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CONTROLLABILITY FOR SOME SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Using the theory of strongly continuous cosine families, we prove controllable of nonlinear second order control system and give an application

1. Introduction

Consider the abstract nonlinear second order control system:

$$
\begin{aligned}
x''(t) &= Ax(t)+f(t,x(t)) + Bu(t) \\
x(0) &= x_0, \quad x'(0) = y_0,
\end{aligned}
\tag{1.1}
$$

where $x_0, y_0 \in X(X$ is a Banach space) and x is a mapping from R to X, A is the infinitesimal generator of a strongly continuous cosine family of linear operators in X, and f is a nonlinear mapping from $R \times X$ to X and the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with U a Banach space. Also, B is a bounded linear operator from U into X. C. C. Travis and G. F. Webb [4] studied existence, uniqueness, continuous dependence, and smoothness of solutions of Eq.(1.1) with $B = 0$. P. Balachandran, J. P. Dauer, and P. Balasubramaniam [1] have shown the controllability of nonlinear integrodifferential systems in Banach space by using the Schauder Fixed-point Theorem. The purpose of the paper is to study

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the controllability of high order differential systems. This paper is organized as follows: In section 2 we give some definitions and lemmas. In section 3 we state the main result. In section 4 we give some application to illustrate our result.

2. Preliminaries

DEFINITION 2.1 [4]. A one parameter family $C(t)$, $t \in R$, of bounded linear operators mapping the Banach space X into itself is called a *strongly continuous cosine family* if

(1) $C(s+t)+C(s-t) = 2C(s)C(t)$ for all $s, t \in R$,
(2) $C(0) = I$,
(3) $C(t)x$ is continuous in t on R for each fixed $x \in X$.

If $C(t), t \in R$, is a strongly continuous cosine family in X, then $S(t)$, $t \in R$, is the one parameter family of operators in X defined by

$$S(t)x = \int_0^t C(s)xds, \quad x \in X, t \in R.$$

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(0)x,$$

where $D(A) = \{x \in X | C(t)x$ is a twice continuously differentiable function of $t\}$. We shall also make use of the set

$E = \{x | C(t)x$ is a once continuously differentiable function of $t\}$.

LEMMA 2.2 [4]. *If $C(t), t \in R$, be a strongly continuous cosine family in X, then*
(i) *there exist constants $K \geq 1$ and $\omega \geq 0$ so that $|C(t)| \leq Ke^{\omega|t|}$ for all $t \in R$ and*

$$|S(t_1) - S(t_2)| \leq K \left| \int_{t_1}^{t_2} e^{\omega|s|} ds \right| \quad \textit{for all} \quad t_1, t_2 \in R.$$

(ii) *if $x \in E$, then $S(t)x \in D(A)$ and $d/dtC(t)x = AS(t)x$.*

LEMMA 2.3 [4]. *Let $C(t), t \in R$, be a strongly continuous cosine family in X with infinitesimal generator A. If $g : R \to X$ is continuous differentiable, $x_1 \in D(A), x_2 \in E$, and*

$$x(t) = C(t)x_1 + S(t)x_2 + \int_0^t S(t-s)g(s)ds, \quad t \in R,$$

then $x(t) \in D(A)$ for $t \in R$, x is twice continuously differentiable, and x satisfies

$$x''(t) = Ax(t) + g(t) \quad x(0) = x_1, \quad x'(0) = x_2.$$

We consider the following mild solution of Eq.(1.1)(see [4])
(2.1)

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s,x(s))ds + \int_0^t S(t-s)Bu(s)ds$$

DEFINITION 2.4 [1]. The system (1.1) is said to be *controllable* on the interval $J = [-T, T]$ if, for every $x_0, y_0, a \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1.1) satisfies $x(T) = a$.

We introduce the following assumptions:

(C_1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators in the Banach space X and the associated sine family $S(t), t \in R$, is compact with $\|S(t)\| \leq M$.

(C_2) The linear operator W from U into X defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

and there exists an bounded invertible operator W^{-1} defind on $L^2(J, U)/\ker W$ and Bu is continuously differentiable.

(C_3) The function $f(t, x(t))$ is a nonlinear continuously differentiable such that

$$\|f(t, x(t))\| \leq N.$$

LEMMA 2.5 [2]. *Suppose that X is a Banach space and g is an integrable function from J into X. Then*

$$(\beta-\alpha)^{-1}\int_{\alpha}^{\beta} g(s)ds \in \overline{co}(\{g(s) : s \in [\alpha,\beta]\})$$

for all $\alpha, \beta \in J$ with $\alpha < \beta$, where J is an interval.

