MORAVA $K$– THEORY OF THE DOUBLE LOOP SPACES OF QUATERNIONIC STIEFEL MANIFOLDS

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ABSTRACT. In this paper we get the Morava $K$– theory of the double loop spaces of quaternionic Stiefel manifolds for an odd prime $p$ by computing the Atiyah – Hirzebruch spectral sequence. We also get the homology with $\mathbb{Z}/(p)$ coefficients and analyze $p$ torsion in the homology with $\mathbb{Z}$ coefficients.

1. Introduction

Let $MU$ be the Thom spectrum for the unitary group. Quillen constructed a multiplicative idempotent map of ring spectra $\epsilon : MU(p) \to MU_{(p)}$ by localizing the spectrum $MU$ at a prime $p$ [5]. Then for a space $X$, the image of $\epsilon_*$ in $MU_*(X)_p$ becomes a natural direct summand of $MU_*(X)_p$, and satisfies all the axioms for a generalized homology theory. So by the Brown’s representability theorem in [2] it has the representing spectrum. This representing spectrum is denoted by $BP$ with $\pi_*(BP) = BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots]$, $\deg v_i = 2(p^i - 1)$. The spectra $k(n)$ can be obtained from the spectrum $BP$ by killing certain bordism classes $(p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots)$ in $BP_*$ via Bass-Sullivan construction in [1]. These $k(n)$ are the spectra for the connective Morava $K$-theories. The spectra $K(n) = \lim \sum_{v_n}^{-2i(p^n-1)} k(n)$ are the representing spectra for Morava $K$–theories where $\pi_*(K(n)) = \mathbb{Z}/(p)[v_n, v_n^{-1}]$.

So there is a sequence of homology theories for each $n$. Morava $K$–theories satisfy many nice properties. Since $K(n)_*$ is the graded field in

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the sense that every graded module over $K(n)_*$ is free, $\text{Tor}^{K(n)}_1(K^*(n)_*(X), K^*(n)_*(Y)) = 0$ for all spaces $X, Y$. Hence from the Küneth formula,

$$K(n)_*(X \times Y) = K(n)_*(X) \otimes K(n)_*(Y).$$

For the case $n = 0$, $K(0)_*(X) = H_*(X; Q)$ and $K(1)_*(X)$ is one of $p - 1$ isomorphic summands of mod $p$ complex $K$–theory for all $p$.

In this paper we study the Morava $K$–theory for an odd prime $p$ of the double loop spaces of the quaternionic Stiefel manifolds by computing the Atiyah–Hirzebruch spectral sequence with the structure of the Morava $K$–theory of the double loop spaces of the spheres in [7].

Besides the rational homology and the mod $p$ complex $K$–theory, we get the homology with $\mathbb{Z}/(p)$ coefficients. Owing to the identification between the Atiyah-Hirzebruch spectral sequence with $E_2 = H_*(X; \mathbb{Z}/(p)) \otimes k(m)_*$ and the Bockstein spectral sequence which analyzes the $v_m$ torsion in $k(m)_*(X)$, we analyze the torsion in the connective Morava $K$–theory and the $p$ torsion in the homology with $\mathbb{Z}$ coefficients from the actions of the higher order Milnor operators and the actions of the higher order Bockstein operators on the homology with $\mathbb{Z}/(p)$ coefficients. As a special case, the Morava $K$–theory of the double loop space of the symplectic group can be obtained from above results.

We consider only the odd primary cases so that the spectra $K(n)$ are commutative ring spectra. Hence in this paper $p$ always denotes an odd prime.

2. Main contents

Let $E(x)$ be the exterior algebra on $x$ and $P(x)$ be the polynomial algebra on $x$ and $\Gamma(x)$ be the divided power algebra on $x$. Let $\Omega^n X$ be the space of all pointed continuous maps from $S^n$ to a space $X$. Let $V_{n, n-k}$ be the space of all $n - k$ frames in $H^n$ where $H$ is the algebra of quaternionic. Then we call $V_{n, n-k}$ the quaternionic Stiefel manifold which can be identified with $Sp(n)/Sp(k)$. Throughout this paper the subscript of an element always means the degree of an element.

