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FIXED POINTS OF A CERTAIN CLASS OF MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we prove in p-uniformly convex space a fixed point theorem for a class of mappings T satisfying: for each x, y in the domain and for $n = 1, 2, 3, \cdots$,

$$\|T^n x - T^n y\| \le a \cdot \|x - y\| + b(\|x - T^n x\| + \|y - T^n y\|) + c(\|x - T^n y\| + \|y - T^n x\|),$$

where a, b, c are nonnegative constants satisfying certain conditions. Further we establish some fixed point theorems for these mappings in a Hilbert space, in L^p spaces, in Hardy spaces H^p and in Sobolev spaces $H^{p,k}$ for $1 < p < \infty$ and $k \ge 0$. As a consequence of our main result, we also extend the results of Goebel and Kirk [7], Lim [8], Lifshitz [12], Xu [20] and others.

1. Introduction

Let K be a nonempty subset of a Banach space E. A mapping $T : K \to K$ is said to be uniformly α-Lipschitzian if

$$\|T^n x - T^n y\| \le \alpha \cdot \|x - y\|$$

for all x, y in K and each $n \ge 1$. This class of mappings have been studied by many authors. Goebel and Kirk [7] proved that such T has a fixed point if K is a bounded closed convex subset of a uniformly

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convex Banach space E and $\alpha < M$, M being the unique solution of the equation $M\cdot(1-\delta_E(\frac{1}{M})) = 1$ and $\delta_E(\cdot)$ the modulus of convexity of E. For a Hirbert space H, $M = \frac{\sqrt{5}}{2}$ and for L^p, $M = (1+\frac{p}{2})^{\frac{1}{p}}$. Lifshitz [12] and Lim [8] extended the Geobel and Kirk's result in the setting of Hilbert space and L^p spaces, respectively. (See also [3, 11, 17 and 19].) Recently, Xu [20] extended these results to p-uniformly convex Banach spaces.

In this paper, we extend all above results for the class of mappings whose nth iterate T^n satisfy

$$\begin{aligned}(1)\qquad \|T^n x - T^n y\| \le a\cdot\|x-y\| &+ b(\|x-T^n x\| + \|y - T^n y\|)\\ &+ c(\|x - T^n y\| + \|y - T^n x\|)\end{aligned}$$

for each $x,\ y \in K$ and $n = 1,\ 2,\ \cdots,$. where $a,\ b,\ c$ are nonnegative constants such that $3b+3c < 1$. By taking $b = c = 0$, it will be seen that this class of mappings is more general than uniformly α-Lipschitzian mappings.

2. Preliminaries

The normal structure coefficient $N(E)$ of E is defined by (cf. Bynum [2])

$$N(E) = \inf\{\frac{diamK}{\gamma_K(K)} : K \text{ is a bounded convex subset of } E \text{ consisting of more than one point}\},$$

where $diamK = \sup\{\|x-y\| : x,\ y\in K\}$ is the diameter of K and $\gamma_K(K) = \inf_{x\in K}(\sup_{y\in K}\|x-y\|)$ is the Chebyshev radius of K relative to itself. E is said to have uniformly normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has uniformly normal structure (cf. Danes [4]) and for a Hilbert space H, $N(H) = \sqrt{2}$. Recently, Pichugov [13] (cf. Prus [15]) calculated that $N(L^p) = \min\{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\}$, $1 < p < \infty$. Some estimates for normal structure coefficient in other Banach spaces may be found in Prus [16].

Let $p > 1$ and denote by λ the number in $[0, 1]$ and by $W_p(\lambda)$ the function $\lambda \cdot (1-\lambda)^p + \lambda^p \cdot (1-\lambda)$.

The functional $\|\cdot\|^p$ is said to be uniformly convex (cf. Zalinescu [21]) on the Banach space E if there exists a positive constant c_p such that for all $\lambda \in [0, 1]$ and $x,\ y \in E$, the following inequality holds:

$$(2)\quad \|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda) \cdot c_p \cdot \|x-y\|^p.$$

Xu [20] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e. there exists a constant $c > 0$ such that the moduli of convexity, $\delta_E(\varepsilon) \geq c \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$.

