GENERALIZED THOM CONJECTURE FOR ALMOST COMPLEX 4-MANIFOLDS

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ABSTRACT. Let $X$ be a closed almost complex 4-manifold with $b_2^+(X) > 1$, and have its canonical line bundle as a basic class. Then the pseudo-holomorphic 2-dimensional submanifolds in $X$ with non-negative self-intersection minimize genus in their homology classes.

1. Introduction

In [8] Kronheimer and Mrowka proved that algebraic curves in $\mathbb{C}P^2$ have minimum genuses in their homology classes. In [10] Morgan, Szabó and Taubes proved that if $\Sigma$ is a smooth holomorphic curve in a compact Kähler surface with non-negative self-intersection number $\Sigma \cdot \Sigma \geq 0$, then $\Sigma$ minimizes genus in its homology class. They used the Seiberg-Witten gauge theory to get their results. In [11] Taubes showed that the canonical complex line bundle of a symplectic 4-manifold has nonzero Seiberg-Witten invariant. The above results extend to symplectic 4-manifolds with almost same proofs. There are non-symplectic 4-manifolds which have nonzero Seiberg-Witten invariants [2].

In this paper we give some almost complex 4-manifolds $X$ constructed by the connected sums of symplectic 4-manifolds $X_1$ and homology 4-spheres. These almost complex 4-manifolds have no symplectic structures. However there are a one-to-one correspondence between Seiberg-Witten invariants of $X$ and ones of $X_1$. If $X$ is an almost complex 4-manifold, then the connected sum $X \# n\mathbb{C}P^2$ of $X$ and $n$ copies
of $\mathbb{C}P^2$ has an almost complex structure. If $X$ is a closed almost complex 4-manifold and its complex canonical line bundle is a basic class, then an embedded pseudo-holomorphic curve $\Sigma$ in $X$ with $\Sigma \cdot \Sigma \geq 0$ minimizes genus in its homology class.

2. Vortex Equations

Let $X$ be an almost complex 4-manifold with canonical complex line bundle $K$ and let $\Sigma$ be an embedded 2-dimensional submanifold of $X$ with zero self-intersection number $\Sigma \cdot \Sigma = 0$. Let $W^+ = E \oplus (K^{-1} \otimes E) \to X$ be a Spin$^c$ structure on $X$, where $E$ is a complex line bundle over $X$. Let $L = \det W^+ = E \otimes (K^{-1} \otimes E)$. For a connection $A$ of $L$ and a section $\phi \in \Gamma(W^+)$ of $W^+$ the Seiberg-Witten equations are

\[
(*) \begin{cases}
D_A \phi = 0 \\
F_A^+ = -\tau(\phi, \bar{\phi}).
\end{cases}
\]

Since $\Sigma \cdot \Sigma = 0$ there is a tubular neighbourhood $N(\Sigma)$ of $\Sigma$ in $X$ such that its boundary $\partial N(\Sigma) \equiv Y$ is diffeomorphic to $\Sigma \times S^1$. Let $(X_R, g_R)$ be the Riemannian manifold obtained from $X$ by cutting along $Y$ and inserting a cylinder $[-R, R] \times Y$, here $g_R$ is a product metric on $[-R, R] \times Y$. In [8] Kronheimer and Mrowka showed the following:

**Proposition 2.1** [8]. Suppose the moduli space $\mathcal{M}(L, g_R)$ is non-empty for all sufficiently large $R$. Then there is a solution of the equation $(*)$ on the cylinder $\mathbb{R} \times Y$ which is translation invariant in a temporal gauge.

In the temporal gauge a connection $A$ and a section $\phi$ on the cylinder $\mathbb{R} \times Y$ can be thought as a path $A(t)$ of connections and a path $\phi(t)$ of sections in the restricted Spin$^c$ structure over 3-manifold $Y$. In this case, the Seiberg-Witten equations become

\[
(**) \begin{cases}
\frac{d\phi}{dt} = -\overline{D}_A \phi \\
\frac{dA}{dt} = -\ast F_A - \tau(\phi, \bar{\phi})
\end{cases}
\]
where $\bar{D}_A$ is the Dirac operator in 3-dimensional Spin$^c$ structure $W$ and $\tau$ is a pairing obtained from Clifford multiplication by using the hermitian metric on $W$.

