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ON A GENERAL HYERS-ULAM STABILITY OF GAMMA FUNCTIONAL EQUATION

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Abstract. In this paper, the Hyers-Ulam stability and the general Hyers-Ulam stability (more precisely, modified Hyers-Ulam-Rassias stability) of the gamma functional equation (3) in the following settings

$$|f(x+1) - xf(x)| \leq \delta \quad \text{and} \quad \left|\frac{f(x+1)}{xf(x)} - 1\right| \leq \frac{\delta}{x^{1+\varepsilon}}$$

shall be proved.

1. Introduction

In 1940, S. M. Ulam [8] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:

> *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

In 1941, D. H. Hyers answered in [4] the question of Ulam for the case when G_1 and G_2 are Banach spaces:

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THEOREM 1. *Let $f : E_1 \to E_2$ be a mapping between Banach spaces satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{1}$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then there exists a unique additive mapping $A : E_1 \to E_2$ for which the inequality

$$\|f(x) - A(x)\| \leq \delta$$

holds true for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the mapping A is linear.

Taking this result into account, the additive Cauchy equation $f(x + y) = f(x)+f(y)$ is said to have the Hyers-Ulam stability on (G, E), where G and E are given spaces, if for every mapping $f : G \to E$ satisfying the inequality (1), for some $\delta \geq 0$ and for all $x, y \in G$, there exists an additive mapping $A : G \to E$ such that $f - A$ is bounded on G.

In 1978, Th. M. Rassias [7] attempted to weaken the condition for the bound of the norm of the Cauchy difference $f(x + y) - f(x) - f(y)$ and proved a considerably generalized result of Hyers. In fact, he proved the following theorem.

THEOREM 2. *Let $f : E_1 \to E_2$ be a mapping between Banach spaces, and let $\theta \geq 0$ and $0 \leq p < 1$ be fixed. If f satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \left(\|x\|^p + \|y\|^p\right) \tag{2}$$

for all $x, y \in E_1$, then there exists a unique additive mapping $A : E_1 \to E_2$ for which the inequality

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$$

holds true for all $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the mapping A is linear.

This result is a significant generalization of that of Hyers (Theorem 1) and stimulated many mathematicians to investigate the stability problems of several functional equations. By regarding a large influence of

Theorem 2 on the study of stability problems of several functional equations, the stability phenomenon of such type is called the Hyers-Ulam-Rassias stability (or a general Hyers-Ulam stability.) If the inequality (2) whose right-hand side is replaced by some suitable mapping $\varphi(x, y)$ is stable, the additive Cauchy equation is said to have the modified Hyers-Ulam-Rassias stability (or a general Hyers-Ulam stability.) These terminologies are similarly applied to the cases of other functional equations.

In 1979, J. Baker, J. Lawrence and F. Zorzitto [1] proved the superstability of the exponential equation $f(x+y) = f(x)f(y)$. In fact, they proved that if f is a mapping from a vector space to the real numbers satisfying the inequality $|f(x+y) - f(x)f(y)| \leq \delta$ for all x and y in the domain, then f is either bounded or exponential. On the other hand, R. Ger and P. Šemrl [3] has recently proved the stability of that equation. More precisely, they proved that if f is a mapping from a cancellative abelian semigroup to the complex numbers without '0' which satisfies

$$\left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \leq \delta$$

for some $0 \leq \delta < 1$, then there exists a unique exponential function F such that

$$\max\left\{\left|\frac{f(x)}{F(x)} - 1\right|, \left|\frac{F(x)}{f(x)} - 1\right|\right\} \leq \varepsilon$$

where $\varepsilon \to 0$ as $\delta \to 0$ (see also [2].)

The following functional equation

$$(3) \qquad f(x+1) = xf(x) \quad \textit{for all} \quad x > 0$$

is called the gamma functional equation. It is well-known that the "gamma function"

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \quad (x > 0)$$

is a solution of the gamma functional equation (3).

The author has treated in [5] the stability problem of the gamma functional equation (3). Really, he treated the inequality $|f(x+1) - xf(x)| \leq \varphi(x)$ for some suitable φ.

