ON THE WEAK INVARIANCE PRINCIPLE
FOR RANGES OF RECURRENT RANDOM
WALKS WITH INFINITE VARIANCE

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1. Introduction

Let \( \{X_k : k = 1, 2, \cdots\} \) be a sequence of independent, identically
distributed integer-valued random variables with common distribution
function \( F \). Throughout this paper we assume that

(A1) \( F \) belongs to the domain of attraction of a strictly \( \alpha \)-stable dis-
    tribution with \( 1 < \alpha \leq 2 \),

(A2) \( EX_1 = 0 \),

(A3) \( E \exp(iuX_1) = 1 \) if and only if \( u \) is a multiple of \( 2\pi \).

We note that \( \{S_n\} \) is an aperiodic recurrent random walk, where \( S_0 = 0 \)
and \( S_n = \sum_{k=1}^n X_k \). Let \( \varphi(u) = E \exp(iuX_1) \). Then it is well-
known that

\[
|\varphi(u)| = \exp\{-|u|^\alpha l(1/|u|)\} \quad \text{for} \quad |u| \leq \pi,
\]

where \( l(x) \) is a slowly varying function at infinity. Furthermore if we
choose \( a_n \) so that

\[
\frac{\alpha}{a_n^\alpha} = \frac{1}{n}
\]

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for each $n$, then $Y^{(n)}(t) = S_{[nt]} / a_n$ converges weakly to a strictly \( \alpha \)-stable process $Y(t)$, where $[x]$ denotes the greatest integer not exceeding $x$ (e.g. see page 345 of [1]).

The range $R_n$ of random walk $\{S_k\}$ and the range $\Lambda(t)$ of stable process $Y$ are defined as follows:

$$R_n = \text{the cardinality of } \{S_0, S_1, \ldots, S_n\}$$

and

$$\Lambda(t) = m \{ Y(s) : 0 \leq s \leq t \},$$

where "$m$" denotes the Lebesgue measure on $\mathbb{R}^1$. Set $\Lambda^{(n)}(t) = R_{[nt]} / a_n$. The aim of the present work is to prove weak convergence of $\Lambda^{(n)}$ to $\Lambda$. In fact, we obtain the existence of $\tilde{\Lambda}^{(n)}$ and $\tilde{\Lambda}$, versions of $\Lambda^{(n)}$ and $\Lambda$, respectively, such that $\tilde{\Lambda}^{(n)}(t)$ converges to $\tilde{\Lambda}(t)$ uniformly on $[0, T]$ in $L^2$-sense for any $T > 0$.

Le Gall and Rosen [6] obtained various limit theorems for the range of $d$-dimensional random walk in the domain of attraction of a stable distribution of index $\alpha$. Their results depend on the value of the ratio $\alpha/d$. That is, for the case $\alpha/d \leq 1$, strong law of large numbers and central limit theorems hold and for the case $\alpha > d = 1$ which we are concerned with in this work, $R_n/a_n$ converges in distribution to $\Lambda(1)$. In this work, we extend their result and prove the weak convergence of $\Lambda^{(n)}$ to $\Lambda$. Borodin [3] obtained a weaker result for the similar question for recurrent random walks with finite variance.

Now we state our main result, whose proof is given in Section 2.

**Theorem.** Under the assumptions (A1), (A2) and (A3), there exist processes $\tilde{Y}^{(n)}$ and $\tilde{Y}$ in $D[0, \infty)$ equipped with Skorokhod metric satisfying the following conditions;

