WEAK SEMICONTINUITY FOR
UNBOUNDED OPERATORS

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ABSTRACT. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. Pedersen and Akemann developed four concepts of lower semicontinuity for elements of A^{**}. Later, Brown suggested using only three classes: strongly lsc, middle lsc, and weakly lsc. In this paper, we generalize the concept of weak semicontinuity [1, 3] to the case of unbounded operators affiliated with A^{**}. Also we consider the generalized version of the conditions of the Brown's theorem [3, Proposition 2.2 & 3.27] for unbounded operators.

1. Introduction and preliminaries

In [1], C. A. Akemann and G. K. Pedersen defined four concepts of semicontinuity for elements of A^{**}, the enveloping von Neumann algebra of a C^*-algebra A. Later, L. G. Brown [3] suggested using only three classes $\overline{A^m_{sa}}$, $\widetilde{A}^m_{sa}$, and $(\widetilde{A}^m_{sa})^{-}$, and named them *strongly lsc*, *middle lsc*, and *weakly lsc*, respectively. Then he made an extensive study on semicontinuity [3]. Recently, the concepts of strong and middle semicontinuity are generalized for unbounded operators in [9, 10]. In this paper we generalize the concept and theory of bounded weak semicontinuous elements. Also we consider the generalized version of the conditions of the Brown's theorem [3, Proposition 2.2 & 3.27] for unbounded operators. Throughout this paper, A will denote a (non unital) C^*-algebra, $S(A)$ the state space of A and $Q(A)$ the quasi-state space of A. Equipped with the weak* topology inherited from A^*, $Q(A)$ is a compact convex set. It is well known that the enveloping von Neumann algebra of A can be identified with the second dual of A, so it will be denoted by

A^{**}. Let H_u denote the universal Hilbert space of A. For $M \subset A^{**}$, let $\overline{M}$ denote the norm closure of M in $B(H_u)$,

$M_{sa} = \{ x \in M \mid x^* = x \}$, and

$M_+ = \{ x \in M \mid x \geq 0 \}$.

For $M \subset A_{sa}^{**}$, M^m (resp. M_m) denotes the set of limits in A^{**} of monotone increasing nets (resp. monotone decreasing nets) of elements of M. Let $\widetilde{A}$ denote the C^*-algebra generated by A and the unit 1 of A^{**}, K_A the Pedersen's ideal of A, $M(A)$ the multiplier algebra of A, and $QM(A)$ the quasi-multipliers of A.

A subset C of a topological space X (not necessarily Hausdorff) is called *relatively (quasi-) compact* if C is contained in a (quasi-) compact subset of X. Throughout this paper Λ will denote the set of all relatively compact open subsets of Prim$A$, the primitive ideal space of A with hull-kernel topology. From [11, 5.39] it follows immediately that Prim(I_a) belongs to Λ for all a in $(K_A)_+$ where I_a is the two sided closed ideal generated by a. Applying [15, Lemma 5], we see that $(C)_\Lambda$ forms an increasing cofinal net where Λ is ordered by set inclusion. For an open subset C of Prim$A$, $I(C)$ denotes the closed two sided ideal of A corresponding to C and p_C the central open projection corresponding to $I(C)$.

2. Definition of $WLSC(A)$ and main results

The generalization of strong semicontinuity was quite smooth due to the cooperation of the quasi-state space $Q(A)$ and the theory of unbounded quadratic forms (see [9]). But for the concept of middle and weak semicontinuity, there are some difficulties even though we have several candidates.

In 1988, N. C. Phillips [15] obtained a new description of the multiplier algebra $\Gamma(K_A)$ of Pedersen's ideal K_A of A as an inverse limit of C^*-algebras (pro C^*-algebra) and derived a number of the results of [11] directly from corresponding facts about inverse limits of C^*-algebras. Inspired by his description Middle semicontinuity was also generalized for unbounded operators as well. We recall the definitions and refer to [9, 10] for the details.

DEFINITION 2.1. Let h be a bounded below selfadjoint operator (not necessarily densely defined) such that $h \, \eta \, A^{**}$.