LEMMA 2.6. *Suppose (C_1) hold. Let Q be a bounded subset of X and $\{h_m : m \in \Gamma\}$ be a set of continuous functions from the finite interval $[\alpha,\beta] \subset (-\infty,\infty)$ to Q. Then $V = \{\int_{\alpha}^{\beta} S(s)h_m(s)ds : m \in \Gamma\}$ is a precompact subset of X.*

Proof. Let $Z = \{S(t)x : t \in [\alpha,\beta], x \in Q\}$. Since X is a Banach space, if we only show that Z is a totally bounded, then Z is a precompact in X. Let $\varepsilon > 0$ and let L is a positive constant such that $\|x\| < L$ for each $x \in Q$. Since $S(t)$ is uniformly continuous on $[\alpha,\beta]$, there exist finite number $t_i \in [\alpha,\beta], 1 \le i \le n$, such that $\alpha = t_1$ and $\beta = t_n, \|S(t_i) - S(t)\| \le \varepsilon/2L$, for $t \in [t_{i-1}, t_i]$. Since $S(t)$ is compact for each $t \in R$, $S(t_i)Q$ is totally bounded. So there exists $\{x_1^i, x_2^i, \cdots, x_{l(t)}^i\} \subset Q$ so that if $x \in Q$, then

$$\|S(t_i)x_j^i - S(t_i)x\| \le \varepsilon/2$$

for some x_j^i. Then, for each $x \in Q$, there exists $x_j^i \in Q$ such that

$$\begin{aligned}\|S(t)x - S(t_i)x_j^i\| &\le \|S(t)x - S(t_i)x\| + \|S(t_i)x - S(t_i)x_j^i\| \\ &\le \frac{\varepsilon}{2M}M + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So Z is a totally bounded. i.e., Z is precompact and therefore so is the convex hull of Z. By Lemma 2.5, since

$$\int_{\alpha}^{\beta} S(s)h_m(s)ds \in (\beta-\alpha)\overline{co}(Z).$$

Hence V is a precompact subset of X. □

Now, in the following section, we prove the result of controllability for Eq.(1.1) with above facts.

3. Main result

THEOREM 3.1. *Suppose* $(C_1) \sim (C_3)$ *hold. Let* D *be an open subset of Banach space* X. *If* $f : J \times D \to X$ *is continuous, then for each* $x \in D$ *such that* $x \in D(A)$ *and for each* $y \in E$, *then the system (1.1) is controllable on* $[-T, T]$.

Proof. Using the assumption (C_2), for an arbitrary function $x(\cdot)$ define the control

$$u(t) = W^{-1}[a - C(T)x_0 - S(T)y_0 - \int_0^T S(T-s)f(s, x(s))ds](t).$$

Also, using this control, we shall show that operator G is defined in below has a fixed point. For $\delta > 0$, consider

$$N_\delta(x) = \{z \in X | \|x - z\| < \delta\}.$$

Put $\phi(t) = C(t)x_0 + S(t)y_0$, then $\phi : R \to X$ is continuous.
Choose $\delta, T > 0$ such that $N_\delta(x) \subset D$ and $\|\phi(t) - x\| < \delta/2$. Let

$$C = C([-T, T], X) \quad \text{and}$$

$$K = \{\psi \in C | \|\psi - \phi\|_C \leq \delta/2\},$$

where $\delta/2 = MNT + MT\|B\|\|W^{-1}\|[\|a\| + Le^{\omega|T|}\|x_0\| + M\|y_0\| + MNT]$.
If $\psi \in K$, then

$$\|\psi(t) - x\| \leq \|\psi(t) - \phi(t)\| + \|\phi(t) - x\| < \delta/2 + \delta/2 = \delta.$$

So, $K \subset N_\delta(x) \subset D$. Define

$$(Gx)(t) = \phi(t) + \int_0^t S(t-s)f(s, x(s))ds + \int_0^t S(t-s)BW^{-1}$$
$$\cdot [a - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\tau)f(\tau, x(\tau))d\tau](s)ds.$$

First of all, we show that G maps K into itself.

$$
\begin{aligned}
&\|(Gx)(t)-\phi(t)\| \\
&\qquad \le \|\int_0^t S(t-s)f(s,x(s))ds\| + \|\int_0^t S(t-s)BW^{-1} \\
&\qquad \cdot [a - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\tau)f(\tau,x(\tau))d\tau](s)ds\| \\
&\qquad \le MNT + MT\|B\|\|W^{-1}\| \\
&\qquad\qquad [\|a\| + Le^{\omega|T|}\|x_0\| + M\|y_0\| + MNT] = \delta/2.
\end{aligned}
$$