We have the following well known fact [6], [7]. For an odd prime $p$,
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\[ K(m)_*(\Omega^2 S^{4n+3}) = E(x_{(4n+2)p^i-1} : 0 \leq i \leq m) \otimes T_m(y_{(4n+2)p^j-2} : j \geq 1) \]

where \( T_m(y) \) denotes the truncated polynomial algebra on \( y \) of height \( p^m \), that is, \( P(y)/(y^{p^m}) \).

First we calculate the homology of the triple loop space of \( Sp/Sp(n) \).

**Theorem 2.1.**

\[ H_*(\Omega^3 Sp/Sp(n+1); Z/(p)) = P(y_{4n+4i+4} : i \geq 0). \]

**Proof.** Consider the following fibration:

\[ Sp(n+1) \longrightarrow Sp \longrightarrow Sp/Sp(n+1) \]

It is well-known that

\[ H^*(Sp(n); Z/(p)) = E(a_{4i+3} : 0 \leq i \leq n-1) \]

\[ \mathcal{P}^j(a_{4i+3}) = (-1)^j(p-1)/2 \binom{2i+1}{j} a_{4i+3+2j(p-1)} \]

where \( \mathcal{P}^j \) is the Steenrod operation. Note that \( \mathcal{P}^n x = x^p \) for \( x \in H^{2n} \).

So we get

\[ H^*(Sp/Sp(n+1); Z/(p)) = E(a_{4n+4i+7} : i \geq 0) \]

\[ \mathcal{P}^j(a_{4i+3}) = (-1)^j(p-1)/2 \binom{2i+1}{j} a_{4i+3+2j(p-1)}. \]

We have the Eilenberg–Moore spectral sequence of the Steenrod module converging to \( H^*(\Omega Sp/Sp(n+1); Z/(p)) \) with

\[ E_2 = \text{Tor}_{H^*(Sp/Sp(n+1); Z/(p))}(Z/(p), Z/(p)) \]

\[ = \text{Tor}_{E(a_{4n+4i+7} : i \geq 0)}(Z/(p), Z/(p)) \]

\[ = \Gamma(b_{4n+4i+6} : i \geq 0). \]

Since \( E_2 \) is even-dimensional, \( E_2 = E_\infty \) and

\[ \mathcal{P}^j(b_{4i+2}) = (-1)^j(p-1)/2 \binom{2i+1}{j} b_{4i+2+2j(p-1)}. \]
Hence $b_{4i+2}^p = p^{2i+1}(b_{4i+2}) = (-1)^{(2i+1)(p-1)/2}b_{(4i+2)p}$. So we have the choices of generators $c_i$ such that

$$H^*(\Omega Sp/Sp(n+1); Z/(p)) = P(c_{4n+4i+6} : i \geq 0).$$

Consider the Eilenberg–Moore spectral sequence again converging to $H^*(\Omega^2 Sp/Sp(n+1); Z/(p))$ with

$$E_2 = \text{Tor}_{H^*}(\Omega Sp/Sp(n+1); Z/(p)) (Z/(p), Z/(p))$$

$$= \text{Tor}_P(c_{4n+4i+6} : i \geq 0) (Z/(p), Z/(p))$$

$$= E(z_{4n+4i+5} : i \geq 0).$$

This spectral sequence is the spectral sequence of a Hopf algebra so that the source of the first non trivial differential should be indecomposable and the target should be primitive. Since the target must be even-dimensional and every primitive element of $E_2$ is odd-dimensional, $E_2 = E_\infty$. Hence we get

$$H^*(\Omega^2 Sp/Sp(n+1); Z/(p)) = E(z_{4n+4i+5} : i \geq 0).$$

Now we apply the Eilenberg–Moore spectral sequence again converging to $H_*(\Omega^3 Sp/Sp(n+1); Z/(p))$ with

