x)-x|^\alpha}{|f(x)|^\alpha} &\geq \frac{|f(x)|^\alpha - R^\alpha}{|f(x)|^\alpha} \\
&= 1 - (\frac{R}{|f(x)|})^\alpha > (1 - \frac{R}{|f(x)|})^\alpha = \frac{(|f(x)| - R)^\alpha}{|f(x)|^\alpha}.
\end{aligned}$$

Consequently, (b$_3$)$'$ is satisfied. □

We formulate now the second main theorem of this note. Its proof is based on a generalization of the Browder-Fan fixed point theorem to arbitrary families of multifunctions (Theorem 5.1 in [3]). Let I be an arbitrary (possibly uncountable) set. Let $\{E_i\}_{i\in I}$ and $\{F_i\}_{i\in I}$ be two families of topological vector spaces. For each $i \in I$, let X_i be a nonempty subset of E_i, let $X := \prod_{i\in I} X_i$, let $\pi_i : \prod_{i\in I} E_i \longrightarrow E_i$ be the ith-projection, and let $B_i : F_i \times E_i \longrightarrow \mathbf{R}, \phi_i : X \longrightarrow F_i$ be mappings. In this context, we will say that ϕ_i is B_i–inward on X if and only if:

$$(3) \qquad \begin{pmatrix} \text{for any } x \in \partial X, \text{ there exists } y_i \in X_i \text{ such that} \\ B_i(\phi_i(x), \pi_i(x) - y_i) < 0 \text{ whenever } \phi_i(x) \neq 0 \text{ in } F_i. \end{pmatrix}$$

THEOREM 3.4. *Assume that for each $i \in I$, the set X_i is convex and the mappings B_i and ϕ_i satisfy the following properties:*

(i) *$B_i(v_i, 0) \geq 0$ for any $v_i \in \phi_i(X)$;*

(ii) *ϕ_i is B_i−inward on X;*

(iii) *for each $y_i \in X_i$, the function $x \longmapsto B_i(\phi_i(x), \pi_i(x) - y_i)$ is upper semicontinuous on X;*

(iv) *for each $v_i \in \phi_i(X)$, the function $B_i(v_i, .)$ is quasiconvex on X_i;*

Assume in addition that any one of the following compactness conditions is satisfied.

(a) *For each $i \in I, X_i$ is compact.*

(b) *There is a compact subset K of X, for each $i \in I$ there is a compact subset C_i of X_i such that: for each $x \in X \setminus K$, there exist $i \in I$ and $y_i \in C_i$ with $B_i(\phi_i(x), \pi_i(x) - y_i) < 0$.*

Then, there exists $x^* \in X$ such that $\phi_i(x^*) = 0$ in F_i for all $i \in I$.

Proof. Clearly, (a) $\Rightarrow$ (b) with $C_i = X_i$ and $K = \emptyset$. Define a family of multifunctions $\Phi_i : X \longrightarrow 2^{X_i}, i \in I$, by putting:

$$\Phi_i(x) = \{y_i \in X_i : B_i(\phi_i(x), \pi_i(x) - y_i) < 0\}, x \in X.$$

Note that in order to prove this theorem, it suffices to show, in view of (ii) and (3), the existence of a common "maximal element" for the family $\{\Phi_i\}_{i \in I}$, i.e. a point $x^* \in X$ with $\Phi_i(x^*) = \emptyset$ for all $i \in I$.

Now, conditions (iii) and (iv) imply that the multifunctions Φ_i have open fibers and convex values. Moreover, hypothesis (i) implies that for $x \in X$ and $i \in I, \pi_i(x) \notin \Phi_i(x)$. All hypotheses of Theorem 5.1 in [3] are thus satisfied. This implies the existence of a common maximal element for the family $\{\Phi_i\}_{i \in I}$. □

REMARK. Assume for simplicity that I reduces to a singleton. Since B−inwardness is equivalent to the non-solvability of an inequality, it follows that the preceding theorem can be formulated as a nonlinear alternative between the solvability of an equation or the solvability of a variational-type inequality. More precisely, under the above hypotheses, one of the following statement holds:

(1) there exists $x^* \in X$ with $\phi(x^*) = 0$, or
(2) there exists $\bar{x} \in X$ such that $\phi(\bar{x}) \neq 0$ and $B(\phi(\bar{x}), \bar{x} - y) \geq 0$ for all $y \in X$.

As a result, one may immediately derive (for suitable choices of the mapping B) classical theorems on variational inequalities.

