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EXISTENCE THEOREMS OF AN OPERATOR-VALUED FEYNMAN INTEGRAL AS AN $\mathcal{L}(L_1, C_0)$ THEORY

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1. Introduction and Preliminaries

The existence of an operator-valued function space integral as an operator on $L_p(\mathbb{R})$ $(1 \le p \le 2)$ was established for certain functionals which involved the Lebesgue measure [1,2,6,7]. Johnson and Lapidus showed the existence of the integral as an operator on $L_2(\mathbb{R})$ for certain functionals which involved any Borel measures [5]. J. S. Chang and Johnson proved the existence of the integral as an operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$ for certain functionals involving some Borel measures [3]. K. S. Chang and K. S. Ryu showed the existence of the integral as an operator from $L_p(\mathbb{R})$ to $L_{p'}(\mathbb{R})$ for certain functionals involving some Borel measures [4].

In this paper, we prove the existence theorem for the integral as an operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$ for the functionals $G(x) = \exp(\int_{(0,t)} \theta(s, x(s)))\, d\eta(s)$ and we express the integral as a simple generalized Dyson series. Also we establish the generalized Dyson series for a functional which involves a sequence of Borel measures and potentials.

Let $\mathbb{R}, \mathbb{C}, \mathbb{C}^+$ and $\tilde{\mathbb{C}}^+$ denote the set of all real numbers, all complex numbers, all complex numbers with positive real part and all nonzero complex numbers with nonnegative real part, respectively. $C_0(\mathbb{R})$ will denote the space of $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ which vanish

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at ∞ with the supremum norm. $L_1(\mathbb{R})$ is the space of Borel measurable, $\mathbb{C}$-valued functions ψ on $\mathbb{R}$ such that $|\psi|$ is integrable with respect to the Lebesgue measure m on $\mathbb{R}$ with the norm $\|\psi\|_1 = \int |\psi|\, dm$. $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ will denote the space of bounded linear operators from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$. Let $\tilde{M}(0,t)$ denote the space of complex Borel measures η on the interval $(0,t)$ which satisfy the following conditions;

(1) If μ is the continuous part of η, the Radon-Nikodym derivative $\frac{d|\mu|}{dm}$ exists and is essentially bounded,where m is the Lebesgue measure on $(0,t)$.

(2) $\eta = \sum_{j=1}^{k} w_j \delta_{\tau_j} + \mu$, where μ is the continuous part of η and δ_{τ_j} is the Dirac measure at $\tau_j \in (0,t)$, $0 < \tau_1 < \cdots < \tau_k < t$ and $w_j \in \mathbb{C}$ for $j = 1, 2, \cdots, k$.

Let $r \in (2, \infty]$ and $\eta \in \tilde{M}(0,t)$. Let $L_{1r:\eta}([0,t] \times \mathbb{R}) \equiv L_{1r:\eta}$ be the space of all Borel measurable $\mathbb{C}$-valued functions θ on $[0,t] \times \mathbb{R}$ such that

$$\| \theta \|_{1r:\eta} \equiv \left\{ \int_{(0,t)} \| \theta(s,\cdot) \|_1^r \, d|\eta|(s) \right\}^{\frac{1}{r}} \tag{1.1}$$

is finite. If θ is in $L_{1r:\eta}$ and $\eta = \mu + \nu$ is the Lebesgue decomposition, it is not difficult to show that $\theta \in L_{1r:\mu} \cap L_{1r:\nu}$. Let $\eta \in \tilde{M}(0,t)$. A Borel measurable $\mathbb{C}$-valued function θ on $[0,t] \times \mathbb{R}$ is said to belong to $L_{\infty 1:\eta}$ if

$$\|\theta\|_{\infty 1:\eta} = \int_{(0,t)} \|\theta(s,\cdot)\|_\infty \, d|\eta|(s) \tag{1.2}$$

is finite. For $\lambda \in \tilde{\mathbb{C}}^+, \psi \in L_1(\mathbb{R})$ and a positive real number s, let

$$(C_{\lambda/s}\psi)(\xi) = \left(\frac{\lambda}{2\pi s}\right) \int_{\mathbb{R}} \psi(u) \exp\left(-\frac{\lambda(u-\xi)^2}{2s}\right) dm(u). \tag{1.3}$$

Then $C_{\lambda/s}$ is in $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ and $\|C_{\lambda/s}\| \le (|\lambda|/2\pi s)^{\frac{1}{2}}$ [7]. And as a function of λ, $C_{\lambda/s}$ is analytic in $\mathbb{C}^+$ and is weakly continuous in $\tilde{\mathbb{C}}^+$

[7]. Let θ be in $L_1(\mathbb{R})$ and let M_θ be the operator of multiplication from $C_0(\mathbb{R})$ to $L_1(\mathbb{R})$ given by $M_\theta\psi = \psi\theta$. Then M_θ is in $\mathcal{L}(C_0(\mathbb{R}), L_1(\mathbb{R}))$ and $\|M_\theta\| \le \|\theta\|_1$ [3]. It will be convenient to let $\theta(s)$ denote $M_{\theta(s,\cdot)}$ for θ in $L_{1r:\eta}$. Let $0 < k < 1$ be given and n in $\mathbb{N}$. For $0 < s_1 < \cdots < s_n < t$, we can easily check that

(1.4)
$$\int_0^t \int_0^{s_n} \cdots \int_0^{s_2} \big(s_1(s_2 - s_1)\cdots(t - s_n)\big)^{-k}\, ds_1 \cdots ds_n$$
$$= \frac{t^{n-(n+1)k}[\Gamma(1-k)]^{n+1}}{\Gamma((n+1)(1-k))},$$

where Γ is the gamma function.

As we continue, we will need to write

$$[w_1\theta(\tau_1, x(\tau_1)) + \cdots + w_m\theta(\tau_m, x(\tau_m)) + \theta(s, x(s))]^n$$

as a product of monomials. However, we will need more refined breakdown of the sum. It will be convenient to introduce a prime notation on sum like $\sum{}'_{q_0+q_1+\cdots+q_{m-k}=n}$: this sum is to be over integers $q_0, q_1, \cdots, q_{m-k}$, where $q_0 \ge 0, q_1 \ge 1, \cdots, q_{m-k} \ge 1$ and $q_0 + \cdots + q_{m-k} = n$. Using this notation, we have the following equality [3].