(i) $\tilde{Y}^{(n)} =_{\mathcal{D}} Y^{(n)}$, $\tilde{Y} =_{\mathcal{D}} Y$,

(ii) $\tilde{Y}^{(n)}$ converges to $\tilde{Y}$ a.s. in $D[0, \infty)$, and

(iii) for each $T > 0$ and positive integer $m$,

$$E \left[ \sup_{0 \leq t \leq T} \left| \tilde{\Lambda}^{(n)}(t) - \tilde{\Lambda}(t) \right|^{2m} \right] \longrightarrow 0 \quad \text{as } n \to \infty,$$

where $\tilde{\Lambda}^{(n)}$ and $\tilde{\Lambda}$ are defined with respect to $\tilde{Y}^{(n)}$ and $\tilde{Y}$, respectively and "$=_{\mathcal{D}}$" means that two processes have the same finite dimensional distributions.
2. Proof of Main Result

Recall that we assume (A1), (A2) and (A3). Throughout this work, we denote $P_0$ and $E_0$ by $P$ and $E$, respectively.

We present the proof of the Theorem in this section. Since the construction of $\tilde{Y}^{(n)}$ and $\tilde{Y}$ satisfying parts (i) and (ii) of the Theorem are well-known (e.g. see chapter 1 of [7]), it suffices to establish part (iii) of the Theorem. Therefore we may abuse our notation and use $Y^{(n)}$, $Y$, $\Lambda^{(n)}$ and $\Lambda$ for $\tilde{Y}^{(n)}$, $\tilde{Y}$, $\tilde{\Lambda}^{(n)}$ and $\tilde{\Lambda}$, respectively throughout the remainder of the work. Essentially, the proof of our assertion amounts to estimating

\begin{equation}
E \left[ \left( \Lambda^{(n)}(t) - \Lambda(t) \right)^{2m} \right],
\end{equation}

since a simple monotonicity argument using also continuity of $\Lambda(t)$ implies our assertion if (2.1) converges to zero. Le Gall and Rosen [6] showed that $\tilde{\Lambda}^{(n)}(t)$ converges to $\tilde{\Lambda}(t)$ in $L^1$-sense, but their technique doesn't work in general. To deal with (2.1), we express the ranges of random walks and stable processes using their local times, respectively. The local time $N(n, x)$ of random walk $\{S_k\}$ is defined by

$$N(n, x) = \text{the number of } \{ 0 \leq k \leq n : S_k = x \}.$$  

Let

$$L^{(n)}(t, x) = \frac{a_n}{n} N([nt], [xa_n])$$

and

$$W_n(t) = \{ x \in \mathbb{R}^1 : L^{(n)}(t, x) > 0 \}.$$  

Then we note that

$$\Lambda^{(n)}(t) = \frac{1}{a_n} \sum_{k \in \mathbb{Z}} \chi_{\{N([nt], k) > 0\}}$$

$$= \int_{\mathbb{R}^1} \chi_{W_n(t)}(x) \, dx.$$  

For a stable process $Y(t)$ of index $1 < \alpha \leq 2$, it is well-known that there exists a version of local time $\{L(t, x)\}$ which is jointly continuous in
(t,x) and satisfies the so-called occupation time density formula, that is, for any Borel set \( B \),

\[
\int_B L(t,x) \, dx = \int_0^t \chi_B(Y(s)) \, ds \quad \text{a.s.}
\]

The existence and joint continuity of \( L(t,x) \) were proved by Trotter [8] for Brownian motion and by Boylan [4] for stable processes of index \( \alpha > 1 \). Moreover, Kang and Wee [5] proved that as \( n \to \infty \),

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^1} \left| L^{(n)}(t,x) - L(t,x) \right| \longrightarrow 0 \quad \text{in} \ L^2.
\]

We provide an useful expression of \( \Lambda(t) \) in terms of local time \( L(t,x) \) in Lemma 2.1, and then apply the result of [5] to estimate (2.1).

**Lemma 2.1.** For each \( t \geq 0 \),

\[
\Lambda(t) = \int_{\mathbb{R}^1} \chi_{W(t)}(x) \, dx \quad \text{a.s.}
\]

where \( W(t) = \{ x \in \mathbb{R}^1 : L(t,x) > 0 \} \).

