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h-STABILITY OF DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

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ABSTRACT. In recent years M. Pinto introduced the notion of h-stability. He extended the study of exponential stability to a variety of reasonable systems called h-systems.

We investigate h-stability for the nonlinear differential systems using the notions of t_∞-similarity and Liapunov functions.

1. Introduction and Basic Notions

We consider the nonlinear nonautonomous differential system

$$x' = f(t,x), \quad x(t_0) = x_0, \tag{1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^+ = [0,\infty)$. We assume that the Jacobian matrix $f_x = \frac{\partial f}{\partial x}$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t,0) = 0$. The symbol $|.|$ denotes arbitrary vector norm on $\mathbb{R}^n$.

Let $x(t) = x(t,t_0,x_0)$ be denoted by the unique solution of (1) through (t_0,x_0) in $\mathbb{R}^+ \times \mathbb{R}^n$ such that $x(t_0,t_0,x_0) = x_0$. Also, we consider the associated variational systems

$$v' = f_x(t,0)v, \quad v(t_0) = v_0 \tag{2}$$

and

$$z' = f_x(t,x(t,t_0,x_0))z, \quad z(t_0) = z_0. \tag{3}$$

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The fundamental matrix solution $\Phi(t, t_0, 0)$ of (2) is given by

$$\Phi(t, t_0, 0) = \frac{\partial}{\partial x_0} x(t, t_0, 0)$$

and the fundamental matrix solution $\Phi(t, t_0, x_0)$ of (3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0).$$

We recall some notions of h-stability [9].

DEFINITION 1.1. The system (1) (or the trivial solution $x = 0$ of (1)) is called

(hS) *h-stable* if there exist $c \geq 1$, $\delta > 0$ and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c|x_0|h(t)h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$,

(GhS) *globally h-stable* if in (hS) the $\delta < \infty$,

(hSV) *h-stable in variation* if (3) (or $z = 0$ of (3)) is h-stable,

(GhSV) *globally h-stable in variation* if (3) (or $z = 0$ of (3)) is globally h-stable.

The notion of h-stability(hS) was introduced by Pinto [9, 10] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems [8].

Pinto studied the important properties about hS for the various differential systems and the nonlinear differential systems [8, 10].

We investigated hS for the nonlinear Volterra integro-differential system [2] and for the nonlinear perturbed systems [3]. Moreover, the concepts of Lipschitz stablity and exponenetial asymptotic stability which are closely related to hS were studied for the nonlinear functional differential systems [4].

Let $\mathfrak{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+ = [0, \infty)$ and $\mathfrak{S}$ be the subset of $\mathfrak{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^2 with the property that $S(t)$ and $S^{-1}(t)$ are bounded.

DEFINITION 1.2. A matrix $A(t) \in \mathfrak{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathfrak{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|ds < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{4}$$

for some $S(t) \in \mathfrak{S}$.

It is not hard to show that the t_∞-similarity is an equivalence relation.

In [5], Hewer introduced the notion of t_∞-similarity and studied the stability properties of the variational equation. This approach includes most types of stability.

In this paper we investigate hS for the nonlinear differential systems using the notions of t_∞-similarity and Liapunov functions.

2. Main Result

For the linear systems, note that

$$\text{GhSV} \Leftrightarrow \text{GhS} \Leftrightarrow \text{hS} \Leftrightarrow \text{hSV}$$

by Theorem 1 in [2]. Also, the linearized system inherits the property of hS from the original nonlinear system, i.e., the solution $v = 0$ of (2) is hS when the solution $x = 0$ of (1) is hS [2, Theorem 3.4]. Further, in the following theorem (its proof is an adaptation of Theorem 4.1 in [5]), we can show that

$$\text{GhSV} \Leftrightarrow \text{GhS} \quad \text{and} \quad \text{hS} \Leftrightarrow \text{hSV}$$

by using the concept of t_∞-similarity.

To do this, we need the following lemma. It is very effective to show hS for the linear systems.

LEMMA 2.1. [9, Lemma 1]. *The linear system*

$$(L) \qquad x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is an $n \times n$ continuous matrix, is hS if and only if there exist a constant $c \geq 1$ and a positive continuous bounded function h defined on $\mathbb{R}^+$ such that for every x_0 in $\mathbb{R}^n$,

$$|\Phi(t, t_0, x_0)| \leq ch(t)h(t_0)^{-1}$$

for all $t \geq t_0 \geq 0$, where $\Phi(t, t_0, x_0)$ is a fundamental matrix of (L).