(1.5)
$$\left[\sum_{j=1}^m w_j\theta(\tau_j, x(\tau_j)) + \theta(s, x(s))\right]^n$$
$$= \sum_{k=0}^m \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum{}'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!}{q_0!\, q_1! \cdots q_{m-k}!}$$
$$[w_{z_1}\theta(\tau_{z_1}, x(\tau_{z_1}))]^{q_1}\cdots[w_{z_{m-k}}\theta(\tau_{z_{m-k}}, x(\tau_{z_{m-k}}))]^{q_{m-k}}[\theta(s, x(s))]^{q_0}.$$

2. A simple Generalized Dyson Series

Let $C[0,t]$ be the space of continuous functions on $[0,t]$ and the Wiener space, $C_0[0,t]$, will consist of those x in $C[0,t]$ such that $x(0) = 0$. Integration over $C_0[0,t]$ will always be with respect to the Wiener measure m_w.

DEFINITION 1. Let F be a functional from $C[0,t]$ to $\mathbb{C}$. Given $\lambda > 0, \psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$(2.1) \qquad (I_\lambda(F)\psi)(\xi) = \int_{C_0[0,t]} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)\, dm_w(x).$$

If $I_\lambda(F)\psi$ is in $C_0(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next suppose that there exists $\lambda_0 (0 < \lambda_0 \le \infty)$ such that $I_\lambda(F)$ exists for all λ in $(0, \lambda_0)$ and further suppose that there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}^+_{\lambda_0} \equiv \{\lambda \in \mathbb{C} \,|\, Re\lambda > 0, |\lambda| < \lambda_0\}$ and agree with $I_\lambda(F)$ on $(0, \lambda_0)$. Then this $\mathcal{L}$-valued function is denoted by $I^{an}_\lambda(F)$ and is called the operator-valued analytic Wiener integral of F associated with λ. Finally, let q be in $\mathbb{R}$ with $0 < |q| < \lambda_0$. Suppose there exists an operator $J^{an}_q(F)$ in $\mathcal{L}$ such that for every ψ in $L_1(\mathbb{R})$, $J^{an}_q(F)\psi$ is the weak limit of $I^{an}_\lambda(F)\psi$ as $\lambda \to -iq$ through $\mathbb{C}^+_{\lambda_0}$. Then $J^{an}_q(F)$ is called the operator-valued Feynman integral of F associated with q.

LEMMA 1. *Let $\eta \in \tilde{M}(0,t)$ and $\theta \in L_{1r:\eta}$. Let*

$$(2.2) \qquad F(y) = \int_{(0,t)} \theta(s, y(s))\, d\eta(s)$$

for any $y \in C[0,t]$ for which the integral exists. Then, for every $\lambda > 0, F(\lambda^{-\frac{1}{2}}x + \xi)$ is defined for $m_w \times m$-a.e. (x,ξ) in $C_0[0,t] \times \mathbb{R}$.

Proof. We first show that for every $\lambda > 0$ and $m_w \times m$ - a.e. (x, ξ), $\theta(s, \lambda^{-\frac{1}{2}}x(s) + \xi)$ is defined. Let $H_\lambda : (0,t) \times C_0[0,t] \times \mathbb{R} \to (0,t) \times \mathbb{R}$ be defined by $H_\lambda(s, x, \xi) = (s, \lambda^{-\frac{1}{2}}x(s) + \xi)$. Then $\theta \circ H_\lambda$ is certainly Borel measurable. Let

$$N := \{(s,v) \in (0,t) \times \mathbb{R} | \, \theta(s,v) \text{ fails to be defined}\}$$

Since $\theta \in L_{1r:\eta}$, N is $|\eta| \times m$-null by Fubini theorem. Let $\lambda > 0$ be given. Then it suffices to show that $H^{-1}_\lambda(N)$ is $|\eta| \times m_w \times m$-null. Accordingly, we consider a (s,ξ)-section $[H^{-1}_\lambda(N)]^{(s,\xi)}$:

$$(2.3) \qquad [H^{-1}_\lambda(N)]^{(s,\xi)} = \{x \in C_0[0,t] | (s, \lambda^{-\frac{1}{2}}x(s) + \xi) \in N\}$$
$$= \{x \in C_0[0,t] | x(s) \in \lambda^{\frac{1}{2}}[N^{(s)} - \xi]\},$$

where $N^{(s)} := \{u \in \mathbb{R} | (s,u) \in N\}$. Now, since N is $|\eta| \times m$-null, it follows that the set $\lambda^{\frac{1}{2}}[N^{(s)} - \xi]$ is m-null (and so for $|\eta| \times m$-a.e. it is m-null). Hence, by the Fubini theorem, $H_\lambda^{-1}(N)$ is $|\eta| \times m_w \times m$-null. And so, for every $\lambda > 0$ and $m_w \times m$-a.e.(x, ξ),

(2.4)

$$\begin{aligned}
&\int_{(o,t)} \left(\int_{C_0[0,t]} |\theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi)| dm_w(x) \right) d|\eta|(s) \\
&\overset{(1)}{\leq} \int_{(0,t)} \left(\frac{\lambda}{2\pi s} \right)^{\frac{1}{2}} \|\theta(s, \cdot)\|_1 d|\eta|(s) \\
&\overset{(2)}{=} \left(\frac{\lambda}{2\pi} \right)^{\frac{1}{2}} \left\{ \int_{(0,t)} s^{-\frac{1}{2}} \|\theta(s, \cdot)\|_1 d|\mu|(s) + \sum_{j=1}^{k} \tau_j^{-\frac{1}{2}} |w_j| \|\theta(\tau_j, \cdot)\|_1 \right\} \\
&\overset{(3)}{\leq} \left(\frac{\lambda}{2\pi} \right)^{\frac{1}{2}} \left\{ \|\theta\|_{1r:\mu} \left\| \frac{d|\mu|}{dm} \right\|_\infty^{\frac{1}{r'}} \left(\int_{(0,t)} s^{-\frac{r'}{2}} ds \right)^{\frac{1}{r'}} \right. \\
&\quad \left. + \sum_{j=1}^{k} \tau_j^{-\frac{1}{2}} |w_j| \|\theta(\tau_j, \cdot)\| \right\} < \infty.
\end{aligned}$$