(a) h is called *unbounded strongly lsc*, denoted by $h \in \mathrm{SLSC}(A)$, if there exist a monotone increasing net (h_i) in $\widetilde{A}_{sa}$, $h_i = a_i + \lambda_i 1$, such that $h_i \nearrow h$ and $\lambda_i \nearrow 0$.

(b) h is called *unbounded middle lower semicontinuous* $(h \in \mathrm{MLSC}(A))$ if there exists x in $\Gamma(K_A)_+$ such that $h + x$ is in $\mathrm{SLSC}(A)_+$.

For h bounded above selfadjoint operator, h is called *unbounded strong, middle upper semicontinuous* $(h \in \mathrm{SLSC}(A), \mathrm{MUSIC}(A))$ if $-h$ is in $\mathrm{SLSC}(A)$, $\mathrm{MLSC}(A)$ respectively.

In order to generalize the weak semicontinuity we are going to use the same kind of considerations as in the middle case. Considering the role of $\Gamma(K_A)$ in the theory of $\mathrm{MLSC}(A)$ we expect the quasicentralizers of K_A to be the elements both weakly lower and upper semicontinuous. It turns out that in some cases an extra condition has to be imposed on the quasicentralizers (see Theorem 2.3 below).

We have the following three conditions on an unbounded self-adjoint operator h (possibly not densely defined) affiliated with A^{**}:

(W1) $\forall \, C \in \Lambda$, hp_C is bounded below, and $a^*ha \in \mathrm{SLSC}(I(C))$ for all a in $I(C)$ where a^*ha denotes the operator that satisfies $(a^*ha, \varphi) = (h, \varphi(a^* \cdot a))$ for all φ in $Q(I(C))$.

  Note: Even if h is densely defined, a^*ha may not be densely defined.

(W2) $\forall \, C \in \Lambda$, hp_C is bounded below, and $(hp_C)\hat{\,}$ is lower semicontinuous on $S(I(C))$.

(W3) $\forall \, C \in \Lambda$, there exists a net (x_i) in $\widetilde{[I(C)^m_{sa}]}^{\,-}$ such that $x_i \nearrow hp_C$.

PROPOSITION 2.2. (W3) $\Rightarrow$ (W2) $\Rightarrow$ (W1).

Proof. (W3)$\Rightarrow$(W2): Let (x_i) be a net in $\widetilde{[I(C)^m_{sa}]}^{\,-}$ such that $x_i \nearrow hp_C$. By [3, Theorem 3.3], $\hat{x}_i$ is lower semicontinuous on $S(I(C))$, and $\hat{x}_i \nearrow (hp_C)\hat{\,}$. Hence $(hp_C)\hat{\,}$ is lower semicontinuous on $S(I(C))$.

(W2)$\Rightarrow$ (W1): Let $\varphi_i \to \varphi$ in the weak* topology of $Q(I(C))$. Then, for any a in $I(C)$, $\varphi_i(a^* \cdot a) \to \varphi(a^* \cdot a)$, and $\|\varphi_i(a^* \cdot a)\| \to \|\varphi(a^* \cdot a)\|$. Since $(hp_C)\hat{\,}$ is lower semicontinuous on $S(I(C))$, this implies $(a^*ha, \varphi) \leq \liminf_{i \to \infty}(a^*ha, \varphi_i)$. Therefore a^*ha is in $\mathrm{SLSC}(I(C))$. $\square$

REMARK. Note that the condition ((M1): $\forall C \in \Lambda$, $\exists \lambda_C > 0$ such that $(h+\lambda_C)p_C \in \mathrm{SLSC}(I(C))_+$) in [10] clearly implies (W3). And each of the conditions (W1)–(W3) yields the same concept of weak continuity. All of them imply continuous elements to be locally bounded; i.e., if h and $-h$ both satisfy any one among (W1)–(W3) and h is densely defined, then hp_C is bounded and hp_C is in $QM(I(C))$ for all C in Λ. Conversely if h satisfies the condition that $hp_C \in QM(I(C))$ for all C in Λ then it is easy to see that h and $-h$ satisfy (W3). Also h is a quasicentralizer of K_A by the operation $h(x,y) = xhy$ for all x and y in K_A.