THEOREM 2.2. *Assume that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. Then the solution $z = 0$ of (3) is a hS provided the solution $v = 0$ of (2) is hS.*

Proof. Since $v = 0$ of (2) is hS, by Lemma 2.1 there exist a constant $c \geq 1$ and a positive continuous bounded function h defined on $\mathbb{R}^+$ such that for every $x_0 \in \mathbb{R}^n$,

$$|\Phi(t, t_0, 0)| \leq ch(t)h(t_0)^{-1}$$

for all $t \geq t_0 \geq 0$, where $\Phi(t, t_0, 0)$ is the fundamental matrix solution of (2). Let $\Phi(t, t_0, x_0)$ denote the fundamental matrix solution of (3). Then it is easily seen by differentiating that the solution $S(t)$ of (4) is

$$S(t) = \Phi(t, t_0, 0)[S(t_0) + \int_{t_0}^{t} \Phi^{-1}(s, t_0, 0)F(s)\Phi(s, t_0, x_0)ds]\Phi^{-1}(t, t_0, x_0)$$

for $t \geq t_0 \geq 0$. Note that

$$\Phi(t, t_0, x_0) = \Phi(t, s, x(s, t_0, x_0))\Phi(s, t_0, x_0)$$

for all $t \geq s \geq t_0$. Thus we have

$$\Phi(t, t_0, x_0) = S^{-1}(t)[\Phi(t, t_0, 0)S(t_0) + \int_{t_0}^{t} \Phi(t, s, 0)F(s)\Phi(s, t_0, x_0)ds].$$

Then from Lemma 2.1 and by virtue of the boundedness of $S(t)$ and $S^{-1}(t)$ there are positive constants c_1 and c_2 such that

$$\begin{aligned}|\Phi(t,t_0,x_0)| &\le |S^{-1}(t)|[|\Phi(t,t_0,0)||S(t_0)| \\ &\quad + \int_{t_0}^t |\Phi(t,s,0)||F(s)||\Phi(s,t_0,x_0)|ds] \\ &\le c_1c_2h(t)h(t_0)^{-1} + c_1c_2\int_{t_0}^t h(t)h(s)^{-1}|F(s)||\Phi(s,t_0,x_0)|ds.\end{aligned}$$

By the well-known Gronwall inequality, we have

$$\begin{aligned}h(t)^{-1}|\Phi(t,t_0,x_0)| &\le c_3h(t_0)^{-1} + c_3\int_{t_0}^t h(s)^{-1}|\Phi(s,t_0,x_0)||F(s)|ds \\ &\le c_3h(t_0)^{-1}\exp(\int_{t_0}^t |F(s)|ds) \\ &\le ch(t_0)^{-1}, \quad c = c_3\exp\int_{t_0}^\infty |F(s)|ds.\end{aligned}$$

Hence we obtain

$$|\Phi(t,t_0,x_0)| \le ch(t)h(t_0)^{-1}, \quad 0 \le t_0 \le t,$$

for some positive constant $c \ge 1$. □

REMARK. Theorem 3.5 in [2] is a corollary of Theorem 2.2, since

$$\int_{t_0}^t |F(s)|ds = \int_{t_0}^t |f_x(s,0) - f_x(s,x(s,t_0,x_0))|ds < \infty,$$

when $S(t) = I$.

COROLLARY 2.3. *Under the same conditions of Theorem 2.2, the solution $x = 0$ of (1) is hSV.*

EXAMPLE. Consider the Ricatti scalar equation

$$\text{(S)} \qquad x' = \lambda(t)(-x + x^2), \quad x(t_0) = x_0, \quad \lambda \in C(\mathbb{R}^+),$$

whose general solution is $x(t, t_0, x_0) = [1 + (\frac{x_0 - 1}{x_0}) \exp \int_{t_0}^t \lambda(s)ds]^{-1}, t \geq t_0 \geq 0$. We claim that $f_x(t, 0)$ and $f_x(t, x(t, t_0, x_0))$ are t_∞-similar if $|x_0| \leq \frac{1}{2}$.