Step (1) results from (1.3). Since $\eta \in \tilde{M}(0,t)$ we obtain step (2). We deduce step (3) directly from the *Hölder's inequality*. Hence, by the Fubini Theorem and (2.3), we have

$$\int_{C_0[0,t]} \left(\int_{(0,t)} |\theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi)| \, d|\eta|(s) \right) dm_w(x) < \infty \quad \text{for } \eta \text{ in } \tilde{M}(0,t). \tag{2.5}$$

Thus, for m_w - a.e.x in $C_0[0,t]$ and for all ξ in $\mathbb{R}$,

$$\int_{(0,t)} |\theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi)| \, d|\eta|(s) \tag{2.6}$$

exists. Hence, for $m_w \times m$-a.e.(x, ξ) in $C_0[0,t] \times \mathbb{R}$,

$$F(\lambda^{-\frac{1}{2}} x + \xi) = \int_{(0,t)} \theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi) \, d\eta(s)$$

is defined. The lemma is proved. □

Througtout this section, let $\eta \in \tilde{M}(0,t)$ and $\theta \in L_{1r:\eta}$. Set

(2.7)
$$F_n(x) := \left(\int_{(0,t)} \theta(s, x(s))\, d\eta(s)\right)^n, \; x \in C[0,t], \quad n = 0, 1, 2, \cdots .$$

Here, if $n = 0$, from the definition, we have $I_\lambda(F_0) = C_{\lambda/t}$.

THEOREM 1. (Finitely supported) *Let* $\eta = \sum_{j=1}^{m} w_j \delta_{\tau_j} + \mu$ *where* δ_{τ_j} *is the Dirac measure at* $\tau_j \in (0,t), 0 < \tau_1 < \cdots < \tau_m < t$ *and* $w_j \in \mathbb{C}$ *for* $j = 1, 2, \cdots, m$. *Suppose that* $\theta(\tau_j, \cdot), j = 1, 2, \cdots, m$, *are essentially bounded. Then the operators* $I_\lambda^{an}(F_n)$ *and* $J_q^{an}(F_n)$ *exist for all* $\lambda \in \mathbb{C}^+$ *and all real* $q \neq 0$, *respectively. Further for* $\lambda \in \mathbb{C}^+$, $\psi \in L_1(\mathbb{R})$ *and* $\xi \in \mathbb{R}$,

(2.8)
$$\begin{aligned}
&(I_\lambda^{an}(F_n)\psi)(\xi) \\
&= \sum_{k=0}^{m} \sum_{1 \le z_1 < \cdots < z_{m-k} \le m} \sum{}'_{q_0 + q_1 + \cdots + q_{m-k} = n} \frac{n! w_{z_1}^{q_1} \cdots w_{z_{m-k}}^{q_{m-k}}}{q_1! \cdots q_{m-k}!} \\
&\quad \Bigg[\sum_{j_1 + \cdots j_{m-k+1} = q_0} \int_{\Delta_{q_0; j_1, \cdots, j_{m-k+1}}^{z_1, \cdots, z_{m-k}}} ((L_0 \circ L_1 \circ \cdots \\
&\qquad\qquad \cdots \circ L_{m-k})\psi)(\xi)\, d \mathop{\times}_{i=1}^{q_0} \mu(s_i) \Bigg],
\end{aligned}$$

where

(2.9)
$$\begin{aligned}
&\Delta_{q_0; j_1, \cdots, j_{m-k+1}}^{z_1, \cdots, z_{m-k}} \\
&= \{(s_1, \cdots, s_{q_0}) \in (0,t)^{q_0} \mid 0 < s_1 < \cdots < s_{j_1} < \tau_{z_1} \\
&\qquad < s_{j_1+1} < \cdots < s_{j_1 + \cdots + j_{m-k}} < \tau_{z_{m-k}} \\
&\qquad < s_{j_1 + \cdots + j_{m-k} + 1} < \cdots < s_{q_0} < t\}
\end{aligned}$$

and for $(s_1, \cdots, s_{q_0}) \in \Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}$ *and* $\alpha \in \{0,1,\cdots,m-k\}$

(2.10)
$$L_\alpha = \theta(\tau_{z_\alpha})^{q_\alpha} \circ C_{\lambda/(s_{j_1+\cdots+j_\alpha+1}-\tau_{z_\alpha})} \circ \theta(s_{j_1+\cdots+j_\alpha+1}) \circ \cdots \circ \theta(s_{j_1+\cdots+j_{\alpha+1}}) \circ C_{\lambda/(\tau_{z_{\alpha+1}}-s_{j_1+\cdots+j_{\alpha+1}})}.$$

(It is convenient to let $\theta(\tau)^q$ *denote the operator of multiplication by* $[\theta(\tau,\cdot)]^q$*, that is,* $\theta(\tau)^q = M_{[\theta(\tau,\cdot)]^q}$*. We use the conventions* $\tau_0 = 0, \tau_{m+1} = t$ *and* $\theta(\tau_0)^{q_0} = 1$*, where 1 is the inclusion map.)*