By the uniformization theorem there is a Riemannian metric on $\Sigma \times S^1 \equiv Y$ such that $\Sigma$ has constant scalar curvature. Then using the Gauss-Bonnet theorem, Kronheimer and Mrowka showed the following:

**Theorem 2.2** [8]. *If there is a solution to the Seiberg-Witten equations on $\mathbb{R} \times Y$ which translation-invariant in a temporal gauge, then $|c_1(L)[\Sigma]| \leq 2g(\Sigma) - 2$.*

We assume that the line bundle $L$ over $\Sigma \times S^1$ is pulled back from a line bundle over $\Sigma$. This may be justified when the cohomology class of $L$ over $\Sigma \times S^1$ has no component in $H^1(\Sigma) \otimes H^1(S^1)$. The equations of $S^1$-invariant solutions of (**) reduce to the following vortex equations (***) over $\Sigma$, for details see [6]. By the symmetry between $L$ and $L^{-1}$, we may suppose that the degree $d = c_1(L^{-1}) \cdot \Sigma \geq 0$ is non-negative.

\[
\begin{cases}
\bar{D}_A \psi = 0 \\
F_A = -|\psi|^2
\end{cases}
\]

where $A$ is a connection of $L^{-1} \to \Sigma$ and $\psi$ is a section of $K^{1/2}_\Sigma \otimes L^{-1/2} \to \Sigma$. In this case the twisted spinor bundle is $W = K^{1/2}_\Sigma \otimes L^{-1/2} \oplus K^{-1/2}_\Sigma \otimes L^{-1/2}$, and the determinant line bundle is $\det W = L^{-1}$ over the Riemann surface $\Sigma$. Suppose that $E \oplus K^{-1} \otimes E \to X$ is a Spin$^c$ structure over an almost complex 4-manifold $X$ and $\Sigma$ is an embedded 2-manifold in $X$ with self-intersection number $\Sigma \cdot \Sigma = 0$.

**Theorem 2.3.** Under above assumptions,

1. If a Spin$^c$ structure $E \oplus (K^{-1} \otimes E) \to X$ has a solution of the Seiberg-Witten equations, then the reduced vortex equations over $\Sigma$ has a solution. In this case $2g(\Sigma) - 2 \geq c_1(L) \cdot \Sigma(= d)$ if $E = K^{1/2}_\Sigma \otimes L^{-1/2}$.

2. If $r = (2g - 2) - d \geq 0$, then the space of solutions of the vortex equations is identified with the symmetric product $s^r(\Sigma)$ of $r$ copies of $\Sigma$.  

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Proof. We will sketch the proof of this theorem, for details see [6] and [13].

(1) The Seiberg-Witten equations (\(*\)) is reduced to the vortex equations (\(\ast \ast \ast\)) under the above conditions. If the vortex equations have a solution, then the degree \(c_1(K^1_\Sigma \otimes L^{-\frac{1}{2}}) \cdot [\Sigma] \geq 0\) is non-negative. So we have \(2g(\Sigma) - 2 \geq c_1(L) \cdot \Sigma\).

(2) If \(r = 2g(\Sigma) - 2 - c_1(L) \cdot \Sigma \geq 0\), then for any set of pairs: 
\((x_1, n_1), \ldots, (x_k, n_k) \in X \times \mathbb{N}\) with \(\sum_{i=1}^{k} n_i = r\), there is a unique up to gauge equivalence, solution to the vortex equations \((\ast \ast \ast)\). □

3. Almost Complex Manifolds

It is a classical result of Wu that a closed 4-manifold \(X\) has an almost complex structure \(J\) with first Chern class \(c \in H^2(X, \mathbb{Z})\) if and only if \(c\) reduces modulo 2 to the second Stiefel-Whitney class \(w_2(X) \in H^2(X, \mathbb{Z}_2)\) and \(c^2 = 2\chi(X) + 3\sigma(X)\). Moreover for each such \(c\) there is unique one isomorphism class of \(J\). The space \(3CP^2\) of 3 copies of \(CP^2\) has an almost complex structure, but the \(2CP^2\) does not. By the vanishing theorem the almost complex space \(3CP^2\) has no basic class.