In this paper, the Hyers-Ulam stability of the gamma functional equation, more precisely, the Hyers-Ulam stability of the functional inequality (4) shall be proved (see Theorem 3). Furthermore, a general Hyers-Ulam stability (modified Hyers-Ulam-Rassias stability) of the functional inequality (9) shall also be investigated in the spirit of R. Ger (see Theorem 5).

2. Hyers-Ulam stability of Gamma functional equation

In the following theorem, the Hyers-Ulam stability of the gamma functional equation (3) is investigated. Throughout section 2, let $\delta > 0$ be fixed and let n_0 be a given non-negative integer.

THEOREM 3. *If a mapping* $f : (0, \infty) \to \mathbb{R}$ *satisfies the following inequality*

$$|f(x+1) - xf(x)| \le \delta \tag{4}$$

for all $x > n_0$ *then there exists a unique solution* $F : (0, \infty) \to \mathbb{R}$ *of the gamma functional equation* (3) *with*

$$|F(x) - f(x)| \le 3\delta/x \tag{5}$$

for all $x > n_0$.

Proof. For any $x > 0$ and for every positive integer n we define

$$P_n(x) = f(x+n) \prod_{i=0}^{n-1} (x+i)^{-1}.$$

By (4) we have

$$|P_{n+1}(x) - P_n(x)| \le \delta \prod_{i=0}^{n} (x+i)^{-1} \tag{6}$$

for $x > n_0$. Now, we use induction on n to prove

$$|P_n(x) - f(x)| \le \delta \sum_{j=0}^{n-1} \prod_{i=0}^{j} (x+i)^{-1} \tag{7}$$

for all $x > n_0$ and for all positive integers n. The inequality (7) for the case of $n = 1$ is an immediate consequence of (4). Assume that (7) holds true for some n. It then follows from (6) and (7) that

$$\begin{aligned} |P_{n+1}(x) - f(x)| &\leq |P_{n+1}(x) - P_n(x)| + |P_n(x) - f(x)| \\ &\leq \delta \sum_{j=0}^{n} \prod_{i=0}^{j} (x+i)^{-1} \end{aligned}$$

which completes the proof of (7). Let m, n be positive integers with $n \geq m$. Suppose x ($> n_0$) is given. In view of (6) we get

$$|P_n(x) - P_m(x)| \leq \sum_{j=m}^{n-1} |P_{j+1}(x) - P_j(x)| \to 0$$

as $m \to \infty$. This fact implies that $\{P_n(x)\}$ is a Cauchy sequence for $x > n_0$ and hence we can define a mapping $F_0 : (n_0, \infty) \to \mathbb{R}$ by

$$F_0(x) = \lim_{n \to \infty} P_n(x)$$

for all $x > n_0$. It is easy to see

$$F_0(x+1) = \lim_{n \to \infty} P_n(x+1) = \lim_{n \to \infty} x P_{n+1}(x) = x F_0(x) \tag{8}$$

for any $x > n_0$. On account of (7), it is clear that F_0 satisfies the inequality (5) for all $x > n_0$. Now, let $G : (n_0, \infty) \to \mathbb{R}$ be another mapping which satisfies (8) as well as (5) for all $x > n_0$. It then follows from (8) and (5) that

$$\begin{aligned} &|F_0(x) - G(x)| \\ &\quad = (x+n-1)^{-1}(x+n-2)^{-1} \cdots x^{-1} |F_0(x+n) - G(x+n)| \\ &\quad \leq 6\delta (x+n)^{-1}(x+n-1)^{-1} \cdots x^{-1} \end{aligned}$$

for all $x > n_0$ and for all positive integers n. This implies the uniqueness of F_0. We can inductively define new mappings $F_i : (n_0 - i, n_0 - i + 1] \to \mathbb{R}$ $(i = 1, 2, ..., n_0)$ by

$$F_i(x) = \frac{1}{x} F_{i-1}(x+1).$$

Further, define a mapping $F : (0, \infty) \to \mathbb{R}$ by $F(x) = F_i(x)$ for $n_0 - i < x \leq n_0 - i + 1$ $(i = 1, 2, ..., n_0)$ and $F(x) = F_0(x)$ for $x > n_0$. From (8) and the definition of F it follows that F is a unique solution of the gamma functional equation (3) and that F satisfies (5) for $x > n_0$. □