For all real $q \neq 0$*,* $(J^{an}_q(F_n)\psi)(\xi)$ *is given by the right hand side of (2.8) with* $\lambda = -iq$*. Finally we have for* $\lambda \in \mathbb{C}^+$*,*

$$\|I^{an}_\lambda(F_n)\| \leq B_n(|\lambda|) \tag{2.11}$$

where

(2.12)
$$\begin{aligned} &B_n(|\lambda|) \\ &:= (n!)^{\frac{1}{r'}} \left[\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}} \vee \left(\frac{|\lambda|}{2\pi}\right)^{\frac{1}{2}}\right] \left[\min_{1\leq j\leq m}(\tau_j - \tau_{j-1})\right]^{-\frac{m+1}{2}} \\ &\Gamma(l(1-\frac{r'}{2}))^{-\frac{m+1}{r'}} \Gamma(1-\frac{r'}{2})^{\frac{m+1}{r'}} \left[\sum_{j=1}^m |w_j|(\|\theta(\tau_j,\cdot)\|_\infty \vee \|\theta(\tau_j,\cdot)\|_1)\right. \\ &\left. + \left(\sum_{j=1}^{m+1}(\tau_j-\tau_{j-1})^{1-\frac{r'}{2}}\right)^{\frac{1}{r'}} \left(\frac{|\lambda|}{2\pi}\right)^{\frac{1}{2}} \|\theta\|_{1r;\mu} \left\|\frac{d|\mu|}{dm}\right\|_\infty^{\frac{1}{r'}} \Gamma(1-\frac{r'}{2})^{\frac{1}{r'}}\right]^n, \end{aligned}$$

where l *is a positive integer such that* $\Gamma(l(1-\frac{r'}{2}))$ *is the minimum value of* $\{\Gamma(i(1-\frac{r'}{2}))|i \in \mathbb{N}\}$*,* Γ *is the Gamma function,* $\frac{1}{r} + \frac{1}{r'} = 1$*, and the notation* $a \vee b$ *means the maximum value of* a *and* b*. The inequality (2.11) also holds for* $J^{an}_q(F_n)$ *with* $|\lambda|$ *replaced by* $|q|$*.*

Proof. The proof of (2.8) is essentially same as that of Theorem 2.1 in [3]. Thus we will only show (2.12). Let λ be in $\mathbb{C}^+$ and $\psi \in L_1(\mathbb{R})$.

Then

$$
\begin{aligned}
&\|I_\lambda^{an}(F)\psi\|_\infty \\
&\overset{(1)}{\le} \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum{}'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!|w_{z_1}^{q_1}|\cdots|w_{z_{m-k}}^{q_{m-k}}|}{q_1!\cdots q_{m-k}!} \\
&\quad \left(\frac{|\lambda|}{2\pi}\right)^{\frac{q_0+m-k+1}{2}} \sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \\
&\quad \prod_{j=1}^{m-k}\Big[\|\theta(\tau_{z_j},\cdot)\|_\infty \vee \|\theta(\tau_{z_j},\cdot)\|_1\Big]^{q_j} \prod_{i=1}^{q_0}\|\theta(s_i,\cdot)\|_1 \\
&\quad [s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{1}{2}}\|\psi\|_1\, d\overset{q_0}{\underset{i=1}{\times}}|\mu|(s_i) \\
&\overset{(2)}{\le} \|\psi\|_1 \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum{}'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!|w_{z_1}^{q_1}|\cdots|w_{z_{m-k}}^{q_{m-k}}|}{q_1!\cdots q_{m-k}!} \\
&\quad \left(\frac{|\lambda|}{2\pi}\right)^{\frac{q_0+m-k+1}{2}} \prod_{j=1}^{m-k}\Big[\|\theta(\tau_{z_j},\cdot)\|_\infty \vee \|\theta(\tau_{z_j},\cdot)\|_1\Big]^{q_j} \left\|\frac{d|\mu|}{dm}\right\|_\infty^{\frac{q_0}{r'}} \\
&\quad \left[\int_{\Delta_{q_0}}\prod_{i=1}^{q_0}\|\theta(s_i,\cdot)\|_1^r d\overset{q_0}{\underset{i=1}{\times}}|\mu|(s_i)\right]^{\frac{1}{r}} \Bigg[\sum_{j_1+\cdots+j_{m-k+1}=q_0}\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \\
&\quad [s_1\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(t-s_{q_0})]^{-\frac{r'}{2}} d\overset{q_0}{\underset{i=1}{\times}} m(s_i)\Bigg]^{\frac{1}{r'}} \\
&\overset{(3)}{\le} \|\psi\|_1\, n! \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum{}'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{|w_{z_1}^{q_1}|\cdots|w_{z_{m-k}}^{q_{m-k}}|}{q_1!\cdots q_{m-k}!} \\
&\quad \left(\frac{|\lambda|}{2\pi}\right)^{\frac{q_0+m-k+1}{2}} \prod_{j=1}^{m-k}\Big[\|\theta(\tau_{z_j},\cdot)\|_\infty \vee \|\theta(\tau_{z_j},\cdot)\|_1\Big]^{q_j} \\
&\quad \left(\frac{1}{q_0!}\right)^{\frac{1}{r}} \|\theta\|_{1r;\mu}^{q_0} \left\|\frac{d|\mu|}{dm}\right\|_\infty^{\frac{q_0}{r'}} \\
&\quad \left[\sum_{j_1+\cdots j_{m-k+1}=q_0} \prod_{l=1}^{m-k+1} \frac{(\tau_{z_l}-\tau_{z_{l-1}})^{j_l-(j_l+1)\frac{r'}{2}}[\Gamma(1-\frac{r'}{2})]^{j_l+1}}{\Gamma[(j_l+1)(1-\frac{r'}{2})]}\right]^{\frac{1}{r'}}
\end{aligned}
$$

$$\overset{(4)}{\leq} \|\psi\|_1 \, (n!)^{\frac{1}{r'}} \sum_{k=0}^{m} \sum_{1\leq z_1<\cdots<z_{m-k}\leq m} {\sum}'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!}{q_0!\cdots q_{m-k}!}$$

$$\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m-k+1}{2}} [\Gamma(1-\frac{r'}{2})]^{\frac{m-k+1}{r'}} \prod_{j=1}^{m-k+1} (\tau_{z_l}-\tau_{z_{l-1}})^{-\frac{1}{2}}$$

$$[\Gamma(l(1-\frac{r'}{2}))]^{-\frac{m-k+1}{r'}} \prod_{j=1}^{m-k} [|w_{z_j}|(\|\theta(\tau_{z_j},\cdot)\|_\infty \vee \|\theta(\tau_{z_j},\cdot)\|_1)]^{q_j}$$

$$\left[\left(\sum_{j=1}^{m-k+1} (\tau_{z_j}-\tau_{z_{j-1}})^{1-\frac{r'}{2}}\right)^{\frac{1}{r'}} \left(\frac{|\lambda|}{2\pi}\right)^{\frac{1}{2}} \|\theta\|_{1r:\mu} \left\|\frac{d|\mu|}{dm}\right\|_\infty^{\frac{1}{r'}} \left[\Gamma(1-\frac{r'}{2})\right]^{\frac{1}{r'}}\right]^{q_0}$$