$$E_2 = \text{Ext}_{H^*}(\Omega^2 Sp/Sp(n+1); Z/(p)) (Z/(p), Z/(p))$$

$$= \text{Ext}_P(c_{4n+4i+5} : i \geq 0) (Z/(p), Z/(p))$$

$$= P(y_{4n+4i+4} : i \geq 0).$$

Since $E_2$ is even-dimensional, $E_2 = E_\infty$ and we get

$$H_*(\Omega^3 Sp/Sp(n+1); Z/(p)) = P(x_{4n+4i+4} : i \geq 0).$$

Now consider the inclusion map $i : Sp(n)/Sp(k) \to Sp/Sp(k)$. We can convert this map to a homotopy equivalent fiber map by using the fact that $Sp(n)/Sp(k)$ is homotopy equivalent to the space of all paths in $Sp/Sp(k)$ with initial points in $Sp(n)/Sp(k)$ under a homotopy which retracts all paths back to their initial points. Let $S_{n,k}$ be the fiber of this fiber map. So we have the following fibration up to homotopy:

$$S_{n,k} \longrightarrow Sp(n)/Sp(k) \longrightarrow Sp/Sp(k)$$

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COROLLARY 2.2.

\[ H_\ast (\Omega^2 S_{n+1,k}; Z/(p)) = P(y_{4n+4i+4} : i \geq 0) \]

Proof. We have the map of the fibrations:

\[
\begin{array}{ccc}
\Omega^2 Sp/Sp(n+1) & \longrightarrow & \Omega Sp(n+1) \\
\downarrow & & \downarrow \\
\Omega S_{n+1,k} & \longrightarrow & \Omega Sp(n+1)/Sp(k) \\
\downarrow & & \downarrow \\
* & \longrightarrow & Sp(k) \\
\end{array}
\]

So we have the fibration:

\[
\Omega^2 Sp/Sp(n+1) \longrightarrow \Omega S_{n+1,k} \longrightarrow *
\]

Hence \(\Omega^2 S_{n+1,k}\) is homotopy equivalent to \(\Omega^3 Sp/Sp(n+1)\).

COROLLARY 2.3.

\[ K(m)_\ast (\Omega^3 Sp/Sp(n+1)) = K(m)_\ast (\Omega^2 S_{n+1,k}) := P(y_{4n+4i+4} : i \geq 0) \]

where \(P(y_{4n+4i+4} : i \geq 0)\) means \(K(m)_\ast [y_{4n+4i+4} : i \geq 0]\).

Proof. We consider the Atiyah–Hirzebruch spectral sequence converging to \(K(n)_\ast (\Omega^3 Sp/Sp(n+1))\) with

\[ E_2 = H_\ast (\Omega^3 Sp/Sp(n+1); K(m)_\ast ) \]

\[ = H_\ast (\Omega^3 Sp/Sp(n+1); Z/(p)) \otimes K(m)_\ast . \]

Since \(H_\ast (\Omega^3 Sp/Sp(n+1); Z/(p))\) is even dimensional, the spectral sequence collapses from the \(E_2\)-term.

From now on, the element \(y_{4n+4i+4}\) in \(H_\ast (\Omega^2 S_{n+1,k})\) or \(K(m)_\ast (\Omega^2 S_{n+1,k})\) will be denoted by \(y_{n+1,4n+4i+4}\).
For each odd number $j$ with $j \neq 0 \mod p$, let $t(n+1,j)$, $n \geq k$, be the non negative integer satisfying the following condition:

$$2k + 1 \leq jp^{t(n+1,j)-1} \leq 2n + 1 < jp^{t(n+1,j)}$$

and let $t(k,j)$ be the smallest non negative integer satisfying the following condition: $2jp^{t(k,j)} - 2 > 4k + 3$ which implies $jp^{t(k,j)} \geq 2k + 1$ and let

$$t(n+1,k,j) = t(n+1,j) - t(k,j).$$

**Theorem 2.4.**

$$K(m)_*(\Omega^2 Sp(n+1)/Sp(k)) =$$

$$E(x_{2jp^{t(k,j)+i-1}} : j : \text{odd, } p \nmid j, \ 0 \leq i < t(n+1,k,j)(m+1))$$

$$\otimes T_{t(n+1,k,j)m} (y_{2jp^{t(n+1,j)-2}} : j : \text{odd, } p \nmid j, \ i \geq 0).$$