Before presenting our main result we need the following:

LEMMA 1 [20]. *Let $p > 1$ and let E be a p-uniformly convex Banach space, K a nonempty closed convex subset of E and let $\{x_n\} \subset E$ be a bounded sequence. Then there exists a unique point z in K such that*

$$(3)\qquad \limsup_{n\to\infty} \|x_n - z\|^p \leq \limsup_{n\to\infty} \|x_n - x\|^p - c_p \cdot \|x - z\|^p$$

for every x in K, where c_p is the constant given in (2).

3. Main results

Now, we are in a position to give our main result.

THEOREM 1. *Let $p > 1$ and let E be a p-uniformly convex Banach space, K a nonempty closed convex subset of E and $T : K \to K$ a mapping whose nth iterate T^n satisfy the inequality (1) with*

$$(\mathrm{i})\qquad \left[\frac{(\alpha+\beta)^p \cdot \{(\alpha+\beta)^p - 1\}}{c_p \cdot N^p}\right]^{\frac{1}{p}} < 1,$$

where

$$\alpha = \frac{a+b+c}{1-b-c}, \qquad \beta = \frac{2b+2c}{1-b-c},$$

N is the normal structure coefficient of E and c_p is the constant given in inequality (2). Suppose that there is an x_0 in K for which $\{T^n x_0\}$ is bounded, then T has a fixed point in K, i.e. there is a z in K such that $T(z) = z$.

Proof. Since $\{T^n x_0\}$ is bounded (and hence $\{T^n x\}$ is bounded for any x in K), by Lemma 1, we can inductively constant a sequence $\{x_n\}_{n\geq 1}$ in K as follows : for each integer $m \geq 0$, x_{m+1} is the asymptotic center of the sequence $\{T^n x_m\}$ in K. Let

$$\gamma_m = \limsup_{n\to\infty} \|T^n x_m - x_{m+1}\| \text{ and } D_m = \sup_{n\geq 1} \|x_m - T^n x_m\|.$$

By using (1) after a simple calculation, we have for each x, y in K

$$\|T^i x - T^j y\| \leq \frac{a+b+c}{1-b-c} \cdot \|x - T^{j-i} y\| + \frac{2b+2c}{1-b-c} \cdot \|T^j y - x\|$$

i.e.,

$$\|T^i x - T^j y\| \leq \alpha \cdot \|x - T^{j-i} y\| + \beta \cdot \|T^j y - x\|. \tag{4}$$

By the result of Lim [9, Theorem 1] and by (4), we have

$$\begin{aligned}\gamma_m &= \limsup_{i\to\infty} \|T^i x_m - x_{m+1}\| \\ &\leq \frac{1}{N} \cdot \limsup_{t\to\infty} \{\|T^i x_m - T^j x_m\| : i,\ j \geq t\} \\ &\leq \frac{1}{N} \cdot \limsup_{t\to\infty} \{\alpha \cdot \|x_m - T^{j-i} x_m\| + \beta \cdot \|x_m - T^j x_m\| : i,\ j \geq t\}\end{aligned}$$

which implies

$$\gamma_m \leq \frac{(\alpha+\beta)}{N} \cdot D_m \tag{5}$$

where N is the normal structure coefficient of E. For each fixed $m \geq 1$ and all $n > k \geq 1$, we have from (2) and (4)

$$\begin{aligned}&\|\lambda x_{m+1} + (1-\lambda) T^k x_{m+1} - T^n x_m\|^p \\ &\qquad + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^k x_{m+1}\|^p \\ &\leq \lambda \|x_{m+1} - T^n x_m\|^p + (1-\lambda) \cdot \|T^k x_{m+1} - T^n x_m\|^p \\ &\leq \lambda \|x_{m+1} - T^n x_m\|^p \\ &\qquad + (1-\lambda) \cdot (\alpha \cdot \|x_{m+1} - T^{n-k} x_m\| + \beta \cdot \|x_{m+1} - T^n x_m\|)^p.\end{aligned}$$

Taking the limit superior as $n \to +\infty$ on each side, by definition of x_m, we get

$$\gamma_m^p + c_p \cdot W_p(\lambda) \cdot \|x_{m+1} - T^k x_{m+1}\|^p \le \{\lambda + (1-\lambda)\cdot(\alpha+\beta)^p\}\gamma_m^p.$$

It then follows that

$$\begin{aligned} D_{m+1}^p &\le \frac{(1-\lambda)\{(\alpha+\beta)^p - 1\}}{c_p \cdot W_p(\lambda)} \cdot \gamma_m^p \\ &\le \frac{(1-\lambda)\{(\alpha+\beta)^p - 1\}}{c_p \cdot W_p(\lambda)} \cdot \frac{(\alpha+\beta)^p}{N^p} \cdot D_m^p. \end{aligned}$$