**Proof.** We write

\[
\Lambda(t) = m(G(t)) + m(W(t)),
\]

where

\[
G(t) = \{ x \in \mathbb{R}^1 : Y(s) = x \text{ for some } s \in [0,t], \ L(t,x) = 0 \}.
\]

Let

\[
\tau_x = \inf\{ s \geq 0 : Y(s) = x \},
\]

\[
\hat{Y}(s) = Y(s + \tau_x) - x,
\]

and \( \hat{L}(s,y) \) be the local time of \( \hat{Y} \). The strong Markov property implies that

\[
E \left[ m(G(t)) \right] = \int_{\mathbb{R}^1} P \left( \tau_x \leq t, \ \hat{L}(t - \tau_x, 0) = 0 \right) \, dx
\]

\[
= \int_{\mathbb{R}^1} \int_0^t P \left( \hat{L}(t-s, 0) = 0 \right) P(\tau_x \in ds) \, dx
\]

\[
= 0,
\]

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where the last equality follows from the definition of $\hat{L}(t,0)$ as a continuous additive functional with support $\{0\}$ (see page 216 of [2]).

**Lemma 2.2.** For each $t \geq 0$ and positive integer $m$,

$$E \left[ \left( \Lambda^{(n)}(t) - \Lambda(t) \right)^{2m} \right] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$ 

**Proof.** Recall that

$$W_n(t) = \{ x \in \mathbb{R}^1 : L^{(n)}(t,x) > 0 \}$$

and

$$W(t) = \{ x \in \mathbb{R}^1 : L(t,x) > 0 \}.$$ 

For each $K > 0$, let

$$\Lambda_K^{(n)}(t) = \int_{-K}^{K} \chi_{W_n(t)}(x) \, dx$$

and

$$\Lambda_K(t) = \int_{-K}^{K} \chi_{W(t)}(x) \, dx.$$ 

Then

$$E \left[ \left( \Lambda^{(n)}(t) - \Lambda_K^{(n)}(t) \right)^{2m} \right]$$

$$\leq E \left[ \Lambda^{(n)}(t)^{2m} \cdot \chi_{\{ \sup_{0 \leq t \leq [nt]} |S_t| > Ka_n \}} \right]$$

$$\leq \left\{ E \left[ \left( \Lambda^{(n)}(t) \right)^{4m} \right] \right\}^{1/2} \cdot \left\{ P \left( \sup_{0 \leq t \leq [nt]} |S_t| > Ka_n \right) \right\}^{1/2}$$

Weak convergence of $Y^{(n)}$ to $Y$ implies that for any $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that

$$\sup_n P \left( \sup_{0 \leq t \leq [nt]} |S_t| > Ka_n \right) < \varepsilon.$$
It follows from [6] that any finite moment of $\Lambda^{(n)}(t)$ is bounded uniformly in $n$. Thus by (2.5), we may choose $K$ large so that (2.4) is sufficiently small for all $n$ large. For $1 < \alpha < 2$, it is easy to see that for any $u > 0$, 

$$E[\exp(u\Lambda(t))] < \infty$$

without assuming the symmetry of $Y(t)$, by modifying the argument in Lemma 4.1 of [9]. Thus

$$E[\Lambda(t)^{2m}] < \infty,$$

which is obvious for $\alpha = 2$. This enables us to have that for any $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that

$$E[(\Lambda(t) - \Lambda_K(t))^{2m}] < \varepsilon.$$

Now we fix $K$ large enough, and observe that

(2.6)

$$E\left[\left(\Lambda_K^{(n)}(t) - \Lambda_K(t)\right)^{2m}\right]$$

$$= E\left[\left(\int_{-K}^K \chi_{W_n(t) \cap W(t)}(x) \, dx - \int_{-K}^K \chi_{W_n(t) \cap W(t)}(x) \, dx\right)^{2m}\right]$$

$$\leq 2^{4m-2}K^{2m-1} E\left[\int_{-K}^K \chi_{W_n(t) \cap W(t)}(x) \, dx\right]^2$$

$$+ 2^{4m-2}K^{2m-1} E\left[\int_{-K}^K \chi_{W_n(t) \cap W(t)}(x) \, dx\right].$$

