FIXED POINTS OF A CERTAIN CLASS OF MAPPINGS
IN UNIFORMLY CONVEX BANACH SPACES

BALWANT SINGH THAKUR AND JONG SOO JUNG

ABSTRACT. In this paper, we prove in $p$-uniformly convex space a fixed point theorem for a class of mappings $T$ satisfying: for each $x, y$ in the domain and for $n = 1, 2, 3, \ldots$,

$$
\|T^nx - T^ny\| \leq a \cdot \|x - y\| + b(\|x - T^nx\| + \|y - T^ny\|)
+ c(\|x - T^ny\| + \|y - T^nx\|),
$$

where $a, b, c$ are nonnegative constants satisfying certain conditions. Further we establish some fixed point theorems for these mappings in a Hilbert space, in $L^p$ spaces, in Hardy spaces $H^p$ and in Sobolev spaces $H^{p,k}$ for $1 < p < \infty$ and $k \geq 0$. As a consequence of our main result, we also extend the results of Goebel and Kirk [7], Lim [8], Lifshitz [12], Xu [20] and others.

1. Introduction

Let $K$ be a nonempty subset of a Banach space $E$. A mapping $T : K \to K$ is said to be uniformly $\alpha$-Lipschitzian if

$$
\|T^nx - T^ny\| \leq \alpha \cdot \|x - y\|
$$

for all $x, y$ in $K$ and each $n \geq 1$. This class of mappings have been studied by many authors. Goebel and Kirk [7] proved that such $T$ has a fixed point if $K$ is a bounded closed convex subset of a uniformly

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convex Banach space $E$ and $\alpha < M$, $M$ being the unique solution of the equation $M \cdot (1 - \delta_E(\frac{1}{M})) = 1$ and $\delta_E(\cdot)$ the modulus of convexity of $E$. For a Hilbert space $H$, $M = \frac{\sqrt{5}}{2}$ and for $L^p$, $M = (1 + \frac{p}{2})^{\frac{1}{p}}$. Lifshitz [12] and Lim [8] extended the Geobel and Kirk’s result in the setting of Hilbert space and $L^p$ spaces, respectively. (See also [3, 11, 17 and 19].) Recently, Xu [20] extended these results to $p$-uniformly convex Banach spaces.

In this paper, we extend all above results for the class of mappings whose $n$th iterate $T^n$ satisfy

$$\|T^n x - T^n y\| \leq a \cdot \|x - y\| + b(\|x - T^n x\| + \|y - T^n y\|)$$

$$+ c(\|x - T^n y\| + \|y - T^n x\|)$$

for each $x, y \in K$ and $n = 1, 2, \cdots$, where $a$, $b$, $c$ are nonnegative constants such that $3b + 3c < 1$. By taking $b = c = 0$, it will be seen that this class of mappings is more general than uniformly $\alpha$-Lipschitzian mappings.

2. Preliminaries

The normal structure coefficient $N(E)$ of $E$ is defined by (cf. Bynum [2])

$$N(E) = \inf \{ \frac{\text{diam}K}{\gamma_K(K)} : K \text{ is a bounded convex subset of } E$$

consisting of more than one point\},

where $\text{diam}K = \sup\{\|x - y\| : x, y \in K\}$ is the diameter of $K$ and $\gamma_K(K) = \inf_{x \in K} (\sup_{y \in K} \|x - y\|)$ is the Chebyshev radius of $K$ relative to itself. $E$ is said to have uniformly normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has uniformly normal structure (cf. Danes [4]) and for a Hilbert space $H$, $N(H) = \sqrt{2}$. Recently, Pichugov [13] (cf. Prus [15]) calculated that $N(L^p) = \min\{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\}, \ 1 < p < \infty$. Some estimates for normal structure coefficient in other Banach spaces may be found in Prus [16].
Let \( p > 1 \) and denote by \( \lambda \) the number in \([0, 1]\) and by \( W_p(\lambda) \) the function \( \lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda) \).