Thus G maps K into itself. Since ϕ, f are continuous functions, so is G. Let $K(t) = \{(Gx)(t) | x \in K\}$ for each $t \in [-T,T]$. Since $S(t)$ is uniformly continuous on $[-T,T]$ and compact, by Lemma 2.6 $K(t)$ is a precompact in X. We want to show that $G(K) = \{Gx | x \in K\}$ is an equicontinuous family of functions.
For $-T \le t_1 \le t_2 \le T$ and $x \in K$,

$$
\begin{aligned}
&\|(Gx)(t_1)-(Gx)(t_2)\| \\
&\le \|\phi(t_1)-\phi(t_2)\| + \|\int_0^{t_1} [S(t_1-s)-S(t_2-s)]f(s,x(s))ds\| \\
&+ \|\int_{t_1}^{t_2} S(t_2-s)f(s,x(s))ds\| + \|\int_0^{t_1} [S(t_1-s)-S(t_2-s)]BW^{-1} \\
&\cdot [a - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\tau)f(\tau,x(\tau))d\tau](s)ds\| \\
&+ \|\int_{t_1}^{t_2} S(t_2-s)BW^{-1} \\
&\cdot [a - C(T)x_0 - S(T)y_0 - \int_0^T S(T-\tau)f(\tau,x(\tau))d\tau](s)ds\| \\
&\le \|\phi(t_1)-\phi(t_2)\| + \int_0^{t_1} \|S(t_1-s)-S(t_2-s)\|\|f(s,x(s))\|ds \\
&+ N\int_{t_1}^{t_2} \|S(t_2-s)\|ds + \int_0^{t_1} \|S(t_1-s)-S(t_2-s)\|\|B\|\|W^{-1}\|
\end{aligned}
$$

$$\cdot [\|a\| + Le^{\omega|T|}\|x_0\| + M\|y_0\| + N\int_0^T \|S(T-\tau)\|d\tau](s)ds$$
$$+\int_{t_1}^{t_2} \|S(t_2-s)\|\|B\|\|W^{-1}\|$$
$$\cdot [\|a\| + Le^{\omega|T|}\|x_0\| + M\|y_0\| + N\int_0^T \|S(T-\tau)\|d\tau](s)ds$$
$$\to 0 \quad as \quad |t_1 - t_2| \to 0.$$

Hence $G(K)$ is an equicontinuous family of functions.
And $G(K)$ is bounded in C. By Arzela-Ascoli Theorem $G(K)$ is pre-compact. Direct application of the Schauder Fixed point Theorem yields the existence of $x \in K$ such that $(Gx)(t) = x(t)$. Since $x_0 \in D(A)$ and $y_0 \in E$, then the solution of (2.1) is a solution of Eq.(1.1) by Lemma 2.3. Therefore every fixed point of G is a mild solution of Eq.(1.1). Consequently, Eq.(1.1) is controllable on $[-T, T]$. □

4. Application

As an example to which we can apply our result of section 3, we cite the following partial differential equation:

(4.1)
$$\frac{\partial^2 z}{\partial t^2}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + g(t, \frac{\partial^2 z}{\partial x^2}(x,t)) + (Bu)(t), \quad 0 < x < \pi, t \in R$$
$$z(0,t) = z(\pi,t), \quad t \in R$$
$$z(x,0) = z_0(x), \quad \frac{\partial z}{\partial t}(x,0) = z_1(x), \quad 0 < x < \pi.$$

Let $X = L^2((0,\pi), R)$ and $B : U \to X$, with $U \subset [-T, T]$, be a bounded linear operator such that Bu be continuously differentiable. Define W defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds$$

and there exists an bounded invertible W^{-1} on $L^2([-T,T], U)/\ker W$. Also, sine family $S(t)$ is compact operator and $g : [-T,T] \times X \to X$ is continuously differentiable and bounded. Let $A : X \to X$ be defined by

$$Aw = w'',$$

where $D(A) = \{w \in X | w, w' \text{ are absolutely continuous}, w'' \in X, w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2 (w, w_n) w_n, \quad w \in D(A)$$

where $w_n(s) = \sqrt{2/\pi} \sin ns, n = 1, 2, \cdots$, is the orthonormal set of eigenvalues of A. And so A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt (w, w_n) w_n, \quad w \in X,$$

and that the associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{\sin nt}{n} (w, w_n) w_n, \quad w \in X.$$

We now define mapping $f : [-T, T] \times X \to X$ as follows:

$$f(t, w)(x) = g(t, w''(x)).$$

The problem (4.1) can be formulated abstractly as

$$x''(t) = Ax(t) + f(t, x(t)) + Bu(t),$$

$$x(0) = x_0, \quad x'(0) = y_0.$$

Then, all the conditions stated in the above theorem are satisfied. So the Eq.(4.1) is controllable on $[-T, T]$.

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