$$\overset{(5)}{\leq} \|\psi\|_1 \, (n!)^{\frac{1}{r'}} \left[\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}} \vee \left(\frac{|\lambda|}{2\pi}\right)^{\frac{1}{2}}\right] [\min_{1\leq j\leq m} (\tau_j-\tau_{j-1})]^{-\frac{m+1}{2}}$$

$$\Gamma(l(1-\frac{r'}{2}))^{-\frac{m+1}{r'}} \left[\Gamma(1-\frac{r'}{2})\right]^{\frac{m+1}{r'}} \left[\left(\sum_{j=1}^{m+1} (\tau_j-\tau_{j-1})^{1-\frac{r'}{2}}\right)^{\frac{1}{r'}} \left(\frac{|\lambda|}{2\pi}\right)^{\frac{1}{2}}\right.$$

$$\left.\|\theta\|_{1r:\mu} \left\|\frac{d|\mu|}{dm}\right\|_\infty^{\frac{1}{r'}} \left[\Gamma(1-\frac{r'}{2})\right]^{\frac{1}{r'}} + \sum_{j=1}^{m} |w_j|(\|\theta(\tau_j,\cdot)\|_\infty \vee \|\theta(\tau_j,\cdot)\|_1)\right]^n.$$

Step (1) is obtained by (1.3). By the *Hölder's inequality* and "simplex trick" [5], we obtain step (2). Step (3) follows from (1.4). Since $0 < 1-\frac{r'}{2} < 1$, $0 < \Gamma(l(1-\frac{r'}{2})) < 1$, we obtain step (4). From equality (1.5), we have step (5). The rest of the proof follows the proof of Theorem 2.1 [3]. Therefore, the theorem is proved. □

From Theorem 1 and dominated convergence theorem, we obtain the following theorem. The function given in (2.13) is in a very important class of functions in Quantum Mechanics.

THEOREM 2. (A simple generalized Dyson series)
Let $\eta = \sum_{j=1}^{m} w_j \delta_{\tau_j} + \mu$, *where* δ_{τ_j} *is the Dirac measure at* $\tau_j \in (0,t), 0 < \tau_1 < \cdots < \tau_m < t$ *and* $w_j \in \mathbb{C}$ *for* $j = 1,2,\cdots,m$. *Suppose that* $\theta(\tau_j,\cdot)\,, j = 1,2,\cdots,m$, *are essentially bounded. Let* G *be defined on*

$C[0,t]$ by

$$G(x) = \exp\left\{ \int_{(0,t)} \theta(s, x(s))\, d\eta(s) \right\}. \tag{2.13}$$

Then the operators $I_\lambda^{an}(G)$ and $J_q^{an}(G)$ exist for all $\lambda \in \mathbb{C}^+$ and all non-zero real q, respectively. Further for $\lambda \in \mathbb{C}^+$,

(2.14)

$$\begin{aligned}
&(I_\lambda^{an}(G)\psi)(\xi) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{m} \sum_{1 \le z_1 < \cdots < z_{m-k} \le m} {\sum}'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{w_{z_1}^{q_1} \cdots w_{z_{m-k}}^{q_{m-k}}}{q_1! \cdots q_{m-k}!} \\
&\quad \Bigg[\sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{m-k+1}}^{z_1,\cdots,z_{m-k}}} ((L_0 \circ L_1 \circ \cdots \\
&\qquad \cdots \circ L_{m-k})\psi)(\xi)\, d \times_{i=1}^{q_0} \mu(s_i) \Bigg].
\end{aligned}$$

Finally, we have the following inequality ; for all $\lambda \in \mathbb{C}^+$

(2.15)

$$\begin{aligned}
&\|I_\lambda^{an}(G)\| \\
&\le \sum_{n=0}^{\infty} (n!)^{\frac{1}{r}} \left[\left(\frac{|\lambda|}{2\pi} \right)^{\frac{m+1}{2}} \vee \left(\frac{|\lambda|}{2\pi} \right)^{\frac{1}{2}} \right] \left[\min_{1 \le j \le m} (\tau_j - \tau_{j-1}) \right]^{-\frac{m+1}{2}} \\
&\quad \Gamma(l(1 - \frac{r'}{2}))^{-\frac{m+1}{r'}} \Gamma(1 - \frac{r'}{2})^{\frac{m+1}{r'}} \Bigg[\sum_{j=1}^{m} |w_j| (\|\theta(\tau_j, \cdot)\|_\infty \vee \|\theta(\tau_j, \cdot)\|_1) \\
&\quad + \left(\sum_{j=1}^{m+1} (\tau_j - \tau_{j-1})^{1-\frac{r'}{2}} \right)^{\frac{1}{r'}} \left(\frac{|\lambda|}{2\pi} \right)^{\frac{1}{2}} \|\theta\|_{1r;\mu} \left\| \frac{d|\mu|}{dm} \right\|_\infty^{\frac{1}{r'}} \Gamma(1 - \frac{r'}{2})^{\frac{1}{r'}} \Bigg]^n.
\end{aligned}$$

For all non-zero real q, $(J_q^{an}(G)\psi)(\xi)$ is given by the right hand side of (2.14) with $\lambda = -iq$. The inequality (2.15) also holds for $J_q^{an}(G)$ with $|\lambda|$ replaced by $|q|$.

3. The Generalized Dyson Series

Let η_u be in $\tilde{M}(0,t)$ and $\eta_u = \mu_u + \nu_u$ be the compositions of η_u into continuous part and discrete part for $u = 1, 2, \cdots, m$. We will write $\nu_u = \sum_{p=1}^{h} w_{p:u}\delta_{\tau_{p:u}}$. Let θ_u be in $L_{1r:\eta_u}$ such that $\theta_u(\tau_{p:u}, \cdot)$ is essentially bounded for $p = 1, 2, \cdots, h$. Let

$$F(y) = \prod_{u=1}^{m} \int_{(0,t)} \theta_u(s, y(s))\, d\eta_u(s) \quad \text{for } y \text{ in } C[0,t]. \tag{3.1}$$

Let A consist of all functionals $F(y)$ in (3.1) as well as $F(x) = 1$. Note that for every $\lambda > 0, F(\lambda^{-\frac{1}{2}}x + \xi)$ is defined for $m_w \times m$ - a.e. (x, ξ) in $C_0[0,t] \times \mathbb{R}$ by Lemma 1.