Let $Y[0; t] = \{Y(s) : 0 \leq s \leq t\}$, $cl(Y[0; t])$ be it’s closure, and $U_\delta(t)$ be the $\delta$-neighborhood of $cl(Y[0; t])$. Then as $\delta \to 0$,

$$m(U_\delta(t)) \to m(cl(Y[0; t])) = m(Y[0; t]) \quad a.s.$$

and part (ii) of the Theorem implies that for fixed $\delta > 0$,

(2.7) 

$$W_n(t) \subset U_\delta(t) \quad a.s.$$
for all sufficiently large $n$. Now fix $\delta > 0$ so that $E \left[ m \left( U_\delta(t) \cap Y[0; t]^c \right) \right]$ is sufficiently small. Then by (2.7) and (2.3), for $n$ large,

\[
E \left[ \int_{-K}^{K} \chi_{W_n(t) \cap W(t)^c}(x) \, dx \right] \\
\leq E \left[ \int_{-K}^{K} \chi_{U_\delta(t) \cap W(t)^c}(x) \, dx \right] \\
\leq E \left[ \int_{-K}^{K} \chi_{Y[0; t] \cap W(t)^c}(x) \, dx \right] + E \left[ m \left( U_\delta(t) \cap Y[0; t]^c \right) \right] \\
\leq E \left[ m \left( G(t) \right) \right] + E \left[ m \left( U_\delta(t) \cap Y[0; t]^c \right) \right] \\
= E \left[ m \left( U_\delta(t) \cap Y[0; t]^c \right) \right].
\]

Hence the first summand of (2.6) can be made arbitrarily small for $n$ sufficiently large. To estimate the second summand of (2.6), observe that for any $\eta > 0$,

\[
P \left( L(t, x) > 0, \ L^{(n)}(t, x) = 0 \right) \leq P \left( 0 < L(t, x) < \eta \right) \\
+ P \left( \left| L^{(n)}(t, x) - L(t, x) \right| \geq \eta \right).
\]

Use (2.2) and bounded convergence theorem to show that

\[
E \left[ \int_{-K}^{K} \chi_{W_n(t) \cap W(t)^c}(x) \, dx \right] \\
= \int_{-K}^{K} P \left( L(t, x) > 0, \ L^{(n)}(t, x) = 0 \right) \, dx \\
\rightarrow 0
\]
as $n$ goes to infinity. \hfill \Box

Proof of the Theorem. Fix $h > 0$, which will be chosen later. Let $0 = t_0 < t_1 < \cdots < t_k \leq t_{k+1} = T$ be a partition of $[0, T]$ such that
\[ t_j - t_{j-1} = h \text{ for all } 1 \leq j \leq k \text{ and } k = \lfloor T/h \rfloor. \text{ Observe that by simple monotonicity of } \Lambda^{(n)} \text{ and } \Lambda, \]
\[
E \left[ \sup_{0 \leq t \leq T} \left| \Lambda^{(n)}(t) - \Lambda(t) \right|^{2m} \right] \\
\leq 3^{2m-1} 2^{2m} E \left[ \max_{0 \leq j \leq k} |\Lambda(t_{j+1}) - \Lambda(t_j)|^2 \right] \\
+ 3^{2m-1} (2^{2m} + 1) \sum_{j=0}^{k+1} E \left[ \left| \Lambda^{(n)}(t_j) - \Lambda(t_j) \right|^2 \right].
\]

By the almost sure continuity of the mapping \( t \mapsto \Lambda(t) \) and dominate convergence theorem, for given \( \varepsilon > 0 \), we can choose \( h \) so that the first term of (2.8) is less than \( \varepsilon/2 \). Then Lemma 2.2 implies that the second term of (2.8) can be made arbitrarily small if \( n \) is large enough, which completes the proof. \qed

References

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