Bull. Korean Math. Soc. **34** (1997), No. 3, pp. 403–409

GENERALIZED THOM CONJECTURE FOR ALMOST COMPLEX 4-MANIFOLDS

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ABSTRACT. Let X be a closed almost complex 4-manifold with $b_2^+(X) > 1$, and have its canonical line bundle as a basic class. Then the pseudo-holomorphic 2-dimensional submanifolds in X with non-negative self-intersection minimize genus in their homology classes.

1. Introduction

In [8] Kronheimer and Mrowka proved that algebraic curves in $\mathbb{C}P^2$ have minimum genuses in their homology classes. In [10] Morgan, Szabó and Taubes proved that if Σ is a smooth holomorphic curve in a compact Kähler surface with non-negative self-intersection number $\Sigma \cdot \Sigma \geq 0$, then Σ minimizes genus in its homology class. They used the Seiberg-Witten gauge theory to get their results. In [11] Taubes showed that the canonical complex line bundle of a symplectic 4-manifold has nonzero Seiberg-Witten invariant. The above results extend to symplectic 4-manifolds with almost same proofs. There are non-symplectic 4-manifolds which have nonzero Seiberg-Witten invariants [2].

In this paper we give some almost complex 4-manifolds X constructed by the connected sums of symplectic 4-manifolds X_1 and homology 4-spheres. These almost complex 4-manifolds have no symplectic structures. However there are a one-to-one correspondence between Seiberg-Witten invariants of X and ones of X_1. If X is an almost complex 4-manifold, then the connected sum $X \sharp n\overline{\mathbb{C}P^2}$ of X and n copies

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Received January 27, 1997.

1991 Mathematics Subject Classification: 57N13, 58B15.

Key words and phrases: Seiberg-Witten invariant, Almost complex 4-manifold, Vortex equation.

This work was supported in part by the Korea Science and Engineering Foundation (KOSEF) through the GARC at Seoul National University and BSRI-97-1424.

of $\overline{\mathbb{C}P^2}$ has an almost complex structure. If X is a closed almost complex 4-manifold and its complex canonical line bundle is a basic class, then an embedded pseudo-holomorphic curve Σ in X with $\Sigma \cdot \Sigma \geq 0$ minimizes genus in its homology class.

2. Vortex Equations

Let X be an almost complex 4-manifold with canonical complex line bundle K and let Σ be an embedded 2-dimensional submanifold of X with zero self-intersection number $\Sigma \cdot \Sigma = 0$. Let $W^+ = E \oplus (K^{-1} \otimes E) \to X$ be a Spin^c structure on X, where E is a complex line bundle over X. Let $L = \det W^+ = E \otimes (K^{-1} \otimes E)$. For a connection A of L and a section $\phi \in \Gamma(W^+)$ of W^+ the Seiberg-Witten equations are

$$(*) \begin{cases} D_A \phi = 0 \\ F_A^+ = -\tau(\phi, \bar{\phi}). \end{cases}$$

Since $\Sigma \cdot \Sigma = 0$ there is a tubular neighbourhood $N(\Sigma)$ of Σ in X such that its boundary $\partial N(\Sigma) \equiv Y$ is diffeomorphic to $\Sigma \times S^1$. Let (X_R, g_R) be the Riemannian manifold obtained from X by cutting along Y and inserting a cylinder $[-R, R] \times Y$, here g_R is a product metric on $[-R, R] \times Y$. In [8] Kronheimer and Mrowka showed the following:

PROPOSITION 2.1 [8]. *Suppose the moduli space $\mathcal{M}(L, g_R)$ is nonempty for all sufficiently large R. Then there is a solution of the equation $(*)$ on the cylinder $\mathbb{R} \times Y$ which is translation invariant in a temporal gauge.*

In the temporal gauge a connection A and a section ϕ on the cylinder $\mathbb{R} \times Y$ can be thought as a path $A(t)$ of connections and a path $\phi(t)$ of sections in the restricted Spin^c structure over 3-manifold Y. In this case, the Seiberg-Witten equations become

$$(**) \begin{cases} \dfrac{d\phi}{dt} = -\overline{D}_A \phi \\ \dfrac{dA}{dt} = - * F_A - \tau(\phi, \bar{\phi}) \end{cases}$$

where $\overline{D}_A$ is the Dirac operator in 3-dimensinal Spin^c structure W and τ is a pairing obtained from Clifford multiplication by using the hermitian metric on W.