Let \(X_1\) be a closed symplectic 4-manifold. Let \(P\) be the Poincaré homology 3-sphere. Surgery on \(P \times S^1\) eliminating the generator of \(\pi_1(S^1)\) yields an integral homology 4-sphere \(X_2\) with \(\pi_1(X_2) = \pi_1(P)\). Let \(X = X_1 \# X_2\) be the connected sum of \(X_1\) and \(X_2\). The collapsing map \(f : X \to X_1\) induces an isomorphism \(f^* : H^2(X_1, \mathbb{Z}) \to H^2(X, \mathbb{Z})\). By the Wu’s theorem the cohomology class \(f^*(c)\) defines an almost complex structure on \(X\) if \(c \in H^2(X_1, \mathbb{Z})\) defines an almost complex structure on \(X_1\). However by [4] the space \(X\) cannot have any symplectic structure. The map \(f : X \to X_1\) induces identification \(\text{Spin}^c\) structures, and preserves the Seiberg-Witten invariants between \(X\) and \(X_1\).

Theorem 3.1. In the above notations, the space \(X = X_1 \# X_2\) has an almost complex structure, but does not have any symplectic struc-
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ture. There are a one-to-one correspondence between Seiberg-Witten invariants of $X$ and ones of $X_1$.

**Lemma 3.2.** Let $X_1$ be an almost complex 4-manifold. Then the connected sum $X \equiv X_1 \# n\overline{CP^2}$ has an almost complex structure.

**Proof.** Let $c_1 \in H^2(X_1, \mathbb{Z})$ define an almost complex structure. Let the $E_i$ ($i = 1, \cdots, n$) be the Poincaré duals of the exceptional spheres of the $\overline{CP^2}$'s. Let $c \equiv c_1 - E_1 - \cdots - E_n \in H^2(X, \mathbb{Z})$. Then the Euler characteristic $\chi(X) = \chi(X_1) + n$, and the signature $\sigma(X) = \sigma(X_1) - n$, and the intersection number $c^2 = c_1^2 - n$. Thus we have

$$2\chi(X) + 3\sigma(X) = 2(\chi(X_1) + n) + 3(\sigma(X_1) - n)$$

$$= (2\chi(X_1) + 3\sigma(X_1)) - n = c_1^2 - n = c^2.$$ 

The class $c$ reduces modulo 2 to the second Stiefel-Whitney class $w_2(X)$. By Wu's theorem $c$ defines an almost complex structure on $X$. \qed

If a cohomology class $c \in H^2(X, \mathbb{Z})$ defines a Spin$^c$ structure on $X$ which has nonzero Seiberg-Witten invariant, then $c$ is called a basic class of $X$.

**Theorem 3.3 (Connected Sum)** [1]. If $K$ is a basic class of a closed, almost complex 4-manifold $X$, then $K \pm E_1 \pm \cdots \pm E_n$ are basic classes of the space $X \# n\overline{CP^2}$, where the $E_i$'s are exceptional spheres of the $\overline{CP^2}$'s.

4. The Main Theorem

Now we are ready to prove the main theorem by using the results of previous sections.