Letting $\lambda \to 1$, we conclude that

$$(6)\quad D_{m+1} = \left[\frac{(\alpha+\beta)^p\{(\alpha+\beta)^p - 1\}}{c_p \cdot N^p}\right]^{\frac{1}{p}} \cdot D_m = A \cdot D_m, \quad m = 1,\ 2, \cdots$$

where $A = \left[\frac{(\alpha+\beta)^p \cdot \{(\alpha+\beta)^p - 1\}}{c_p \cdot N^p}\right]^{\frac{1}{p}} < 1$, by the assumption of the theorem. So, in general

$$D_{m+1} \le A \cdot D_m \le \cdots \le A^m D_1.$$

Since

$$\|x_{m+1} - x_m\| \le \|x_{m+1} - T^n x_m\| + \|T^m x_m - x_m\|,$$

taking the limit superior as $n \to +\infty$ on each side, we have

$$\begin{aligned} \|x_{m+1} - x_m\| \le \gamma_m + D_m \le 2 \cdot D_m \le \cdots &\le 2 \cdot A^{m-1} D_1, \\ &\to 0 \end{aligned}$$

as $m \to +\infty$. It then follows that $\{x_m\}$ is a Cauchy sequence. Let $z = \lim_{m\to\infty} x_m$. Then we have from triangle inequality and by (4)

$$\begin{aligned} \|z - Tz\| &\le \|z - x_m\| + \|x_m - Tx_m\| + \|Tx_m - Tz\| \\ &\le \|z - x_m\| + \|x_m - Tx_m\| + \alpha \cdot \|x_m - z\| + \beta \cdot \|Tz - x_m\| \end{aligned}$$

and so

$$\begin{aligned} \|z - Tz\| \le \frac{1+\alpha+\beta}{1-\beta} &\cdot \|z - x_m\| + \frac{1}{1-\beta} \cdot \|x_m - Tx_m\| \\ \to 0 & \end{aligned}$$

as $m \to +\infty$. Hence $Tz = z$. This completes the proof. □

If we put $b = c = 0$ in Theorem 1, then we will have the following result:

COROLLARY 1 [20, Theorem 3]. *Let $p > 1$ and let E and K be as in Therem 1 and $T : K \to K$ is a uniformly α-Lipschitzian mapping. Suppose that there is an x_0 in K for which $\{T^n x_0\}$ is bounded and that*

$$\alpha < \left[\frac{1}{2}(1 + \sqrt{1 + 4 \cdot c_p \cdot N^p})\right]^{\frac{1}{p}}.$$

Then T has a fixed point in K.

Now we give applications of the established inequalities analogous to (2) in some Banach spaces. Let us begin with the following:

LEMMA 2. (i) *In a Hilbert space H, the following inequality holds:*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{7}$$

for all x, y in H and $\lambda \in [0,1]$.

(ii) *If $1 < p \leq 2$, then we have for all x, y in L^p and $\lambda \in [0,1]$,*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\cdot(p-1)\cdot\|x-y\|^2. \tag{8}$$

(The inequality (8) is contained in Lim, Xu and Xu [11] and Smarzewski [18]).

(iii) *Assume that $2 < p < \infty$ and t_p is the unique zero of the functions $g(x) = -x^{p-1} + (p-1)x + p - 2$ in the interval $(1, \infty)$. Let $c_p = (p-1)\cdot(1+t_p)^{2-p} = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}$. Then we have the following inequality*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)\cdot c_p \cdot \|x-y\|^p \tag{9}$$

for all x, y in L^p and $\lambda \in [0,1]$. (The inequality (9) is essentially due to Lim, Xu and Xu [11] and Xu [20].)