Proof. We obtain two variational systems on the solution $x(t, t_0, x_0)$ of (S) as the following :

$$\text{(V-1)} \qquad v' = f_x(t, 0)v = -\lambda(t)v$$

and

(V-2)

$$z' = f_x(t, x(t, t_0, x_0))z = \lambda(t)\{-1 + 2[1 + (\frac{x_0 - 1}{x_0}) \exp \int_{t_0}^t \lambda(s)ds)]^{-1}\}z.$$

Thus the fundamental matrix solution $\Phi(t, t_0, 0)$ of (V-1) is given by

$$\Phi(t, t_0, 0) = -\exp \int_{t_0}^t -\lambda(s)ds$$

and the fundamental matrix solution $\Phi(t, t_0, x_0)$ of (V-2) is given by

$$\begin{aligned}\Phi(t, t_0, x_0) &= \frac{\partial}{\partial x_0} x(t, t_0, x_0) \\ &= \frac{-\exp \int_{t_0}^t \lambda(s)ds}{[x_0(1 + \exp \int_{t_0}^t \lambda(s)ds) - \exp \int_{t_0}^t \lambda(s)ds]^2}.\end{aligned}$$

Then we have

$$|\Phi(t, t_0, 0)| \leq \exp(-\int_{t_0}^t \lambda(s)ds) = h(t)h(t_0)^{-1},$$

where $h(t) = \exp(-\int_0^t \lambda(s)ds)$. Hence $v = 0$ of (V-1) is hS. Also, there exists $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(s)|ds < \infty$$

such that

$$\dot{S}(t) + S(t)f_x(t,0) - f_x(t, x(t,t_0,x_0))S(t) = F(t)$$

for some $S(t) = \exp(-\int_0^t \lambda(s)ds)$ and $\lambda(t) \in L_1(\mathbb{R}^+)$, since

$$\int_0^\infty |F(s)|ds \le \int_0^\infty 5\lambda(s)\exp(-\int_0^s \lambda(\tau)d\tau)ds$$
$$= [-5\exp(-\int_0^t \lambda(s)ds)]_0^\infty < \infty.$$

Thus $f_x(t,0)$ and $f_x(t,x(t,t_0,x_0))$ are t_∞-similar. Therefore $z = 0$ of (V-2) is hS by Theorem 2.2. □

Now, we shall prove Massera type converse theorem for hS by using Theorem 2.2 and Liapunov functions. The techniques and results are similar to those of [6].

We define the Liapunov functions

$$D^+V_{(1)}(t,x) = \lim_{\delta \to 0+} \sup \frac{1}{\delta}[V(t+\delta, x+\delta f(t,x)) - V(t,x)]$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ and for the solution $x(t) = x(t,t_0,x_0)$ of (1),

$$D^+V(t,x(t)) = \lim_{\delta \to 0+} \sup \frac{1}{\delta}[V(t+\delta, x(t+\delta)) - V(t,x)].$$

Then it is well-known that

$$D^+V_{(1)}(t,x) = D^+V(t,x(t))$$

if $V(t,x)$ is Lipschitzian in x for each t.

THEOREM 2.4. *If $x = 0$ of (1) is GhS and $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$. Suppose further that $h'(t)$ exists and is continuous on $\mathbb{R}^+$. Then there exists a function $V(t,x)$ satisfying the following properties :*

(i) *$V \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ and $V(t,x)$ is Lipschitzian in x for each $t \in \mathbb{R}^+$.*

(ii) $|x| \le V(t,x) \le c|x|$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

(iii) $D^+V_{(1)}(t,x) \le h'(t)h(t)^{-1}V(t,x)$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Proof. Define the Liapunov function

$$V(t,x) = \sup_{\tau \geq 0} |x(t+\tau,t,x)| h(t+\tau)^{-1} h(t)$$

where $x(t+\tau,t,x)$ is a solution of (1) for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$. From GhS of (1) we have

$$|x(t,t_0,x_0)| \leq c|x_0| h(t) h(t_0)^{-1}, \quad t \geq t_0 \geq 0, \ |x_0| < \infty.$$

Furthermore, we obtain

$$\sup_{\tau \geq 0} |x(t+\tau,t,x)| h(t+\tau)^{-1} h(t) \geq |x(t,t,x)| = |x|$$

and

$$V(t,x) \leq c|x| h(t+\tau) h(t)^{-1} h(t+\tau)^{-1} h(t) = c|x|.$$

Therefore (ii) is satisfied. From the definition of hS and uniqueness of solutions of (1) it follows that $V(t,x)$ is defined on $\mathbb{R}^+ \times \mathbb{R}^n$.