COROLLARY 4. *Let $f : (0, \infty) \to \mathbb{R}$ be a mapping. If f satisfies the functional inequality (4) asymptotically, more precisely, if there exist a positive number δ and a positive integer n_0 such that the functional inequality (4) holds true for all $x > n_0$, then f satisfies the gamma functional equation (3) asymptotically.*

Proof. Let n $(> n_0)$ be an arbitrary integer. Since f satisfies the inequality (4) for all $x > n$, according to Theorem 3, there exists a unique solution $F_n : (0, \infty) \to \mathbb{R}$ of the gamma functional equation (3) which satisfies the inequality (5) for any $x > n$. Now, let m $(> n)$ be an integer. According to Theorem 3 again, there exists a unique solution $F_m : (0, \infty) \to \mathbb{R}$ of (3) which satisfies (5) for all $x > m$. Since F_n is a unique solution of (3) which satisfies (5) for all $x > m$, we conclude $F_m = F_n$. Let $F = F_n$ for any $n > n_0$. Then, (5) implies that $f(x) \to F(x)$ as $x \to \infty$. □

3. General Hyers-Ulam stability of Gamma functional equation

In this section, a general Hyers-Ulam stability (modified Hyers-Ulam-Rassias stability) of the gamma functional equation (3) shall be investigated in the spirit of R. Ger. Throughout this section, let $\delta, \varepsilon > 0$ be given and define

$$\alpha(x) = \prod_{i=0}^{\infty} \left[1 - \delta(x+i)^{-(1+\varepsilon)}\right], \quad \beta(x) = \prod_{i=0}^{\infty} \left[1 + \delta(x+i)^{-(1+\varepsilon)}\right]$$

for any $x > \delta^{1/(1+\varepsilon)}$. Let $n_0 \geq 0$ be any integer.

THEOREM 5. *If a mapping $f : (0, \infty) \to (0, \infty)$ satisfies the inequality*

$$\left| \frac{f(x+1)}{xf(x)} - 1 \right| \leq \frac{\delta}{x^{1+\varepsilon}} \tag{9}$$

for all $x > n_0$, then there exists a unique solution $F : (0, \infty) \to [0, \infty)$ of the gamma functional equation (3) with

$$\alpha(x) \leq F(x)/f(x) \leq \beta(x) \tag{10}$$

for any $x > \max\{n_0, \delta^{1/(1+\varepsilon)}\}$.

Proof. Let $P_n(x)$ be defined as in the proof of Theorem 3. For any $x > 0$ and for all positive integers m, n with $n > m$, it holds

$$\frac{P_n(x)}{P_m(x)} = \frac{f(x+m+1)}{(x+m)f(x+m)} \frac{f(x+m+2)}{(x+m+1)f(x+m+1)} \cdots \frac{f(x+n)}{(x+n-1)f(x+n-1)}.$$

If m ($> n_0$) is so large that $1 - \delta(x+m)^{-(1+\varepsilon)} > 0$, we then obtain

$$\prod_{i=m}^{n-1} \left[1 - \delta(x+i)^{-(1+\varepsilon)}\right] \leq P_n(x)/P_m(x) \leq \prod_{i=m}^{n-1} \left[1 + \delta(x+i)^{-(1+\varepsilon)}\right]$$

or

$$\sum_{i=m}^{n-1} \ln\left[1 - \delta(x+i)^{-(1+\varepsilon)}\right] \leq \ln P_n(x) - \ln P_m(x) \leq \sum_{i=m}^{n-1} \ln\left[1 + \delta(x+i)^{-(1+\varepsilon)}\right].$$