The functional \( \| \cdot \|^p \) is said to be uniformly convex (cf. Zalinescu [21]) on the Banach space \( E \) if there exists a positive constant \( c_p \) such that for all \( \lambda \in [0, 1] \) and \( x, y \in E \), the following inequality holds:

\[
(2) \quad \| \lambda x + (1 - \lambda)y \|^p \leq \lambda \| x \|^p + (1 - \lambda) \| y \|^p - W_p(\lambda) \cdot c_p \cdot \| x - y \|^p.
\]

Xu [20] proved that the functional \( \| \cdot \|^p \) is uniformly convex on the whole Banach space \( E \) if and only if \( E \) is \( p \)-uniformly convex, i.e. there exists a constant \( c > 0 \) such that the moduli of convexity, \( \delta_E(\varepsilon) \geq c \cdot \varepsilon^p \) for all \( 0 \leq \varepsilon \leq 2 \).

Before presenting our main result we need the following:

**Lemma 1** [20]. Let \( p > 1 \) and let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty closed convex subset of \( E \) and let \( \{ x_n \} \subset E \) be a bounded sequence. Then there exists a unique point \( z \) in \( K \) such that

\[
(3) \quad \lim_{n \to \infty} \sup \| x_n - z \|^p \leq \lim_{n \to \infty} \sup \| x_n - x \|^p - c_p \cdot \| x - z \|^p
\]

for every \( x \) in \( K \), where \( c_p \) is the constant given in (2).

3. Main results

Now, we are in a position to give our main result.

**Theorem 1.** Let \( p > 1 \) and let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty closed convex subset of \( E \) and \( T : K \to K \) a mapping whose \( n \)th iterate \( T^n \) satisfy the inequality (1) with

\[
(i) \quad \left[ \frac{(\alpha + \beta)^p \cdot ((\alpha + \beta)^p - 1)}{c_p \cdot N^p} \right]^{\frac{1}{p}} < 1,
\]

where

\[
\alpha = \frac{a + b + c}{1 - b - c}, \quad \beta = \frac{2b + 2c}{1 - b - c},
\]

\( N \) is the normal structure coefficient of \( E \) and \( c_p \) is the constant given in inequality (2). Suppose that there is an \( x_0 \) in \( K \) for which \( \{ T^n x_0 \} \) is bounded, then \( T \) has a fixed point in \( K \), i.e. there is a \( z \) in \( K \) such that \( T(z) = z \).
Proof. Since \( \{T^n x_0\} \) is bounded (and hence \( \{T^n x\} \) is bounded for any \( x \) in \( K \)), by Lemma 1, we can inductively constant a sequence \( \{x_n\}_{n \geq 1} \) in \( K \) as follows: for each integer \( m \geq 0 \), \( x_{m+1} \) is the asymptotic center of the sequence \( \{T^n x_m\} \) in \( K \). Let

\[
\gamma_m = \limsup_{n \to \infty} \|T^n x_m - x_{m+1}\| \quad \text{and} \quad D_m = \sup_{n \geq 1} \|x_m - T^n x_m\|.
\]

By using (1) after a simple calculation, we have for each \( x, y \) in \( K \)

\[
\|T^i x - T^j y\| \leq \frac{a + b + c}{1 - b - c} \cdot \|x - T^{j-i} y\| + \frac{2b + 2c}{1 - b - c} \cdot \|T^j y - x\|
\]

i.e.,

\[
(4) \quad \|T^i x - T^j y\| \leq \alpha \cdot \|x - T^{j-i} y\| + \beta \cdot \|T^j y - x\|.
\]