**Proof.** Consider the following map of the fibrations:

$$\begin{array}{ccc}
\Omega^2 S_{n,k} & \longrightarrow & \Omega^2 Sp(n)/Sp(k) \\
\downarrow f_{n+1} & & \downarrow \\
\Omega^2 S_{n+1,k} & \longrightarrow & \Omega^2 Sp(n+1)/Sp(k)
\end{array}$$

$$\begin{array}{ccc}
\Omega^2 S_{n+1,k} & \longrightarrow & \Omega^2 Sp(n+1)/Sp(k) \\
\downarrow & & \downarrow \\
\Omega^2 S^{4n+3} & \longrightarrow & \Omega^2 S^{4n+3}
\end{array}$$

Now we will illustrate the behavior of the composite of $f_n$'s. Consider the following map of the fibrations:

$$\begin{array}{ccc}
\Omega^2 S_{l,k} & \longrightarrow & * \\
\downarrow f_{l+1} & & \downarrow \\
\Omega^2 S_{l+1,k} & \longrightarrow & \Omega^2 S^{4l+3}
\end{array}$$

$$\begin{array}{ccc}
\Omega^2 S_{l+1,k} & \longrightarrow & \Omega^2 S^{4l+3} \\
\downarrow & & \downarrow \\
\Omega S_{l,k} & \longrightarrow &
\end{array}$$

From Morava $K$-theory $K(m)$ of $\Omega^2 S^{4l+3}$, we get

$$K(m)_*(f_{l+1})(y_{l,(4l+2)p^{j+m-2}}) = v_m(y_{l+1,(4l+2)p^{j-2}})^{p^m}, \ j \geq 1$$

$$K(m)_*(f_{l+1})(y_{l,i}) = y_{l+1,i} \text{ for the other degrees.}$$

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Consider the map of the fibrations:

\[
\begin{array}{ccc}
\Omega^2 S_{l_p + \frac{p-1}{2}, k} & \longrightarrow & \ast \longrightarrow \Omega S_{l_p + \frac{p-1}{2}, k} \\
\downarrow \quad f_{l_p + \frac{p+1}{2}} & & \downarrow \\
\Omega^2 S_{l_p + \frac{p+1}{2}, k} & \longrightarrow & \Omega^2 S^{(4l+2)p+1} \longrightarrow \Omega S_{l_p + \frac{p-1}{2}, k}
\end{array}
\]

From Morava $K$-theory $K(m)$ of $\Omega^2 S^{(4l+2)p+1}$, we get

\[
K(m)_*(f_{l_p + \frac{p+1}{2}})(y_{l_p + \frac{p-1}{2}, (4l+2)p^j+m-2}) = v_m(y_{l_p + \frac{p+1}{2}, (4l+2)p^j-2})^p^m, \ j \geq 2.
\]

Then

\[
\begin{align*}
K(m)_* & (f_{l_p + \frac{p+1}{2}} \circ \cdots \circ f_{l+2} \circ f_{l+1})(y_{l,(4l+2)p^j+2m-2}) \\
& = v_m(v_m(y_{l_p + \frac{p+1}{2}, (4l+2)p^j-2})^p^m)^p^m \\
& = v_m^{p^m+1}(y_{l_p + \frac{p+1}{2}, (4l+2)p^j-2})^p^{2m}, \ j \geq 2.
\end{align*}
\]

Similarly we have

\[
\begin{align*}
K(m)_* & (f_{l_p^2 + \frac{p+1}{2}} \circ \cdots \circ f_{l+1})(y_{l,(4l+2)p^j+3m-2}) \\
& = v_m^{p^{2m}+p^{m+1}}(y_{l_p^2 + \frac{p+1}{2}, (4l+2)p^j-2})^p^m, \ j \geq 3.
\end{align*}
\]

Note that in the image of the composite map, the power of $v_m$ depends on the number $j$ of the lowest dimensional element of form $y_{l_p + \frac{p+1}{2}, (4l+2)p^j-2}$.