Given k between 0 and m, $[k : m]$ will denote the collection of all subsets of size k of the set of integers $\{1, 2, \cdots, m\}$. If $\{\alpha_1, \cdots, \alpha_k\}$ is in $[k : m]$, we shall always write

$$\{\alpha_{k+1}, \cdots, \alpha_m\} = \{1, 2, \cdots, m\} - \{\alpha_1, \cdots, \alpha_k\}.$$

Then

(3.2)

$$\begin{aligned} &F(y) \\ &= \prod_{u=1}^{m} \left[\int_{(0,t)} \theta_u(s, y(s))\, d\mu_u(s) + \sum_{p=1}^{h} w_{p:u}\theta_u(\tau_{p:u}, y(\tau_{p:u})) \right] \\ &= \sum_{k=0}^{m} \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1},\cdots,p_m=1}^{h} \prod_{u=1}^{k} \left[\int_{(0,t)} \theta_{\alpha_u}(s_u, y(s_u))\, d\mu_{\alpha_u}(s_u) \right] \\ &\qquad \left[\prod_{u=k+1}^{m} w_{p_u:\alpha_u}\theta_{\alpha_u}(\tau_{p_u:\alpha_u}, y(\tau_{p_u:\alpha_u})) \right] \\ &= \sum_{k=0}^{m} \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1},\cdots,p_m=1}^{h} \left[\prod_{u=k+1}^{m} w_{p_u:\alpha_u}\theta_{\alpha_u}(\tau_{p_u:\alpha_u}, y(\tau_{p_u:\alpha_u})) \right] \\ &\qquad \left[\int_{(0,t)^k} \prod_{u=1}^{k} \theta_{\alpha_u}(s_u, y(s_u))\, d \mathop{\times}_{u=1}^{k} \mu_{\alpha_u}(s_u) \right]. \end{aligned}$$

We want to calculate the Wiener integral defining $I_\lambda(F)$. For this purpose, we will need to order the time variables. We begin by ordering the $\tau's$ that appear within a given term of the series in (3.3). For fixed $k, \{\alpha_1, \cdots, \alpha_k\}$ in $[k:m]$ and $p_{k+1}, \cdots, p_m$, let σ be a permutation of $\{k+1, \cdots, m\}$ such that

$$\tau_{p_{\sigma(k+1)}:\alpha_{\sigma(k+1)}} \le \tau_{p_{\sigma(k+2)}:\alpha_{\sigma(k+2)}} \le \cdots \le \tau_{p_{\sigma(m)}:\alpha_{\sigma(m)}}. \tag{3.3}$$

(If the τ's involved in (3.3) are distinct, the permutation σ is unique).

THEOREM 3. *Let F be defined by (3.1). Then operators $I_\lambda^{an}(F)$ and $J_q^{an}(F)$ exist for λ in $\mathbb{C}_{2\pi t,\infty}^+ \equiv \{\lambda \in \mathbb{C} : Re\lambda > 0, 2\pi t < |\lambda| < \infty\}$. Moreover, for all λ in $\mathbb{C}_{2\pi t,\infty}^+, \psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$,*

(3.4)

$$\begin{aligned}
&(I_\lambda^{an}(F)\psi)(\xi) \\
&= \sum_{k=0}^{m} \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1},\cdots,p_m=1}^{h} \\
&\sum_{\rho\in S_k} \sum_{\substack{j_1+\cdots+j_{m-k+1}=k \\ 0\le j_1,\cdots,j_{m-k+1}\le k}} \left(\prod_{u=k+1}^{m} w_{p_{\sigma(u)}:\alpha_{\sigma(u)}}\right) \\
&\left(\int_{\Delta_{k:j_1,\cdots,j_{m-k+1}}(\rho)} ((L_k \circ \cdots \circ L_m)\psi)(\xi)\, d \mathop{\times}_{u=1}^{k} \mu_{\alpha_{\rho(u)}}(s_{\rho(u)})\right),
\end{aligned}$$

where ρ ranges through the group S_k of the permutations of $\{1, 2, \cdots, k\}$ and

(3.5)

$$\begin{aligned}
&\Delta_{k:j_1,\cdots,j_{m-k+1}}(\rho) \\
&= \{(s_1, \cdots, s_k) \in (0,t)^k | 0 < s_{\rho(1)} < \cdots < s_{\rho(j_1)} < \tau_{p_{\sigma(k+1)}:\alpha_{\sigma(k+1)}} \\
&\qquad < s_{\rho(j_1+1)} < \cdots < s_{\rho(j_1+j_2)} < \tau_{p_{\sigma(k+2)}:\alpha_{\sigma(k+2)}} \\
&\qquad < s_{\rho(j_1+j_2+1)} < \cdots < s_{\rho(j_1+\cdots+j_{m-k})} < \tau_{p_{\sigma(m)}:\alpha_{\sigma(m)}} \\
&\qquad < s_{\rho(j_1+\cdots+j_{m-k}+1)} < \cdots < s_{\rho(k)} < t\}.
\end{aligned}$$

Also for $(s_1, \cdots, s_k) \in \Delta_{k:j_1,\cdots,j_{m-k+1}}(\rho)$ *and* $n = k, \cdots, m$