We show that $V(t,x)$ is Lipschitzian in x for each $t \in \mathbb{R}^+$. Let $(t,x),(t,y) \in \mathbb{R}^+ \times \mathbb{R}^n$. Then we have

$$\begin{aligned} |V(t,x) - V(t,y)| &\leq |\sup_{\tau \geq 0} |x(t+\tau,t,x)| h(t+\tau)^{-1} h(t) \\ &\quad - \sup_{\tau \geq 0} |x(t+\tau,t,y)| h(t+\tau)^{-1} h(t)| \\ &\leq \sup_{\tau \geq 0} |x(t+\tau,t,x) - x(t+\tau,t,y)| h(t+\tau)^{-1} h(t). \end{aligned}$$

Since for each x_0 and y_0 in a convex subset D of $\mathbb{R}^n$

$$|x(t,t_0,x_0) - x(t,t_0,y_0)| \leq |x_0 - y_0| \sup_{\eta \in D} |\Phi(t,t_0,\eta)|$$

[6], we have

$$|V(t,x) - V(t,y)| \leq |x-y| \sup_{\eta \in D} |\Phi(t+\tau,t,\eta)| h(t+\tau)^{-1} h(t).$$

Now, by Theorem 2.2 and Lemma 2.1, we obtain

$$\begin{aligned}|V(t,x)-V(t,y)| &\le |x-y|ch(t+\tau)h(t)^{-1}h(t+\tau)^{-1}h(t)\\ &\le c|x-y|.\end{aligned}$$

This implies that $V(t,x)$ is Lipschitzian in x for each t.

Next, the continuity of $V(t,x)$ can be proved as in Theorem 3.6.1 of [6].

We can compute the following by the uniqueness of solutions and the definition of hS.

$$\begin{aligned}
&D^+V(t,x(t))\\
&= \lim_{\delta\to 0+}\sup \frac{1}{\delta}[V(t+\delta, x(t+\delta,t,x)) - V(t,x)]\\
&= \lim_{\delta\to 0+}\sup \frac{1}{\delta}[\sup_{\tau\ge 0}|x(t+\delta+\tau, t+\delta, x(t+\delta,t,x))|h(t+\delta+\tau)^{-1}h(t+\delta)\\
&- \sup_{\tau\ge 0}|x(t+\tau,t,x)|h(t+\tau)^{-1}h(t)]\\
&= \lim_{\delta\to 0+}\sup \frac{1}{\delta}[\sup_{\tau\ge \delta}|x(t+\tau,t,x)|h(t+\tau)^{-1}h(t+\delta)\\
&- \sup_{\tau\ge 0}|x(t+\tau,t,x)|h(t+\tau)^{-1}h(t)]\\
&\le \lim_{\delta\to 0+}\sup \frac{1}{\delta}[\sup_{\tau\ge 0}|x(t+\tau,t,x)|h(t+\tau)^{-1}h(t)(h(t+\delta)h(t)^{-1}-1)]\\
&\le \lim_{\delta\to 0+}\sup \frac{1}{\delta}[h(t+\delta)h(t)^{-1}-1]V(t,x)\\
&\le \frac{h'(t)}{h(t)}V(t,x).
\end{aligned}$$

Since, for small $\delta > 0$,

$$\begin{aligned}V(t+\delta, x+\delta f(t,x)) - V(t,x) &\le |V(t+\delta, x+\delta f(t,x)) - V(t+\delta, x(t+\delta,t,x))|\\ &+ |V(t+\delta, x(t+\delta,t,x)) - V(t,x)|\\ &\le c|x+\delta f(t,x) - x(t+\delta,t,x)|\\ &+ |V(t+\delta, x(t+\delta,t,x)) - V(t,x)|,\end{aligned}$$

it easily follows that

$$D^+V_{(1)}(t,x) \le \frac{h'(t)}{h(t)}V(t,x). \qquad \square$$

REMARK. For the system (1), Martynyuk [7] defined exponential x_1-stability by splitting the vector $x \in \mathbb{R}^n$ into two subvector $x_i \in \mathbb{R}^{n_i}$, $i = 1,2$, $n_1 + n_2 = 2$. If we adapt his conditions in Theorem 1 [7], we can obtain the following.

THEOREM 2.5. *Suppose that there exists a function $V(t,x) \in C(\mathbb{R}^+ \times\mathbb{R}^n, \mathbb{R}^+)$ which is locally Lipschitzian in x and a positive bounded continuously differentiable function $h(t)$ on $\mathbb{R}^+$ satisfying the following properties :*

(i) *There exist a strictly increasing function b on $\mathbb{R}^+$ with $b(0) = 0$ and two positive constants N, γ such that*

$$N|x|^\gamma \le V(t,x) \le b(|x|), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

(ii) *For the solution $x(t)$ of (1) through (t_0, x_0) we suppose that*

$$D^+V_{(1)}(t,x) \le h'(t)h(t)^{-1}V(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then $x = 0$ of (1) is hS.