Let $f = f_{n+1} \circ f_n \cdots \circ f_{k+1}$. We have the map of the fibrations:

\[
\begin{array}{ccc}
\Omega^3 Sp/Sp(k) & \longrightarrow & \ast \longrightarrow \Omega^2 Sp/Sp(k) \\
\downarrow \quad f & & \downarrow \\
\Omega^2 S_{n+1,k} & \longrightarrow & \Omega^2 Sp(n+1)/Sp(k) \longrightarrow \Omega^2 Sp/Sp(k)
\end{array}
\]

Consider the Atiyah–Hirzebruch spectral sequence converging to $K(n)_*(\Omega^2 Sp(n+1)/Sp(k))$ with

\[
E_2 = H_*((\Omega^2 Sp/Sp(k); K(m)_*(\Omega^2 S_{n+1,k}))) \\
= H_*((\Omega^2 Sp/Sp(k); Z/(p)) \otimes K(m)_*(\Omega^2 S_{n+1,k})).
\]

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Note that $\Omega^2 S_{n+1,k}$ is $4n+3$ connected and $H_*(\Omega^2 Sp/Sp(k); Z/(p)) = E(x_{4k+4i+1} : i \geq 0)$. Let $j$ be the odd number and $j \neq 0 \mod p$ and $i$ be the non-negative integer such that $2jp^i - 1 \geq 4k+1$, that is, $jp^i \geq 2k+1$ so $i \geq t(k,j)$.

Note that $t(n+1,j)$ is defined to satisfy the following condition:

$$4k \leq 2jp^{t(n+1,j)} - 2 \leq 4n + 3 < 2jp^{t(n+1,j)} - 2,$$

which implies $2k + 1 \leq jp^{t(n+1,j)-1} \leq 2n + 1 < jp^{t(n+1,j)}$. Since $\Omega^2 S_{n+1,k}$ is $4n+3$ connected, $y_{n+1,2jp^{t(n+1,j)}-2}$ is the lowest degree element of the form $y_{n+1,2jp^{t-2}}$ in $K(m)_*(\Omega^2 S_{n+1,k})$. Then we have

$$K(m)_*(f)(y_{k,2jp^{t(n+1,j)+t(n+1,k,j)m}}) = v_m^{p^{t(n+1,k,j)-1} + p^{t(n+1,k,j)-2} + \cdots + p^{n+1}},$$

and $y_{n+1,2jp^{t(n+1,j)-2}}^{p^{t(n+1,k,j)m}}$, $i \geq 0$

$$K(m)_*(f)(y_{k,i}) = y_{n+1,i} \text{ for the other degrees}.$$

By the naturality, we have the following differentials

$$d(x_{2jp^{t(n+1,k,j)+t(n+1,k,j)m-1}}) = v_m^{p^{t(n+1,k,j)+t(n+1,k,j)-1} + p^{t(n+1,k,j)-2} + \cdots + p^{n+1}},$$

and $(y_{n+1,2jp^{t(n+1,j)-2}})^{p^{t(n+1,k,j)m}}$, $i \geq 0$,

$$d(x_{2jp^{i-1}}) = 0 \text{ for } 0 \leq i < t(n+1,j) + t(n+1,k,j)m, i \geq t(k,j)$$

$$d(x_i) = y_{n+1,i-1} \text{ for the other degrees}.$$

Then $x_{2jp^{i-1}}$ survives permanently for $t(k,j) \leq i < t(n+1,j) + t(n+1,k,j)m$ and $(y_{n+1,2jp^{t(n+1,j)-2}})^{\ell}$ also survives permanently for $i \geq 0$ and $0 \leq \ell < p^{t(n+1,k,j)m}$. Since $t(n+1,k,j) = t(n+1,j) - t(k,j)$, $x_{2jp^{t(k,j)+i-1}}$ survive for $0 \leq i < t(n+1,k,j)(m+1)$.