By Theorem 1 and Lemma 2, we immediately obtain the following results:

THEOREM 2. *Let K be a nonempty closed convex subset of a Hilbert space H. If $T : K \to K$ be a mapping whose nth iterate T^n satisfy the inequality (1) with $\left[\frac{(\alpha+\beta)^2\{(\alpha+\beta)^2-1\}}{2}\right]^{\frac{1}{2}} < 1$, where α, β as in Theorem 1. Suppose that there is an x_0 in K for which $\{T^n x_0\}$ is bounded. Then T has a fixed point in K.*

THEOREM 3. *Let K be a nonempty closed convex subset of $L^p(1 < p < \infty)$. If $T : K \to K$ be a mapping whose nth iterate T^n satisfy the inequality* (1) *with*

$$\left[\frac{(\alpha+\beta)^2\{(\alpha+\beta)^2-1\}}{(p-1)\cdot 2^{\frac{p-1}{p}}}\right]^{\frac{1}{2}} < 1 \quad \textit{for} \quad 1 < p \leq 2$$

and

$$\left[\frac{(\alpha+\beta)^p\cdot\{(\alpha+\beta)^p-1\}}{c_p\cdot 2}\right]^{\frac{1}{p}} < 1 \quad \textit{for} \quad 2 < p < \infty.$$

Suppose that there is an x_0 in K for which $\{T^n x_0\}$ is bounded. Then T has a fixed point in K.

If we put $b = c = 0$ in Theorem 3, then we will have the following result:

COROLLARY 2 [20, Corollary 4]. *Let K be a nonempty closed convex subset of $L^p(1 < p < \infty)$. If $T : K \to K$ is a uniformly α-Lipschitzian mapping. Suppose that there is an x_0 in K for which $\{T^n x_0\}$ is bounded and that*

$$\alpha < \left[\frac{1}{2}\left(1+\sqrt{1+4\cdot(p-1)\cdot 2^{\frac{p-1}{p}}}\right)\right]^{\frac{1}{2}} \quad \textit{if } 1 < p \leq 2$$

and

$$\alpha < \left[\frac{1}{2}\left(1+\sqrt{1+8\cdot c_p}\right)\right]^{\frac{1}{p}} \qquad \textit{if } 2 < p < \infty,$$

where c_p is as in (2). *Then T has a fixed point in K.*

4. Additional results

Using the results of Prus and Smarzewski [14], Smarzewski [17] and Xu [20], we can obtain from Theorem 1 the fixed point theorems, for example, for Hardy and Sobolev spaces.

Let H^p, $1 < p < \infty$, denote the Hardy space [6] of all functions x analytic in the unit disc $|x| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r \to 1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty$$

Now, let Ω be an open subset of R^n. Denote by $H^{r,p}(\Omega)$, $r \geq 0$, $1 < p < \infty$ the Sobolev space [1, p.149] of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm

$$\|x\| = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha x(\omega)|^p d\omega \right)^{\frac{1}{p}}.$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha), \alpha \in \wedge$, be a sequence of positive measure spaces, where index set $\wedge$ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, we denote by $L_{q,p}, 1 < p < \infty$ and $q = \max(2, p)$ [10], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \wedge\}$ equipped with the norm

$$\|x\| = \left(\sum_{\alpha \in \wedge} (\|x_\alpha\|_{p,\alpha})^q \right)^{\frac{1}{q}},$$

where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \sum_1, \mu_1)$ and $L_q = L^q(S_2, \sum_2, \mu_2)$, where $1 < p < \infty$, $q = \max(2, p)$ and $(S_i, \sum_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [5, III. 2.10] of all measurable L_p-value function x on S_2 such that

$$\|x\| = \left(\int_{S_2} (\|x(s)\|_p)^q \mu_2(ds) \right)^{\frac{1}{q}}.$$

These spaces are q-uniformly convex with $q = \max(2, p)$ [14, 17] and the norm in these spaces satisfies

$$\|\lambda x + (1 - \lambda) y\|^q \leq \lambda \|x\|^q + (1 - \lambda)\|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q$$

with a constant

$$d = d_p = \begin{cases} \dfrac{p-1}{8} & \text{if } 1 < p \leq 2 \\ \dfrac{1}{p \cdot 2^p} & \text{if } 2 < p < \infty. \end{cases}$$

Hence from Theorem 1, we have the following result:

THEOREM 4. *Let K be a nonempty closed convex subset of the space E, where $E = H^p$, or $E = H^{r,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max(2,p)$, $r \geq 0$. If $T : K \to K$ be a mapping whose nth iterate T^n satisfy the inequality (1) with*

$$\left[\frac{(\alpha+\beta)^q\{(\alpha+\beta)^q - 1\}}{d \cdot N^q}\right]^{\frac{1}{q}} < 1.$$

Suppose that there is an x_0 in K for which $\{T^n x_0\}$ is bounded. Then T has a fixed point in K.

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