Let $Q\Gamma(K_A)$ denote the set of quasicentralizers of K_A and

$$Q\Gamma_0(K_A) = \{\, h\,\eta\, A^{**} \mid hp_C \in QM(I(C)),\ \forall C \in \Lambda \,\}.$$

In general, $Q\Gamma(K_A)$ is strictly bigger than $Q\Gamma_0(K_A)$. For example, let $A = \mathcal{K}$ then K_A is the set of finite rank operators and $M(A) = QM(A) = \Gamma(K_A) = Q\Gamma_0(K_A) = B(H)$. But $Q\Gamma(K_A)$ is the set of everywhere defined quadratic forms on H. Such forms may not even be represented by linear operators on H. Nevertheless, the following theorem follows from Brown [4].

THEOREM 2.3. *If A cannot be decomposed in the form $A_1 \oplus E$, where $E \cong \mathcal{K}(H)$ for an infinite dimensional Hilbert space H, then $Q\Gamma(K_A)$ can be identified with $Q\Gamma_0(K_A)$.*

We do not know whether the conditions (W1)–(W3) are equivalent. In order to consider a more general situation, we will take (W1) as our definition for unbounded weak semicontinuity.

DEFINITION 2.4. Let h be a self-adjoint operator (not necessarily densely defined) such that $h\,\eta\, A^{**}$. Then h is called *unbounded weakly lower semicontinuous* ($h \in \mathrm{WLSC}(A)$) if h satisfies (W1); i.e., for all C in Λ, hp_C is bounded below and a^*ha is in SLSC(A) for all a in $I(C)$. Also h is called *unbounded weakly upper semicontinuous* ($h \in \mathrm{WUSC}(A)$) if $-h$ is in WLSC(A). As a special case, we denote by $\mathrm{WLSC}^d(A)$ (resp. $\mathrm{WUSC}^d(A)$) the set of all densely defined h in WLSC(A) (resp. WUSC(A)).

PROPOSITION 2.5. *Let the superscript ^{d} stands for densely defineness. Then*

(a) $h \in (\widetilde{A}_{sa}^m)^- \iff h \in WLSC(A)$ *and h is bounded.*

(b) $WLSC^d(A) \cap WUSC^d(A) = Q\Gamma_0(K_A)_{sa}.$

(c) $\Gamma(K_A)_{sa} = MLSC^d(A) \cap WUSC^d(A) = WLSC^d(A) \cap MUSC^d(A).$

Proof. (a) Combine Propositions 2.4 and 2.24 of [3].

(b) See the remark after Proposition 2.2.

(c) By [3, Proposition 2.3],

$$M(I(C))_{sa} = \widetilde{I(C)}_{sa}^m \cap [(\widetilde{I(C)}_{sa})_m]^- = (\widetilde{I(C)}_{sa}^m)^- \cap (\widetilde{I(C)}_{sa})_m$$

for each C in Λ. This implies the result by [10, Corollary 1.2]. $\qquad\square$

PROPOSITION 2.6. *Let I be an ideal of A with open central projection z.*

(a) $h \in WLSC(A) \Rightarrow zh \in WLSC(I).$

(b) $h \in WLSC(A)_+ \Rightarrow zh \in WLSC(A)_+$ *and $zh \in WLSC(I)_+.$*

Proof. (a) follows from the definition of $WLSC(A)$ and [9, Proposition 3.12].

(b) It remains to prove $zh \in WLSC(A)_+$. Since $h \in WLSC(A)_+$, $\forall\, C \in \Lambda$, $a^*ha \in SLSC(I(C))$ for all a in $I(C)$. By [3, Proposition 2.18], $a^*zha = za^*ha \in SLSC(I(C))_+$ for all a in $I(C)$. Therefore $zh \in WLSC(I)_+$. $\qquad\square$

PROPOSITION 2.7. *If (I_α) is an increasing net of ideals with open central projections z_α such that $A = (\bigcup I_\alpha)^-$ and $h\,\eta\,A_{sa}^{**}$, then*

$$h \in WLSC(A) \iff z_\alpha h \in WLSC(I_\alpha), \text{ for all } \alpha.$$

Proof. Assume $z_\alpha h \in WLSC(I_\alpha)$, for all α. Note that $(\text{Prim} I_\alpha)$ is an increasing net and forms an open cover of $\text{Prim} A$. Thus for each C in Λ, there exists α_0 such that $\text{Prim} I_{\alpha_0} \supset C$. This implies that $z_{\alpha_0} \geq p_C$ and $I_{\alpha_0} \supset I(C)$. Hence $hp_C = z_{\alpha_0} hp_C$ and $a^*ha \in SLSC(I(C))$ for all $a \in I(C)$. Therefore $h \in WLSC(A)$.