By the uniformization theorem there is a Riemannian metric on $\Sigma \times S^1 \equiv Y$ such that Σ has constant scalar curvature. Then using the Gauss-Bonnet theorem, Kronheimer and Mrowka showed the following:

THEOREM 2.2 [8]. *If there is a solution to the Seiberg-Witten equations on $\mathbb{R} \times Y$ which translation-invariant in a temporal gauge, then $|c_1(L)[\Sigma]| \leq 2g(\Sigma) - 2$.*

We assume that the line bundle L over $\Sigma \times S^1$ is pulled back from a line bundle over Σ. This may be justified when the cohomology class of L over $\Sigma \times S^1$ has no component in $H^1(\Sigma) \otimes H^1(S^1)$. The equations of S^1-invariant solutons of $(**)$ reduce to the following vortex equations $(***)$ over Σ, for details see [6]. By the symmetry between L and L^{-1}, we may suppose that the degree $d = c_1(L^{-1}) \cdot \Sigma \geq 0$ is non-negative.

$$(***) \begin{cases} \bar{\partial}_A \psi = 0 \\ F_A = -|\psi|^2 \end{cases}$$

where A is a connection of $L^{-1} \to \Sigma$ and ψ is a section of $K_\Sigma^{\frac{1}{2}} \otimes L^{-\frac{1}{2}} \to \Sigma$. In this case the twisted spinor bundle is $W = K_\Sigma^{\frac{1}{2}} \otimes L^{-\frac{1}{2}} \oplus K_\Sigma^{-\frac{1}{2}} \otimes L^{-\frac{1}{2}}$, and the determinant line bundle is $\det W = L^{-1}$ over the Riemann surface Σ. Suppose that $E \oplus K^{-1} \otimes E \to X$ is a Spin^c structure over an almost complex 4-manifold X and Σ is an embedded 2-manifold in X with self-intersection number $\Sigma \cdot \Sigma = 0$.

THEOREM 2.3. *Under above assumptions,*

(1) *If a Spin^c structure $E \oplus (K^{-1} \otimes E) \to X$ has a solution of the Seiberg-Witten equations, then the reduced vortex equations over Σ has a solution. In this case $2g(\Sigma) - 2 \geq c_1(L) \cdot \Sigma (\equiv d)$ if $E = K_\Sigma^{-\frac{1}{2}} \otimes L^{-\frac{1}{2}}$.*
(2) *If $r = (2g - 2) - d \geq 0$, then the space of solutions of the vortex equations is identified with the symmetric product $s^r(\Sigma)$ of r copies of Σ.*

Proof. We will sketch the proof of this theorem, for details see [6] and [13].

(1) The Seiberg-Witten equations $(*)$ is reduced to the vortex equations $(***)$ under the above conditions. If the vortex equations have a solution, then the degree $c_1(K_\Sigma^{\frac{1}{2}} \otimes L^{-\frac{1}{2}}) \cdot [\Sigma] \geq 0$ is non-negative. So we have $2g(\Sigma) - 2 \geq c_1(L) \cdot \Sigma$.