(3.6)
$$
\begin{aligned}
&L_n \\
&= \theta_{\alpha_{\sigma(n)}}(\tau_{p_{\sigma(n)}:\alpha_{\sigma(n)}}) \circ C_{\lambda/(s_{\rho(j_1+\cdots+j_{n-k}+1)}-\tau_{p_{\sigma(n)}:\alpha_{\sigma(n)}})} \\
&\circ \theta_{\alpha_{\rho(j_1+\cdots+j_{n-k}+1)}}(s_{\rho(j_1+\cdots+j_{n-k}+1)}) \\
&\circ C_{\lambda/(s_{\rho(j_1+\cdots+j_{n-k}+2)}-s_{\rho(j_1+\cdots+j_{n-k}+1)})} \\
&\circ \theta_{\alpha_{\rho(j_1+\cdots+j_{n-k}+2)}}(s_{\rho(j_1+\cdots+j_{n-k}+2)}) \cdots \\
&\circ C_{\lambda/(s_{\rho(j_1+\cdots+j_{n-k+1})}-s_{\rho(j_1+\cdots+j_{n-k+1}-1)})} \\
&\circ \theta_{\alpha_{\rho(j_1+\cdots+j_{n-k+1})}}(s_{\rho(j_1+\cdots+j_{n-k+1})}) \\
&\circ C_{\lambda/(\tau_{p_{\sigma(n+1)}:\alpha_{\sigma(n+1)}}-s_{\rho(j_1+\cdots+j_{n-k+1})})}.
\end{aligned}
$$

Here, σ *is a permutation of* $\{k+1, k+2, \cdots, m\}$ *as defined in (3.3). In addition, we adopt the conventions* $\tau_{p_{\sigma(k)}:\alpha_{\sigma(k)}} = 0, \tau_{p_{\sigma(m+1)}:\alpha_{\sigma(m+1)}} = t$ *and* $\theta(\tau_{p_{\sigma(k)}:\alpha_{\sigma(k)}}) = 1$, *the inclusion map on* $C_0(\mathbb{R})$. *Further, we take* $j_0 = 0$; *then, when* $n = k$, *it is reasonable to interpret* $j_1 + \cdots + j_{n-k} + 1$ *as 1 and we also get* $j_{n-k} = j_0 = 0$. *The* s' *between two equal* $\tau's$ *are omitted in (3.5). The series in (3.4) converges in the operator norm. Further, for all* λ *in* $\mathbb{C}^+_{2\pi t,\infty}$.

$$\|I^{an}_\lambda(F)\| \le b_n(\lambda), \tag{3.7}$$

where

$$
\begin{aligned}
&b_n(\lambda) \\
&:= \sum_{k=0}^{m} \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1},\cdots,p_m=1}^{h} \left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}} \\
&\quad \prod_{u=k+1}^{m} (|w_{p_u:\alpha_u}| \|\theta_{\alpha_u}(\tau_{p_u:\alpha_u}, \cdot)\|_1)
\end{aligned}
$$

$$\prod_{u=1}^{k}\left(\|\theta_{\alpha_u}\|_{1r:\mu_u}\left\|\frac{d|\mu_u|}{dm}\right\|_\infty^{\frac{1}{r'}}\right)(k!)^{\frac{1}{r'}}\Bigg\{\sum_{\substack{j_1+\cdots+j_{m-k+1}=k\\0\le j_1,\cdots,j_{m-k+1}\le k}}\prod_{u=k+1}^{m}$$

$$\left[\frac{\Gamma(1-\frac{r'}{2})^{j_u+1}}{\Gamma[(j_u+1)(1-\frac{r'}{2})]}(\tau_{p_u:\alpha_u}-\tau_{p_{u-1}:\alpha_{u-1}})^{j_u-(j_u+1)\frac{r'}{2}}\right]\Bigg\}^{\frac{1}{r'}}.$$

For all real q with $|q|>2\pi t$, $(J_q^{an}(F)\psi)(\xi)$ is given by the right hand side of (3.6) with $\lambda=-iq$ and the inequality of (3.7) also holds for $J_q^{an}(F)$ with $|\lambda|$ replaced by $|q|$.

Proof. Once we have the norm estimates, the rest of the proof proceeds similarly to the proof of Theorem 2.1 in [3].
For $\lambda\in\mathbb{C}^+_{2\pi t,\infty}$, $\psi\in L_1(\mathbb{R})$,

$$\|I_\lambda^{an}(F)\psi\|_\infty$$

$$\overset{(1)}{\le}\sum_{k=0}^{m}\sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]}\sum_{p_{k+1},\cdots,p_m=1}^{h}\sum_{\rho\in S_k}\sum_{\substack{j_1+\cdots+j_{m-k+1}=k\\0\le j_1,\cdots,j_{m-k+1}\le k}}\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}}$$

$$\int_{\Delta_{k:j_1,\cdots,j_{m-k+1}}(\rho)}\left(\prod_{u=k+1}^{m}|w_{p_{\sigma(u)}:\alpha_{\sigma(u)}}|\right)\prod_{u=1}^{k}\|\theta_{\alpha_{\rho(u)}}(s_{\rho(u)},\cdot)\|_1$$

$$[s_{\rho(1)}(s_{\rho(2)}-s_{\rho(1)})\cdots(\tau_{p_{\sigma(k+1)}:\alpha_{\sigma(k+1)}}-s_{\rho(j_1)})\cdots(t-s_{\rho(m)})]^{-\frac{1}{2}}$$

$$\prod_{u=k+1}^{m}\|\theta_{\alpha_{\sigma(u)}}(\tau_{p_{\sigma(u)}:\alpha_{\sigma(u)}},\cdot)\|_1\|\psi\|_1 d\bigtimes_{u=1}^{k}|\mu_{\alpha_{\rho(u)}}|(s_{\rho(u)})$$

$$\overset{(2)}{\le}\|\psi\|_1\sum_{k=0}^{m}\sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]}\sum_{p_{k+1},\cdots,p_m=1}^{h}\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}}$$

$$\prod_{u=k+1}^{m}(|w_{p_{\sigma(u)}:\alpha_{\sigma(u)}}|\|\theta_{\alpha_{\sigma(u)}}(\tau_{p_{\sigma(u)}:\alpha_{\sigma(u)}},\cdot)\|_1)$$

$$\left(\int_{(0,t)^k}\prod_{u=1}^{k}\|\theta_{\alpha_u}(s_u,\cdot)\|_1^r d\underset{u=1}{\overset{k}{\times}}|\mu_{\alpha_u}|(s_u)\right)^{\frac{1}{r}}$$

$$\left[k!\sum_{\substack{j_1+\cdots+j_{m-k+1}=k\\0\le j_1,\cdots,j_{m-k+1}\le k}}\int_{\Delta_{k:j_1,\cdots,j_{m-k+1}}}[s_1(s_2-s_1)\cdots\right.$$