Since

$$\lim_{m\to\infty} \sum_{i=m}^{\infty} \left|\ln\left[1 - \delta(x+i)^{-(1+\varepsilon)}\right]\right| = \lim_{m\to\infty} \sum_{i=m}^{\infty} \ln\left[1 + \delta(x+i)^{-(1+\varepsilon)}\right] = 0,$$

we conclude that $\{\ln P_n(x)\}$ is a Cauchy sequence for all $x > 0$. Hence, we can define

$$L(x) = \lim_{n\to\infty} \ln P_n(x)$$

and

$$F(x) = e^{L(x)}$$

for all $x > 0$. In fact, the above definition of F is equivalent to

$$F(x) = \lim_{n\to\infty} P_n(x).$$

It is easy to see

$$F(x+1) = \lim_{n\to\infty} P_n(x+1) = \lim_{n\to\infty} xP_{n+1}(x) = xF(x)$$

for any $x > 0$. Now, let $x > \max\{n_0, \delta^{1/(1+\varepsilon)}\}$. It then holds $1 - \delta(x+i)^{-(1+\varepsilon)} > 0$ for $i = 0, 1, \cdots$. Therefore, it follows from (9) that

$$\prod_{i=0}^{n-1}\left[1 - \delta(x+i)^{-(1+\varepsilon)}\right] \le P_n(x)/f(x) \le \prod_{i=0}^{n-1}\left[1 + \delta(x+i)^{-(1+\varepsilon)}\right],$$

since

$$P_n(x) = \frac{f(x+1)}{xf(x)}\frac{f(x+2)}{(x+1)f(x+1)}\cdots\frac{f(x+n)}{(x+n-1)f(x+n-1)}f(x).$$

This implies the validity of (10). Now, it remains only to prove the uniqueness of F. Assume that $G : (0,\infty) \to [0,\infty)$ is another solution of the gamma functional equation (3) and satisfies (10). Since both F and G are solutions of (3), it follows

$$\frac{F(x)}{G(x)} = \frac{F(x+n)}{G(x+n)} = \frac{F(x+n)}{f(x+n)}\frac{f(x+n)}{G(x+n)}$$

for any $x > 0$. Hence, we have

$$\frac{\alpha(x+n)}{\beta(x+n)} \le \frac{F(x)}{G(x)} \le \frac{\beta(x+n)}{\alpha(x+n)}$$

for all sufficiently large n. It is clear that the infinite products $\alpha(x)$ and $\beta(x)$ converge for all $x > 0$. Therefore, by using the relations

$$\alpha(x) = \lim_{n\to\infty} \alpha(x+n) \lim_{n\to\infty} \prod_{i=0}^{n-1}\left(1 - \frac{\delta}{(x+i)^{1+\varepsilon}}\right) = \lim_{n\to\infty} \alpha(x+n)\alpha(x)$$

and

$$\beta(x) = \lim_{n\to\infty} \beta(x+n) \lim_{n\to\infty} \prod_{i=0}^{n-1} \left(1 + \frac{\delta}{(x+i)^{1+\varepsilon}}\right) = \lim_{n\to\infty} \beta(x+n)\beta(x),$$

we conclude that $\alpha(x+n) \to 1$ and $\beta(x+n) \to 1$ as $n \to \infty$. Hence, it is obvious that $F(x) = G(x)$ holds true for all $x > 0$. □

Similarly to Corollary 4, the following corollary can be easily proved by using Theorem 5. Hence, we omit the proof.

COROLLARY 6. *Let $f : (0, \infty) \to (0, \infty)$ be a mapping. If f satisfies the functional inequality* (9) *asymptotically, then f satisfies the gamma functional equation* (3) *asymptotically.*

REMARK. Even though a mapping $F : (0, \infty) \to [0, \infty)$ is a solution of the gamma functional equation (3), F does not necessarily equal to the gamma function Γ on $(0, \infty)$. However, if F is logarithmically convex on $(0, \infty)$ and is a solution of the gamma functional equation (3) and $F(1) = 1$ then F necessarily equals to the gamma function Γ on $(0, \infty)$ (see [6].)

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