\[\square\]

**Corollary 2.5.**

$$H_*(\Omega^2 Sp(n+1)/Sp(k); Q) = K(0)_*(\Omega^2 Sp(n+1)/Sp(k)) = E(x_{2j-1} : 2k + 1 \leq j \leq 2n + 1, j \text{ odd}, j \neq 0 \mod p).$$
Proof. Since \( m = 0 \), the truncated polynomial algebra \( T_0 \) disappear. Let \( n \) be fixed. Note that Theorem 2.4 holds for any odd prime number \( p \). Since \( t(n + 1, j) \) is the integer satisfying the condition:

\[
2k + 1 \leq jp^{t(n + 1, j) - 1} \leq 2n + 1 \leq j p^{t(n + 1, j)},
\]

\( t(n + 1, j) \) is 1 for a sufficiently large odd prime number \( p \). Hence \( 2k + 1 \leq j \leq 2n + 1 \).

Now we turn to the mod \( p \) homology.

**Theorem 2.6.**

\[
H_*(\Omega^2 Sp(n + 1)/Sp(k); Z/(p)) = E(x_{2jp^{t(k,j) + i - 1}} : j : \text{odd}, p \nmid j, i \geq 0) \\
\otimes P(y_{2jp^{t(n + 1, j) - 2}} : j : \text{odd}, p \nmid j, i \geq 0).
\]

**Proof.** In the proof of Theorem 2.4, we computed the Atiyah–Hirzebruch spectral sequence converging to \( K(m)_*(\Omega^2 Sp(n + 1))/Sp(k) \) with

\[
E_2 = H_*(\Omega^2 Sp/Sp(k)) \otimes K(m)_*(\Omega^2 S_{r+1,k}) \\
= H_*(\Omega^2 Sp/Sp(k)) \otimes H_*(\Omega^2 S_{n+1,k}) \otimes K(m)_*
\]

and we got the following differentials

\[
\begin{cases}
    d(x_{2jp^{t(n + 1, j) + t(n + 1, k, j)m - 1}}) \\
    = v_m^{p^{t(n + 1, k, j) - 1}m + p^{t(n + 1, k, j) - 2}m + \ldots p^m - 1} \\
    (y_{n+1,2jp^{t(n + 1, j) - 2}})^{p^{t(n + 1, k, j)m}}, i \geq 0, \\
    d(x_{2jp^{t - 1}}) = 0, t(k, j) \leq i \leq t(n + 1, j) + t(n + 1, k, j)m \\
    d(x_i) = y_{n+1,i-1} \text{ for the other degrees}.
\end{cases}
\]

For each element \( x_{2jp^{t - 1}} \) of any fixed \( j \) and \( t \), if we choose \( m \) to be large enough that \( t(n + 1, j) + t(n + 1, k, j)m \) is larger than \( t \), then we have \( d(x_{2jp^{t - 1}}) = 0 \). Now we consider the Serre spectral sequence converging to \( H_*(\Omega^2 Sp(n + 1))/Sp(k) \) with

\[
E_2 = H_*(\Omega^2 Sp/Sp(k)) \otimes H_*(\Omega^2 S_{n+1,k}),
\]

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then we have
\[
\begin{aligned}
&\begin{cases}
  d(x_{2jp^i−1}) = 0, \ i \geq t(k, j) \\
  d(x_i) = y_{n+1, i−1} \text{ for the other degrees.}
\end{cases}
\end{aligned}
\]
So \(x_{2jp^{t(k,j)+i−1}}\) survives permanently for all \(i \geq 0\) and \((y_{n+1,2jp^i−2})\) survives permanently for \(i \geq t(n+1, j)\). Therefore we get the conclusion. \(\square\)

Now we consider the connective Morava \(K\) theory.