By the result of Lim [9, Theorem 1] and by (4), we have

\[
\gamma_m = \limsup_{i \to \infty} \|T^i x_m - x_{m+1}\|
\]

\[
\leq \frac{1}{N} \cdot \limsup_{t \to \infty} \{\|T^i x_m - T^j x_m\| : i, j \geq t\}
\]

\[
\leq \frac{1}{N} \cdot \limsup_{t \to \infty} \{\alpha \cdot \|x_m - T^{j-i} x_m\| + \beta \cdot \|x_m - T^j x_m\| : i, j \geq t\}
\]

which implies

\[
(5) \quad \gamma_m \leq \frac{(\alpha + \beta)}{N} \cdot D_m
\]

where \( N \) is the normal structure coefficient of \( E \). For each fixed \( m \geq 1 \) and all \( n > k \geq 1 \), we have from (2) and (4)

\[
\|\lambda x_{m+1} + (1 - \lambda)T^k x_{m+1} - T^n x_m\|^p
\]

\[
+ c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^k x_{m+1}\|^p
\]

\[
\leq \lambda \|x_{m+1} - T^n x_m\|^p + (1 - \lambda) \cdot \|T^k x_{m+1} - T^n x_m\|^p
\]

\[
\leq \lambda \|x_{m+1} - T^n x_m\|^p + (1 - \lambda) \cdot (\alpha \cdot \|x_{m+1} - T^{n-k} x_m\| + \beta \cdot \|x_{m+1} - T^n x_m\|)^p.
\]

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Taking the limit superior as \( n \to +\infty \) on each side, by definition of \( x_m \), we get
\[
\gamma_m^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^k x_{m+1}\|^p \leq \{\lambda + (1 - \lambda) \cdot (\alpha + \beta)^p\} \gamma_m^p.
\]
it then follows that
\[
D_{m+1}^p \leq \frac{(1 - \lambda)\{(\alpha + \beta)^p - 1\}}{c_p \cdot W_p(\lambda)} \cdot \gamma_m^p
\leq \frac{(1 - \lambda)\{(\alpha + \beta)^p - 1\}}{c_p \cdot W_p(\lambda) \cdot N_p} \cdot (\alpha + \beta)^p \cdot D_m^p.
\]
Letting \( \lambda \to 1 \), we conclude that
\[
(6) \quad D_{m+1} = \left[ \frac{(\alpha + \beta)^p \{(\alpha + \beta)^p - 1\}}{c_p \cdot N_p} \right]^{\frac{1}{p}} \cdot D_m = A \cdot D_m, \quad m = 1, 2, \ldots
\]
where \( A = \left[ \frac{(\alpha + \beta)^p \cdot \{(\alpha + \beta)^p - 1\}}{c_p \cdot N_p} \right]^{\frac{1}{p}} \leq 1 \), by the assumption of the theorem. So, in general
\[
D_{m+1} \leq A \cdot D_m \leq \cdots \leq A^m D_1.
\]
Since
\[
\|x_{m+1} - x_m\| \leq \|x_{m+1} - T^m x_m\| + \|T^m x_m - x_m\|,
\]
taking the limit superior as \( n \to +\infty \) on each side, we have
\[
\|x_{m+1} - x_m\| \leq \gamma_m + D_m \leq 2 \cdot D_m \leq \cdots \leq 2 \cdot A^{m-1} D_1,
\]
\[
\to 0
\]
as \( m \to +\infty \). It then follows that \( \{x_m\} \) is a Cauchy sequence. Let
\[
z = \lim_{m \to \infty} x_m.
\]
Then we have from triangle inequality and by (4)
\[
\|z - Tz\| \leq \|z - x_m\| + \|x_m - Tx_m\| + \|Tx_m - Tz\|
\leq \|z - x_m\| + \|x_m - Tx_m\| + \alpha \cdot \|x_m - z\| + \beta \cdot \|Tz - x_m\|
\]
and so
\[
\|z - Tz\| \leq \frac{1 + \alpha + \beta}{1 - \beta} \cdot \|z - x_m\| + \frac{1}{1 - \beta} \cdot \|x_m - Tx_m\|
\]
\[
\to 0
\]
as \( m \to +\infty \). Hence \( Tz = z \). This completes the proof. \( \square \)

If we put \( b = c = 0 \) in Theorem 1, then we will have the following result:
Corollary 1 [20, Theorem 3]. Let \( p > 1 \) and let \( E \) and \( K \) be as in Theorem 1 and \( T : K \to K \) is a uniformly \( \alpha \)-Lipschitzian mapping. Suppose that there is an \( x_0 \) in \( K \) for which \( \{ T^n x_0 \} \) is bounded and that

\[
\alpha < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot c_p \cdot N^p} \right) \right]^\frac{1}{p}.
\]

Then \( T \) has a fixed point in \( K \).