$$\left.(\tau_{p_{\sigma(k+1)}:\alpha_{\sigma(k+1)}}-s_{j_1})\cdots(t-s_m)]^{-\frac{r'}{2}}d\underset{u=1}{\overset{k}{\times}}|\mu_{\alpha_u}|(s_u))\right]^{\frac{1}{r'}}$$

$$\overset{(3)}{\le}\|\psi\|_1\sum_{k=0}^{m}\sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]}\sum_{p_{k+1},\cdots,p_m=1}^{h}\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}}$$

$$\prod_{u=k+1}^{m}(|w_{p_u:\alpha_u}|\|\theta_{\alpha_u}(\tau_{p_u:\alpha_u},\cdot)\|_1)\prod_{u=1}^{k}\|\theta_{\alpha_u}\|_{1r:\mu_{\alpha_u}}(k!)^{\frac{1}{r'}}$$

$$\left[\sum_{\substack{j_1+\cdots+j_{m-k+1}=k\\0\le j_1,\cdots,j_{m-k+1}\le k}}\int_{\Delta_{k:j_1,\cdots,j_{m-k+1}}}[s_1(s_2-s_1)\right.$$

$$\left.\cdots(\tau_{p_{\sigma(k+1)}:\alpha_{\sigma(k+1)}}-s_{j_1})\cdots(t-s_m)]^{-\frac{r'}{2}}d\underset{u=1}{\overset{k}{\times}}|\mu_{\alpha_u}|(s_u)\right]^{\frac{1}{r'}}$$

$$\overset{(4)}{\le}\|\psi\|_1\sum_{k=0}^{m}\sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]}\sum_{p_{k+1},\cdots,p_m=1}^{h}\left(\frac{|\lambda|}{2\pi}\right)^{\frac{m+1}{2}}$$

$$\prod_{u=k+1}^{m}(|w_{p_u:\alpha_u}|\|\theta_{\alpha_u}(\tau_{p_u:\alpha_u},\cdot)\|_1)$$

$$\prod_{u=1}^{k}\left(\|\theta_{\alpha_u}\|_{1r:\mu_{\alpha_u}}\left\|\frac{d|\mu_{\alpha_u}|}{dm}\right\|_\infty^{\frac{1}{r'}}\right)(k!)^{\frac{1}{r'}}\left\{\sum_{\substack{j_1+\cdots+j_{m-k+1}=k\\0\le j_1,\cdots,j_{m-k+1}\le k}}\prod_{u=k+1}^{m}\right.$$

$$\left.\left[\frac{\Gamma(1-\frac{r'}{2})^{j_u+1}}{\Gamma[(j_u+1)(1-\frac{r'}{2})]}(\tau_{p_{\sigma(u)}:\alpha_{\sigma(u)}}-\tau_{p_{\sigma(u-1)}:\alpha_{\sigma(u-1)}})^{j_u-(j_u+1)\frac{r'}{2}}\right]\right\}^{\frac{1}{r'}}.$$

For λ in $\mathbb{C}^+_{2\pi t}$, $\|C_{\lambda/s_1+s_2}\|\le\left(\frac{|\lambda|}{2\pi(s_1+s_2)}\right)^{\frac{1}{2}}\le\left(\frac{|\lambda|}{2\pi s_1}\right)^{\frac{1}{2}}\cdot\left(\frac{|\lambda|}{2\pi s_2}\right)^{\frac{1}{2}}$, we obtain step (1). Step (2) follows from the *Hölder's inequality* and "simple trick" [3]. Step (3) is obtained by (1.1) and step (4) is obtained by (1.4). □

From Theorem 3, directly we have the following Theorem 4.

THEOREM 4. (Generalized Dyson Series) *Let $\{F_n\}$ be a sequence of functionals such that*

$$F_n(y) = \prod_{u=1}^{m_n} \int_{(0,t)} \theta_{n,u}(s, y(s))\, d\eta_{n,u}(s) \tag{3.8}$$

for y in $C[0,t]$ where $\eta_{n,u}$ is in $\tilde{M}(0,t)$ and $\theta_{n,u}$ is in $L_{1r:\eta_{n:u}}$. (Note that if $m_n = 0$, we take $F_n = 1$.) For a discrete point τ of $\eta_{n,u}$, we assume that $\theta_{n,u}(\tau,\cdot)$ are essentially bounded for all u and n. Let $\lambda_0 > 2\pi t$. Suppose that $\sum_{n=0}^{\infty} b_n(|\lambda|) < \infty$ for λ in $\mathbb{C}^+_{2\pi t,\lambda_0}$. Then for $\lambda \in (2\pi t, \lambda_0)$ and $\xi \in \mathbb{R}$, $\sum_{n=0}^{\infty} F_n(\lambda^{-\frac{1}{2}}x + \xi)$ converges absolutely for a.e. $x \in C_0[0,t]$. Let

$$F(y) = \sum_{n=0}^{\infty} F_n(y). \tag{3.9}$$

Then the operators $I_\lambda^{an}(F)$ and $J_q^{an}(F)$ exist for all $\lambda \in \mathbb{C}^+_{2\pi t,\lambda_0}$ and all real q with $2\pi t < |q| < \lambda_0$, respectively. Further for $\lambda \in \mathbb{C}^+_{2\pi t,\lambda_0}$

$$I_\lambda^{an}(F) = \sum_{n=0}^{\infty} I_\lambda^{an}(F_n) \tag{3.10}$$

and

$$J_q^{an}(F) = \sum_{n=0}^{\infty} J_q^{an}(F_n), \tag{3.11}$$

where F_n is the functional defined in (3.8). Moreover, for λ in $\mathbb{C}^+_{2\pi t,\lambda_0}$, the series in (3.10) and (3.11) satisfy

$$\|I_\lambda^{an}(F)\| \le \sum_{n=0}^{\infty} b_n(|\lambda|) \tag{3.12}$$

and

$$\|J_q^{an}(F)\| \le \sum_{n=0}^{\infty} b_n(|q|) \tag{3.13}$$

and both of them converge in the operator norm.

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