The converse follows from the definition of $WLSC(A)$ and Proposition 2.6. $\qquad\square$

THEOREM 2.8. *Assume* $0 \leq h \, \eta \, A_{sa}^{**}$. *Then*

$$h \in WUSC(A) \Longleftrightarrow h^{-1} \in SLSC(A) \text{ and } hp_C \text{ is bounded, } \forall C \in \Lambda.$$

Proof. If $h \in \text{WUSC}(A)$ then hp_C is bounded, and hence hp_C is in $[(\widetilde{I(C)}_{sa})_m]^{\overline{}}$ by definition and [3, Proposition 2.4]. Applying [9, Theorem 3.18], we have $(hp_C)^{-1} \in \text{SLSC}(I(C))_+$ for all C in Λ. Therefore $h^{-1} \in \text{SLSC}(A)_+$ by [9, Theorem 3.19].

For the converse, we apply the Theorem 3.18 and 3.19 of [9] to get $hp_C \in [(\widetilde{I(C)}_{sa})_m]^{\overline{}}$ for all C in Λ. Hence $h \in \text{WUSC}(A)$ by definition. □

In the proof of the following theorems we will use a certain Möbius map. Let

$$f_\delta(x) = \frac{x}{1 + \delta x} \qquad \text{on} \quad \begin{cases} (-\dfrac{1}{\delta}, \infty) & \text{if } \delta > 0 \\ (-\infty, -\dfrac{1}{\delta}) & \text{if } \delta < 0. \end{cases}$$

Note that f_δ is operator monotone on its domain such that $f_\delta \cdot f_{-\delta} = f_{-\delta} \cdot f_\delta = id$ and $f_\delta \cdot f_\epsilon = f_{\delta+\epsilon}$ where defined. For a selfadjoint operator h which is bounded below by $\alpha < 0$, $\tilde{f}_\delta(h)$ ($0 < \delta < -\frac{1}{\alpha}$) denotes the bounded selfadjoint operator $f_\delta(h) \oplus \frac{1}{\delta}(1 - p_h)$. For k bounded above by $\beta > 0$, we write $\tilde{f}_{-\delta}(k) = f_{-\delta}(k) \oplus (-\frac{1}{\delta})(1 - p_h)$, $0 < \delta < \frac{1}{\beta}$.

Let $\mathcal{U}$ denote the set of universally measurable elements of A^{**} and z_{at} the central projection in A^{**} which corresponds to the atomic part of A^{**} (See [14]). The following proposition shows that all kinds of semicontinuous elements are completely determined by their atomic parts.

PROPOSITION 2.9. *For h in WLSC(A), $z_{at}h$ determines h completely.*

Proof. Assume h_1 and h_2 are in WLSC(A) and $z_{at}h_1 = z_{at}h_2$. We will show that $h_1 p_C = h_2 p_C$, $\forall C \in \Lambda$; that is, $(h_1 v, v) = (h_2 v, v)$, $\forall v \in p_C H_u$. Note that $p_C A^{**} \cong I(C)^{**}$ and $p_C H_u$ is quasi-equivalent to

the universal Hilbert space of $I(C)$. By the Cohen-Hewitt factorization theorem (see [7]), $\forall v \in p_C H_u$, $\exists a \in I(C)$ and $w \in p_C H_u$ such that $aw = v$. Since $h_1, h_2 \in \text{WLSC}(A)$, $a^* h_1 a$, $a^* h_2 a \in \text{SLSC}(A)$. Therefore $\tilde{f}_\delta(a^* h_i a) \in \overline{A^m_{sa}} \subset \mathcal{U}$, $i = 1, 2$, for sufficiently small positive δ. Since the atomic representation is faithful on $\mathcal{U}$ ([14]), this implies $\tilde{f}_\delta(a^* h_1 a) = \tilde{f}_\delta(a^* h_2 a)$, and hence $a^* h_1 a = a^* h_2 a$. Therefore $(h_1 v, v) = (h_1 aw, aw) = (a^* h_1 aw, w) = (a^* h_2 aw, w) = (h_2 v, v)$, and we are done. $\qquad\square$