Now we give applications of the established inequalities analogous to (2) in some Banach spaces. Let us begin with the following:

Lemma 2. (i) In a Hilbert space \( H \), the following inequality holds:

\[
\| \lambda x + (1 - \lambda) y \|^2 = \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda(1 - \lambda) \| x - y \|^2
\]

for all \( x, y \) in \( H \) and \( \lambda \in [0, 1] \).

(ii) If \( 1 < p \leq 2 \), then we have for all \( x, y \) in \( L^p \) and \( \lambda \in [0, 1] \),

\[
\| \lambda x + (1 - \lambda) y \|^2 \leq \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda(1 - \lambda) \cdot (p-1) \cdot \| x - y \|^2.
\]

(The inequality (8) is contained in Lim, Xu and Xu [11] and Smarzewski [18]).

(iii) Assume that \( 2 < p < \infty \) and \( t_p \) is the unique zero of the functions \( g(x) = -x^{p-1} + (p-1)x + p - 2 \) in the interval \((1, \infty)\). Let \( c_p = (p-1) \cdot (1 + t_p)^{2-p} = \frac{1 + t_p^{-1}}{(1 + t_p)^{p-1}} \). Then we have the following inequality

\[
\| \lambda x + (1 - \lambda) y \|^p \leq \lambda \| x \|^p + (1 - \lambda) \| y \|^p - W_p(\lambda) \cdot c_p \cdot \| x - y \|^p
\]

for all \( x, y \) in \( L^p \) and \( \lambda \in [0, 1] \). (The inequality (9) is essentially due to Lim, Xu and Xu [11] and Xu [20].)

By Theorem 1 and Lemma 2, we immediately obtain the following results:

Theorem 2. Let \( K \) be a nonempty closed convex subset of a Hilbert space \( H \). If \( T : K \to K \) be a mapping whose \( n \)th iterate \( T^n \) satisfy the inequality (1) with \( \left( \frac{(\alpha + \beta)^2 (\alpha + \beta)^2 - 1}{2} \right)^\frac{1}{2} < 1 \), where \( \alpha, \beta \) as in Theorem 1. Suppose that there is an \( x_0 \) in \( K \) for which \( \{ T^n x_0 \} \) is bounded. Then \( T \) has a fixed point in \( K \).
Theorem 3. Let $K$ be a nonempty closed convex subset of $L^p(1 < p < \infty)$. If $T : K \rightarrow K$ be a mapping whose $n$th iterate $T^n$ satisfy the inequality (1) with

$$\left[ \frac{(\alpha + \beta)^2 \{(\alpha + \beta)^2 - 1\}}{(p - 1) \cdot 2^{\frac{p-1}{p}}} \right]^{\frac{1}{2}} < 1 \quad \text{for} \quad 1 < p \leq 2$$

and

$$\left[ \frac{(\alpha + \beta)^p \cdot \{(\alpha + \beta)^p - 1\}}{c_p \cdot 2} \right]^{\frac{1}{p}} < 1 \quad \text{for} \quad 2 < p < \infty.$$ 

Suppose that there is an $x_0$ in $K$ for which $\{T^n x_0\}$ is bounded. Then $T$ has a fixed point in $K$.