(2) If $r = 2g(\Sigma) - 2 - c_1(L) \cdot \Sigma \geq 0$, then for any set of pairs: $(x_1, n_1), \cdots, (x_k, n_k) \in X \times \mathbb{N}$ with $\sum_{i=1}^{k} n_i = r$, there is a ,unique up to gauge equvalence, solution to the vortex equations $(***)$. □

3. Almost Complex Manifolds

It is a classical result of Wu that a closed 4-manifold X has an almost complex structure J with first Chern class $c \in H^2(X, \mathbb{Z})$ if and only if c reduces modulo 2 to the second Stiefel-Whitney class $w_2(X) \in H^2(X, \mathbb{Z}_2)$ and $c^2 = 2\chi(X) + 3\sigma(X)$. Moreover for each such c there is unique one isomorphism class of J. The space $3\mathbb{C}P^2$ of 3 copies of $\mathbb{C}P^2$ has an almost complex structure, but the $2\mathbb{C}P^2$ does not. By the vanishing theorem the almost complex space $3\mathbb{C}P^2$ has no basic class.

Let X_1 be a closed symplectic 4-manifold. Let P be the Poincaré homology 3-sphere. Surgery on $P \times S^1$ eliminating the generator of $\pi_1(S^1)$ yields an integral homology 4-sphere X_2 with $\pi_1(X_2) = \pi_1(P)$. Let $X = X_1 \sharp X_2$ be the connected sum of X_1 and X_2. The collapsing map $f : X \to X_1$ induces an isomorphism $f^* : H^2(X_1, \mathbb{Z}) \to H^2(X, \mathbb{Z})$. By the Wu's theorem the cohomology class $f^*(c)$ defines an almost complex structure on X if $c \in H^2(X_1, \mathbb{Z})$ defines an almost complex structure on X_1. However by [4] the space X cannot have any symplectic structure. The map $f : X \to X_1$ induces identification Spin^c structures, and preserves the Seiberg-Witten invariants between X and X_1.

THEOREM 3.1. *In the above notations, the space $X = X_1 \sharp X_2$ has an almost complex structure, but does not have any symplectic struc-*

ture. There are a one-to-one correspondence between Seiberg-Witten invariants of X and ones of X_1.

LEMMA 3.2. *let X_1 be an almost complex 4-manifold. Then the connected sum $X \equiv X_1 \sharp n\overline{\mathbb{C}P^2}$ has an almost complex structure.*

Proof. Let $c_1 \in H^2(X_1, \mathbb{Z})$ define an almost complex structure. Let the E_i $(i = 1, \cdots, n)$ be the Poincaré duals of the exceptional spheres of the $\overline{\mathbb{C}P^2}$'s. Let $c \equiv c_1 - E_1 - \cdots - E_n \in H^2(X, \mathbb{Z})$. Then the Euler characteristic $\chi(X) = \chi(X_1) + n$, and the signature $\sigma(X) = \sigma(X_1) - n$, and the intersection number $c^2 = c_1^2 - n$. Thus we have

$$\begin{aligned} 2\chi(X) + 3\sigma(X) &= 2(\chi(X_1) + n) + 3(\sigma(X_1) - n) \\ &= (2\chi(X_1) + 3\sigma(X_1)) - n = c_1^2 - n = c^2. \end{aligned}$$

The class c reduces modulo 2 to the second Stiffel-Whitney class $w_2(X)$. By Wu's theorem c defines an almost complex structure on X. □

If a cohomology class $c \in H^2(X, \mathbb{Z})$ defines a Spin^c structure on X which has nonzero Seiberg-Witten invariant, then c is called a basic class of X.

THEOREM 3.3 (CONNECTED SUM) [1]. *If K is a basic class of a closed, almost complex 4-manifold X, then $K \pm E_1 \pm \cdots \pm E_n$ are basic classes of the space $X \sharp n\overline{\mathbb{C}P^2}$, where the E_i's are exceptional spheres of the $\overline{\mathbb{C}P^2}$'s.*

4. The Main Theorem

Now we are ready to prove the main theorem by using the results of previous sections.