COROLLARY 3.5. *Assume that E is a Banach space, and that $B : E \times E \longrightarrow \mathbf{R}$ is a symmetric continuous and coercive bilinear form. Assume that X is a nonempty closed convex subset of E and that $\phi : X \longrightarrow E$ is a weakly inward mapping on X.*

If any one of the following conditions is satisfied:

(a) *X is compact and ϕ is $B-$demicontinuous on X, i.e.:*

$$x_n \to_X x \Longrightarrow B(\phi(x_n), y) \to B(\phi(x), y) \text{ for any } y \in E);$$

(b) *ϕ is strongly continuous on X, i.e.:*

$$x_n \rightharpoonup_X x \Longrightarrow \phi(x_n) \to \phi(x)),$$

and $\limsup_{|x| \to \infty, x \in X} B(\phi(x), x - y_0) < 0$ *for some* $y_0 \in X$.

Then ϕ has a zero in X.

REMARK. Weak inwardness in the preceding corollary could be replaced by any one of the tangency conditions (b_1)-(b_4) of Example 8.

We end this note with a common fixed point theorem for families of real functions. Let I be an arbitrary set. For each $i \in I$, let $[a_i, b_i]$ be an interval in $\mathbf{R}$, and let $[a, b] = \prod_{i \in I} [a_i, b_i]$ be equipped with the order relations:

$$x = (x_i)_{i \in I} \leq (< \text{ resp.}) \; y = (y_i)_{i \in I} \Leftrightarrow x_i \leq (< \text{ resp.}) \; y_i \text{ for all } i \in I.$$

Denote by π_i the $ith-$projection of $[a, b]$ onto $[a_i, b_i]$.

COROLLARY 3.6. *Let* $f = (f_i)_{i \in I}$ *be a mapping from* $[a, b]$ *into* $\mathbf{R}^I$ *satisfying the following conditions:*

(i) *for each* $i \in I$ *and* $y_i \in [a_i, b_i]$, *the function* $x \longmapsto (f_i(x) - \pi_i(x))(\pi_i(x) - y_i)$ *is upper semicontinuous on* $[a, b]$;

(ii) $a \leq f(a)$ *and* $f(b) \leq b$.

Then f *has a fixed point.*

Proof. We apply Theorem 3.4 with $E_i = \mathbf{R}$, $X_i = [a_i, b_i], X = [a, b]$, $\phi_i = f_i - \pi_i$, and $B_i(\phi_i(x), \pi_i(x) - y_i) = \phi_i(x)(\pi_i(x) - y_i), x \in [a, b], y_i \in [a_i, b_i], i \in I$. By assumption (i), for each $i \in I$ and $y_i \in [a_i, b_i]$, the function $x \longmapsto B_i(\phi_i(x), \pi_i(x) - y_i)$ is upper semicontinuous on $[a, b]$. By definition, for any $x \in [a, b]$, the function $y_i \longmapsto B_i(\phi_i(x), \pi_i(x) - y_i)$ is linear hence quasiconvex on $[a_i, b_i]$.

To see that condition (ii) of Theorem 3.4 is satisfied, for any given $i \in I$, let $x \in [a, b]$ be such that $f_i(x) \neq \pi_i(x)$. If $x = a$, assumption (ii) implies that $a_i < f_i(a)$. Choose any $y_i \in (a_i, \min\{f_i(a), b_i\})$, so that $(f_i(a) - a_i)(a_i - y_i) < 0$. Similarily, if $x = b$ then $f_i(b) < b_i$. Choose any $y_i \in (\max\{f_i(b), a_i\}, b_i)$, so that $(f_i(b) - b_i)(b_i - y_i) < 0$. Finally, if $a < x < b$, then choose any $y_i \in (\max\{f_i(x), a_i\}, \pi_i(x))$ provided $f_i(x) < \pi_i(x)$, or choose any $y_i \in (\pi_i(x), \min\{f_i(x), b_i\})$ provided $f_i(x) > \pi_i(x)$. In both cases $(f_i(x) - \pi_i(x))(\pi_i(x) - y_i) < 0$. This ends the proof. □

4. Concluding Remarks

The results presented above extend in a natural way to the multivalued case. Also, one could very well formulate similar theorems based on classical fixed point theorems for multifunctions other than the Browder-Fan fixed point theorem (e.g. the Fan-Glicksberg-Kakutani fixed point theorem) or other classical coincidence properties.

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