If we put $b = c = 0$ in Theorem 3, then we will have the following result:

Corollary 2 [20, Corollary 4]. Let $K$ be a nonempty closed convex subset of $L^p(1 < p < \infty)$. If $T : K \rightarrow K$ is a uniformly $\alpha$-Lipschitzian mapping. Suppose that there is an $x_0$ in $K$ for which $\{T^n x_0\}$ is bounded and that

$$\alpha < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4 \cdot (p - 1) \cdot 2^{\frac{p-1}{p}}} \right) \right]^{\frac{1}{2}} \quad \text{if} \quad 1 < p \leq 2$$

and

$$\alpha < \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 8 \cdot c_p} \right) \right]^{\frac{1}{p}} \quad \text{if} \quad 2 < p < \infty,$$

where $c_p$ is as in (2). Then $T$ has a fixed point in $K$.

4. Additional results

Using the results of Prus and Smarzewski [14], Smarzewski [17] and Xu [20], we can obtain from Theorem 1 the fixed point theorems, for example, for Hardy and Sobolev spaces.
Let $H^p$, $1 < p < \infty$, denote the Hardy space [6] of all functions $x$ analytic in the unit disc $|x| < 1$ of the complex plane and such that

$$
\|x\| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty
$$

Now, let $\Omega$ be an open subset of $R^n$. Denote by $H^{r,p}(\Omega)$, $r \geq 0$, $1 < p < \infty$ the Sobolev space [1, p.149] of distributions $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm

$$
\|x\| = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha x(\omega)|^p \, d\omega \right)^{1/p}.
$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha), \alpha \in \Lambda$, be a sequence of positive measure spaces, where index set $\Lambda$ is finite or countable. Given a sequence of linear subspaces $X_\alpha$ in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, we denote by $L_{q,p}, 1 < p < \infty$ and $q = \max(2, p)$ [10], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$
\|x\| = \left( \sum_{\alpha \in \Lambda} (\|x_\alpha\|_{p,\alpha})^q \right)^{1/q},
$$

where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \sum_1, \mu_1)$ and $L_q = L^q(S_2, \sum_2, \mu_2)$, where $1 < p < \infty$, $q = \max(2, p)$ and $(S_i, \sum_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [5, III. 2.10] of all measurable $L_p$-value function $x$ on $S_2$ such that

$$
\|x\| = \left( \int_{S_2} (\|x(s)\|_p)^q \mu_2(ds) \right)^{1/q}.
$$

These spaces are $q$-uniformly convex with $q = \max(2, p)$ [14, 17] and the norm in these spaces satisfies

$$
\|\lambda x + (1 - \lambda)y\|^q \leq \lambda\|x\|^q + (1 - \lambda)\|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q
$$

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with a constant

\[ d = d_p = \begin{cases} 
\frac{p - 1}{8} & \text{if } 1 < p \leq 2 \\
\frac{1}{p \cdot 2^p} & \text{if } 2 < p < \infty.
\end{cases} \]

Hence from Theorem 1, we have the following result:

**Theorem 4.** Let \( K \) be a nonempty closed convex subset of the space \( E \), where \( E = H^p \), or \( E = H^{r,p}(\Omega) \), or \( E = L_{q,p} \), or \( E = L_q(L_p) \), and \( 1 < p < \infty \), \( q = \max(2, p) \), \( r \geq 0 \). If \( T : K \to K \) be a mapping whose \( n \)th iterate \( T^n \) satisfy the inequality (1) with

\[
\left[ \frac{(\alpha + \beta)^q \{(\alpha + \beta)^q - 1\}}{d \cdot N^q} \right]^{\frac{1}{q}} < 1.
\]

Suppose that there is an \( x_0 \) in \( K \) for which \( \{T^n x_0\} \) is bounded. Then \( T \) has a fixed point in \( K \).

**References**


Balwant Singh Thakur  
Govt. B. H. S. S. Gariaband, Dist. Raipur, M. P. 493889, India

Jong Soo Jung  
Department of Mathematics, Dongs A University, Pusan 607-714, Korea  
E-mail: jungjs@daunet.donga.ac.kr