PROPOSITION 2.10. *If h is in WLSC(A) and h is affiliated with the set $\mathcal{Z}$ of central projections in A^{**}, then h satisfies (M1).*

Proof. Assume $h \in \text{WLSC}(A)$ and $h \eta \mathcal{Z}$. For any C in Λ, there is $\lambda_C > 0$ s.t. $(h + \lambda_C) p_C \geq 0$ and $a^* ha \in \text{SLSC}(I(C))$ for all a in $I(C)$. Let (e_α) be an increasing approximate identity of $I(C)$. Then $e_\alpha^{\frac{1}{2}}(h + \lambda_C) p_C e_\alpha^{\frac{1}{2}} \in \text{SLSC}(I(C))_+$ and note that, for φ_v in $Q(I(C))$,

$$\begin{aligned}
(e_\alpha^{\frac{1}{2}}(h + \lambda_C) p_C e_\alpha^{\frac{1}{2}})\widehat{}(\varphi_v) &= ((h + \lambda_C) p_C, \varphi_v(e_\alpha^{\frac{1}{2}} \cdot e_\alpha^{\frac{1}{2}})) \\
&= ((h + \lambda_C) p_C e_\alpha^{\frac{1}{2}} v, e_\alpha^{\frac{1}{2}} v) \\
&= \|((h + \lambda_C) p_C)^{\frac{1}{2}} e_\alpha^{\frac{1}{2}} v\|^2 \\
&= \|e_\alpha^{\frac{1}{2}}((h + \lambda_C) p_C)^{\frac{1}{2}} v\|^2 \quad \text{(since } h \eta \mathcal{Z}) \\
&\nearrow \|((h + \lambda_C) p_C)^{\frac{1}{2}} v\|^2 \\
&= (((h + \lambda_C) p_C v, v) \\
&= ((h + \lambda_C) p_C)\widehat{}(\varphi_v).
\end{aligned}$$

Therefore $((h + \lambda_C) p_C)\widehat{}$ is lower semicontinuous on $Q(I(C))$, and hence $(h + \lambda_C) p_C \in \text{SLSC}(I(C))_+$ by [9, Theorem 3.6]. $\qquad\square$

REMARK. For $h \eta \mathcal{Z}$, the above theorem implies that (W1)–(W3) and (M1) are all equivalent. In this case h is also q-LSC by [10, Proposition 3.3].

Now, we will investigate and generalize the conditions in the following theorem for unbounded operators.

(b)$\Rightarrow$ (c): Assume that there exists a net (h_i) in $\left[\widetilde{I(C)}_{sa}^m\right]^-$ such that $h_i \nearrow h_C$. Let $\lambda = \|h_{i_0}\|$ for some i_0, then $0 \le h_i + \lambda p_C \in [\widetilde{I(C)}_{sa}^m]$ for $i \ge i_0$. Since given (b) implies the condition (ii) in [3, Proposition 2.2] for each $I(C)$, applying the proposition, we have $0 \le h_i + \lambda p_C \in [\widetilde{I(C)}_{sa}^m] \subset \mathrm{MLSC}(I(C))$, for $i \ge i_0$. Then by (b), $0 \le h_i + \lambda p_C \in \mathrm{SLSC}(I(C))_+$ for $i \ge i_0$ and hence $h_C + \lambda p_C \in \mathrm{SLSC}(I(C))_+$ as the limit of monotone increasing net. This means $h_C \in \mathrm{MLSC}(I(C))$.

The converse is obvious.