THOEREM 4.1. *Let X be a closed almost complex 4-manifold with $b_2^+(X) > 1$, and have its canonical bundle as a basic class. Then the pseudo-holomorphic 2-dimensional submanifolds in X with non-negative self-intersection minimize genus in their homology classes.*

Proof. Let $u : \Sigma \to X$ represent a pseudo-holomorphic 2-dimensional submanifold in X with non-negative self-intersection. The almost complex structure on X descends to an almost complex structure on Σ, here we identify Σ with its image $u(\Sigma)$ in X. Thus we have the adjunction formula:

$$2g(\Sigma) - 2 = \Sigma \cdot \Sigma + K \cdot \Sigma$$

here we use the same notation K as the Poincaré dual of the first Chern class of the canonical line bundle K to X.

(1) Suppose that the self-intersection number $\Sigma \cdot \Sigma = 0$ is zero. The boundary of a tubular neighbourhood of Σ in X is diffeomorphic to the 3-manifold $\Sigma \times S^1$. For some positive number $r > 0$, embed $\Sigma \times S^1 \times [-r, r]$ into X as an isometry. Since the canonical class K is a basic class, by the theorem [8] the restriction of K on $(\Sigma \times S^1) \times \mathbb{R}$ has a solution to the Seiberg-Witten equations which is a translation-invariant in a temporal gauge. Thus we have a solution to the 3-dimensional Seiberg-Witten equations for the restricted bundle $K \to \Sigma \times S^1$. As in [6] or [13] the equations descend to the vortex equations over Σ for the line bundle of degree $d = K \cdot \Sigma$. The vortex equations has a solution if and only if the degree d is less than or equal to $2g(\Sigma) - 2$. Thus we have the inequality $2g(\Sigma) - 2 \geq K \cdot \Sigma$.

(2) Suppose that the self-intersection number $\Sigma \cdot \Sigma > 0$ is positive. Then we can reduce to the case of zero self-intersection number by n-times connected sum of $\overline{\mathbb{C}P^2}$. Let $\bar{X} = X \sharp n \overline{\mathbb{C}P^2}$ be the connected sum of X and n copies of $\overline{\mathbb{C}P^2}$, where we can think of the connected sums as being made at n points of Σ. Let E_i $(i = 1, \cdots, n)$ be the exceptional spheres in the corresponding $\overline{\mathbb{C}P^2}$'s. Let $\bar{\Sigma}$ be the surface in $\bar{X}$ obtained by taking a internal connected sum with the E_i's in the $\overline{\mathbb{C}P^2}$'s. Then the surface $\bar{\Sigma}$ has the form $\Sigma - E_1 - \cdots - E_n$ and self-intersection number $\bar{\Sigma} \cdot \bar{\Sigma} = 0$. By the connected sum formula the classes $K \pm E_1 \pm \cdots \pm E_n$ are basic classes in $\bar{X}$. Let $\bar{K} \equiv K + E_1 + \cdots + E_n$ be the sum of K and the E_i's. Then the degree of the line bundle for $\bar{K}$ over $\bar{\Sigma}$ is

$$\begin{aligned} \bar{K} \cdot \bar{\Sigma} &= (K + E_1 + \cdots + E_n) \cdot (\Sigma - E_1 \cdot \ \cdot - E_n) \\ &= K \cdot \Sigma + n = K \cdot \Sigma + \Sigma \cdot \Sigma. \end{aligned}$$

As in (1) we have $2g(\bar{\Sigma}) - 2 \geq \bar{K} \cdot \bar{\Sigma}$. Thus we have the required inequality

$$2g(\Sigma) - 2 \geq K \cdot \Sigma + \Sigma \cdot \Sigma$$

which is equal when Σ is a pseudo-holomorphic curve in X. □

REMARK. If $b_2^+(X) = 1$, then the Seiberg-Witten invariants on X depend on the metric on X. In [10] they define and use the negative Seiberg-Witten invariant for X to get the same result for compact Kähler surfaces. We may use the negative Seiberg-Witten invariant for almost complex 4-manifold with $b_2^+ = 1$.

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