**Corollary 2.7.**

\[
k(m)_*(\Omega^2 Sp(n + 1)/Sp(k))/(v_m^\infty) = \\
E(x_{2jp^{t(k,j)+i−1}} : j : odd, \ p \nmid j, \ 0 \leq i < t(n+1, k, j)(m + 1)) \\
\otimes T_{t(n+1,k,j)m}(y_{2jp^{t(n+1,j)+i−2}} : j : odd, \ p \nmid j, \ i \geq 0).
\]

Here we denote \((v_m^\infty) = \bigcup_{i \geq 1}(v_m^i)\) and \((v_m^i) = \{x \in k(m)_*(\Omega^2 Sp(n + 1)/Sp(k))|v_m^ix = 0\}\).

In the Atiyah–Hirzebruch spectral sequence which converges to \(k(m)_*(X)\) with
\[
E_2 = H_*(X; k(m)_*)
\]
as the classical \(K\)-theory, the first non-trivial differential in \(k(m)\) theory is determined by the Milnor operation \(Q_m\) in [4], where \(Q_m\) is defined inductively as the commutator for
\[
Q_0 = \beta, \\
Q_{k+1} = [Q_k, P_k^{p^k}]
\]
where \(\beta\) is the mod \(p\) homology Bockstein operation. Let \(Q_m^{(r)}\) be the \(r\)-th order Milnor operation defined by the relations \(Q_mQ_m^{(r−1)} = 0\) where \(deg Q_m^{(r)} = −2r(p^m − 1) − 1\).

In particular, the differentials in the Atiyah–Hirzebruch spectral sequence for \(k(m)_*(X)\) are given by the \(k\)-invariants of \(k(m)\) [3], so all the higher order non-trivial differentials are determined by the higher order Milnor operations given by \(d_{2r(p^m−1)+1}(x \otimes v_m^i) = cQ_m^{(r)}x \otimes v_m^{i+r}, \ c \neq 0 \text{ mod } p\). That means that there is the identification between the Atiyah-Hirzebruch spectral sequence with \(E_2 = H_*(X; Z/(p)) \otimes k(m)_*\) and the Bockstein spectral sequence which analyzes the \(v_m\) torsion in \(k(m)_*(X)\).
Corollary 2.8. Let \( s = p^{(t(n+1,k,1)-1)m + p^{(t(n+1,k,1)-2)m + \ldots p^m + 1}} \). Then \( v_m^s \) annihilates all the \( v_m \) torsions in \( k(m)_* (\Omega^2 Sp(n+1)/Sp(k)) \).

Proof. As we mentioned, there is the identification between the Atiyah-Hirzebruch spectral sequence with \( E_2 = H_*(X; Z/(p)) \otimes k(m)_* \) and the Bockstein spectral sequence. We can interpret the Atiyah-Hirzebruch spectral sequence for Theorem 2.4 as the Atiyah-Hirzebruch spectral sequence with \( E_2 = H_*(\Omega^2 Sp(n+1))/Sp(k); Z/(p)) \otimes k(m)_* \). From the proof of Theorem 2.4, the sets \( (v_m^{p^{(t(n+1,k,1)-1)m + p^{(t(n+1,k,1)-2)m + \ldots p^m + 1}}}, t \geq t(n+1, j) \) of \( v_m \) torsion of order \( p^{(t(n+1,k,1)-1)m + p^{(t(n+1,k,1)-2)m + \ldots p^m + 1}} \). Since \( \text{max} \{ t(n+1,k, j) : j \} = t(n+1,k,1) \), \( v_m^{p^{(t(n+1,k,1)-1)m + p^{(t(n+1,k,1)-2)m + \ldots p^m + 1}} } \) annihilates all the \( v_m \) torsions. \( \square \)

Corollary 2.9. There exist nontrivial actions of the higher order Milnor operators \( Q_m^{(r)} \) on \( H_*(\Omega^2 Sp(n+1)/Sp(k); Z/(p)) \) such that

\[
Q_m^{p^{(t(n+1,k,1)-1)m + p^{(t(n+1,k,1)-2)m + \ldots p^m + 1}}(x_{2jp^{i+t(n+1,j)+t(n+1,k, j)m - 1}}^{i+2jp^{i+t(n+1,j)-2})^{p^{(t(n+1,k,1)m}}}, i \geq 0.
\]

From above information we analyze the \( p \) torsion in the homology with \( Z \) coefficients.