Proof. Let $x(t, t_0, x_0)$ be any solution of (1). As a consequence of (ii), we obtain

$$V(t, x(t, t_0, x_0)) \le V(t_0, x_0) \exp \int_{t_0}^t \frac{h'(s)}{h(s)} ds = V(t_0, x_0)h(t)h(t_0)^{-1}.$$

From the condition (i) we have

$$|x(t, t_0, x_0)| \le N^{\frac{-1}{\gamma}} b(|x_0|)^{\frac{1}{\gamma}} h(t)^{\frac{1}{\gamma}} h(t_0)^{-\frac{1}{\gamma}}, \quad t \ge t_0 \ge 0.$$

For every $\varepsilon > 0$ we can choose

$$\delta = b^{-1}(N\varepsilon^\gamma) \ge |x_0| \ge \frac{\varepsilon}{c}, \ c \ge 1.$$

Then we have

$$|x(t, t_0, x_0)| \le \varepsilon h(t)^{\frac{1}{\gamma}} h(t_0)^{-\frac{1}{\gamma}} \le c|x_0| H(t)H(t_0)^{-1}, \ |x_0| \le \delta,$$

where $H(t) = h(t)^{\frac{1}{\gamma}}$ is a positive bounded continuous function on $\mathbb{R}^+$. $\square$

Now, the following theorem can be easily obtained.

THEOREM 2.6. *Suppose that $h(t)$ is a positive bounded continuously differentiable function on $\mathbb{R}^+$. Furthermore assume that there exists a function $V(t,x)$ satisfying the following properties :*

(i) $V \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+)$ *and* $V(t,x)$ *is Lipschitzian in* x *for each* $t \in \mathbb{R}^+$.

(ii) $|x| \le V(t,x) \le c|x|, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

(iii) $D^+V_{(1)}(t,x) \le h'(t)h(t)^{-1}V(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then $x = 0$ of (1) is GhS.

Proof. As in the proof of Theorem 2.8, we obtain

$$|x(t,t_0,x_0)| \le c|x_0|h(t)h(t_0)^{-1}, \quad t \ge t_0 \ge 0,$$

whenever $|x_0| < \infty$.

REMARK. If $h'(t) = 0$ in Theorem 2.6, then $x = 0$ of (1) is uniformly Lipschitz stable, i.e., there exist $M > 0$ and $\delta > 0$ such that

$$|x(t)| \le M|x_0| \text{ whenever } |x_0| \le \delta \text{ and } t \ge t_0 \ge 0.$$

We consider the perturbed system of (1)

$$\text{(P)} \qquad y' = f(t,y) + g(t,y)$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t,0) = 0$. The following theorem is motivated by [1, Theorem 3.2] and can be proved by Liapunov's second method and the comparison principle. □

THEOREM 2.7. *Let $x = 0$ of (1) be GhS. Suppose that the perturbation term satisfies*

$$|g(t,y)| \le \lambda(t)|y|, \quad t \ge t_0 \ge 0, \ |y| < \infty,$$

where $\lambda \in L_1(\mathbb{R}^+)$.

Then $y = 0$ of (P) is GhS.

Proof. By Theorem 2.4 there exist functions $V(t,x)$ and $h(t)$ having the three properties in that theorem. We have

$$\begin{aligned} D^+V_{(p)}(t,y) &\le D^+V_{(1)}(t,y) + c|g(t,y)| \\ &\le \frac{h'(t)}{h(t)}V(t,y) + c\lambda(t)|y| \\ &\le [\frac{h'(t)}{h(t)} + \lambda(t)]c|y|. \end{aligned}$$

We apply the comparison principle, where

$$w(t,u) = [\frac{h'(t)}{h(t)} + \lambda(t)]cu.$$

Let $y(t,t_0,y_0)$ be a solution of (P) such that $V(t_0,y_0) \le 2cu_0,\ u_0 \ge 0$. Then the maximal solution of the scalar equation

$$u' = w(t,u) = [\frac{h'(t)}{h(t)} + \lambda(t)]cu, \quad u(t_0) = 2cu_0$$

is

$$\begin{aligned} u(t,t_0,u_0) &= u_0 \exp(c\int_{t_0}^t \frac{h'(s)}{h(s)}ds)\exp(c\int_{t_0}^t \lambda(s)ds) \\ &= u_0(h(t)h(t_0)^{-1})^c \exp(c\int_{t_0}^t \lambda(s)ds) \\ &\le C|y_0|H(t)H(t_0)^{-1},\ H(t) = h(t)^c,\ C = \exp(c\int_{t_0}^\infty \lambda(s)ds). \end{aligned}$$

Hence $y = 0$ of (P) is GhS. □

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