**Theorem 4.1.** Let $X$ be a closed almost complex 4-manifold with $b_2^+(X) > 1$, and have its canonical bundle as a basic class. Then the pseudo-holomorphic 2-dimensional submanifolds in $X$ with non-negative self-intersection minimize genus in their homology classes.
Proof. Let \( u : \Sigma \to X \) represent a pseudo-holomorphic 2-dimensional submanifold in \( X \) with non-negative self-intersection. The almost complex structure on \( X \) descends to an almost complex structure on \( \Sigma \), here we identify \( \Sigma \) with its image \( u(\Sigma) \) in \( X \). Thus we have the adjunction formula:

\[
2g(\Sigma) - 2 = \Sigma \cdot \Sigma + K \cdot \Sigma
\]

here we use the same notation \( K \) as the Poincaré dual of the first Chern class of the canonical line bundle \( K \) to \( X \).

(1) Suppose that the self-intersection number \( \Sigma \cdot \Sigma = 0 \) is zero. The boundary of a tubular neighbourhood of \( \Sigma \) in \( X \) is diffeomorphic to the 3-manifold \( \Sigma \times S^1 \). For some positive number \( r > 0 \), embed \( \Sigma \times S^1 \times [-r, r] \) into \( X \) as an isometry. Since the canonical class \( K \) is a basic class, by the theorem [8] the restriction of \( K \) on \( (\Sigma \times S^1) \times \mathbb{R} \) has a solution to the Seiberg-Witten equations which is a translation-invariant in a temporal gauge. Thus we have a solution to the 3-dimensional Seiberg-Witten equations for the restricted bundle \( K \to \Sigma \times S^1 \). As in [6] or [13] the equations descend to the vortex equations over \( \Sigma \) for the line bundle of degree \( d = K \cdot \Sigma \). The vortex equations has a solution if and only if the degree \( d \) is less than or equal to \( 2g(\Sigma) - 2 \). Thus we have the inequality \( 2g(\Sigma) - 2 \geq K \cdot \Sigma \).

(2) Suppose that the self-intersection number \( \Sigma \cdot \Sigma > 0 \) is positive. Then we can reduce to the case of zero self-intersection number by \( n \)-times connected sum of \( \overline{CP^2} \). Let \( \bar{X} = X \#_n \overline{CP^2} \) be the connected sum of \( X \) and \( n \) copies of \( \overline{CP^2} \), where we can think of the connected sums as being made at \( n \) points of \( \Sigma \). Let \( E_i \) \((i = 1, \ldots, n)\) be the exceptional spheres in the corresponding \( \overline{CP^2} \)'s. Let \( \bar{\Sigma} \) be the surface in \( \bar{X} \) obtained by taking a internal connected sum with the \( E_i \)'s in the \( \overline{CP^2} \)'s. Then the surface \( \bar{\Sigma} \) has the form \( \Sigma - E_1 - \cdots - E_n \) and self-intersection number \( \bar{\Sigma} \cdot \bar{\Sigma} = 0 \). By the connected sum formula the classes \( K \pm E_1 \pm \cdots \pm E_n \) are basic classes in \( \bar{X} \). Let \( \bar{K} \equiv K + E_1 + E_n \) be the sum of \( K \) and the \( E_i \)'s. Then the degree of the line bundle for \( \bar{K} \) over \( \bar{\Sigma} \) is

\[
\bar{K} \cdot \bar{\Sigma} = (K + E_1 + \cdots + E_n) \cdot (\Sigma - E_1 - \cdots - E_n) = K \cdot \Sigma + n = K \cdot \Sigma + \Sigma \cdot \Sigma.
\]

As in (1) we have \( 2g(\bar{\Sigma}) - 2 \geq \bar{K} \cdot \bar{\Sigma} \). Thus we have the required
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inequality

\[ 2g(\Sigma) - 2 \geq K \cdot \Sigma + \Sigma \cdot \Sigma \]

which is equal when \( \Sigma \) is a pseudo-holomorphic curve in \( X \).

\[ \square \]

Remark. If \( b_2^+(X) = 1 \), then the Seiberg-Witten invariants on \( X \) depend on the metric on \( X \). In [10] they define and use the negative Seiberg-Witten invariant for \( X \) to get the same result for compact Kähler surfaces. We may use the negative Seiberg-Witten invariant for almost complex 4-manifold with \( b_2^+ = 1 \).

References


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