Corollary 2.10. \( p^{(n+1,k,1)} \) annihilates all the \( p \) torsion in \( H_*(\Omega^2 Sp(n+1)/Sp(k); Z) \). That is, if we let \( p^{r-1} \leq 2n + 1 < p^r \) and \( s \) be the smallest integer satisfying the condition: \( p^{s-1} < 2k + 1 \leq p^s \), then \( p^{r-s} \) annihilates all the \( p \) torsions in \( H_*(\Omega^2 Sp(n+1)/Sp(k); Z) \).

Proof. \( k(0)_*(\Omega^2 Sp(n+1)/Sp(k)) \) consists of the torsion free part and the torsion part. The torsion free part is that \( k(0)_*(\Omega^2 Sp(n+1)/Sp(k))/(v_0^{\infty}) = E(x_{2j-1} : 2k + 1 \leq j \leq 2n + 1, j \text{ odd}, j \neq 0 \mod p) \). We have already computed in Theorem 2.4 :

\[
\begin{align*}
K(m)_*(f)(y_{k,2jp^{i+t(n+1,j)+t(n+1,k, j)m}}) &= v_m^{p^{(t(n+1,k,1)-1)m + p^{(t(n+1,k,1)-2)m + \ldots p^m + 1}}}
\end{align*}
\]

\[
(y_{n+1,2jp^{i+t(n+1,j)-2})^{p^{(t(n+1,k,1)m}}}, i \geq 0
\]

\[
K(m)_*(f)(y_{k,i}) = y_{n+1,i} \text{ for the other degrees.}
\]
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We know that \( k(0)_*(X) = H_*(X; Z)_{(p)} \) with \( v_0 = p \). From above we get that

\[
\begin{aligned}
  k(0)_*(\tilde{f})(y_{k,2jp^{i+t(n+1,j)}}) &= v_0^{t(n+1,k,j)}(y_{n+1,2jp^{i+t(n+1,j)}-2}), \quad i \geq 0 \\
  &= p^{t(n+1,k,j)}(y_{n+1,2jp^{i+t(n+1,j)}-2}), \quad i \geq 0 \\
  k(0)_*(\tilde{f})(y_{k,i}) &= y_{n+1,i} \quad \text{for the other degrees.}
\end{aligned}
\]

Owing to the identification between the Atiyah-Hirzebruch spectral sequence and the Bockstein spectral sequence, we have the nontrivial higher order differentials \( \beta^{t(n,j)} \) in \( H_*(\Omega^2 Sp(n+1)/Sp(k); Z/(p)) \) such that

\[
\beta^{t(n+1,k,j)}(x_{2jp^{i-1}}) = y_{2jp^{i-2}} \quad \text{for } i \geq t(n+1, j).
\]

Hence in \( H_*(\Omega^2 Sp(n+1)/Sp(k); Z) \) we have the elements of degree \( (2jp^{i-2}) \) for \( i \geq t(n+1, j) \) of the \( p \) torsion of order \( t(n+1, k, j) \). Since \( \max\{t(n+1, k, j) : j\} = t(n+1, k, 1), t(n+1, k, 1) \) annihilates all the \( p \) torsions in \( H_*(\Omega^2 Sp(n+1)/Sp(k); Z) \).

\[
\text{COROLLARY 2.11. There exist nontrivial higher order Bockstein actions } \beta^{t(n,j)} \text{ on } H_*(\Omega^2 Sp(n+1)/Sp(k); Z/(p)) \text{ such that } \beta^{t(n+1,k,j)}(x_{2jp^{i-1}}) = y_{2jp^{i-2}} \text{ for } i \geq t(n+1, j).
\]

References


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