(c)$\Rightarrow$ (c'): Let $h_C \in \left[\widetilde{I(C)}_{sa}^m\right]^-$. Then, since the lower semicontinuity is preserved under monotone increasing limits, there exists a net (h_i) in $\left[\widetilde{I(C)}_{sa}^m\right]^-$ such that $h_i \nearrow h_C$. By the given condition (c), we have h_C in $\mathrm{MLSC}(I(C))$. Since h_C is bounded and C is relatively compact, this implies that $h_C \in \widetilde{I(C)}_{sa}^m$.

(c')$\Rightarrow$ (a): Assume that $0 < \epsilon \le h_C \in \mathrm{SLSC}(I(C))$. Then, by [9, Theorem 3.18 (a)] and (c'), $h_C^{-1} \in [((\widetilde{I(C)}_{sa})_m]^- = ((\widetilde{I(C)}_{sa})_m$. Applying [9, Theorem 3.18 (b)], we obtain that there exists $\delta > 0$ such that $h_C - \delta p_C \in \mathrm{SLSC}(I(C))$.

(c')$\Rightarrow$ (d): This follows easily from [1] and [14].

If A is σ-unital, then $I(C)$ is σ-unital such that (d) is equivalent to (c') by [3, Theorem 3.27]. $\qquad \square$

The conditions in the above Proposition are sort of local generalization of those in [3, Proposition 2.2] for all ideal $I(C)$ of A where C is a relatively compact open subset of $\mathrm{Prim}A$. And we obtained the same kind of implications as in Proposition 2.11. However, it seems not very smooth with the global generalization of the conditions. The result that we have obtained so far is as follows.

PROPOSITION 2.13. *Consider the following conditions:*

(a) *For $\forall 0 < \epsilon \le h \in SLSC(A)$, $\exists \delta > 0$ such that $h - \delta 1 \in SLSC(A)$.*
(b) *$0 \le h \in MLSC(A) \Rightarrow h \in SLSC(A)_+$.*
(c) *$h \in MLSC(A) \Leftrightarrow h \in WLSC(A)$.*
(d) *$\Gamma(K_A) = Q\Gamma_0(K_A)$.*

Then [Prop 2.12 (b)] $\Rightarrow$ (b) $\Rightarrow$ (a) $\Leftrightarrow$ [Prop 2.11 (a)] and (c) $\Rightarrow$ (d) $\Leftrightarrow$ [Prop 2.12 (d)] .

Proof. [Prop 2.12 (b)] $\Rightarrow$ (b): Let $0 \leq h \in \text{MLSC}(A)$. Then it is easy to see that $0 \leq hp_C \in \text{MLSC}(I(C))$ for all $C \in \Lambda$ by [9, Theorem 3.19] and [10, Corollary 1.2]. Therefore $hp_C \in SLSC(I(C))_+$ for all $C \in \Lambda$ by given condition, and this implies that $h \in SLSC(I(C))_+$.

(b) $\Rightarrow$ (a): Let $0 < \epsilon \leq h \in \text{SLSC}(A)$. Then $0 \leq h - \epsilon 1 \in \mathbf{R} + SLSC(A) \subset MLSC(A)$. By the given condition, $h - \epsilon 1$ belongs to $SLSC(A)_+$.

(a) $\Leftrightarrow$ [Prop 2.11 (a)] : Let $0 < \epsilon \leq h \in \text{SLSC}(A)$. Then $0 < f_1(\epsilon) \leq f_1(h) \in \overline{A_{sa}^m}$ by [9, Theorem 3.6]. Applying the condition (a), we have that there exists $\delta > 0$ such that $f_1(h) - \delta 1 \in \overline{A_{sa}^m}$. Choose a δ small enough such that $f_{-1}(\delta) < \epsilon$ and $0 < f_1(h) - \delta 1$. Then a little computation shows that

$$h - f_{-1}(\delta) 1 \leq \tilde{f}_{-1}(f_1(h)) - f_{-1}(\delta) 1$$
$$= \frac{1}{1 - \delta} \tilde{f}_{-1}(\frac{1}{1 - \delta}(f_1(h) - \delta 1)) \in \text{SLSC}(A).$$

The converse is obvious.

(c) $\Rightarrow$ (d) and (d) $\Leftrightarrow$ [Prop 2.12 (d)] follow from [10, Corollary 1.2] and the *Remark* after Proposition 2.2. $\qquad\square$

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