Bull. Korean Math. Soc. **34** (1997), No. 4, pp. 509–518

DEVELOPING MAPS OF AFFINELY FLAT LIE GROUPS

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ABSTRACT. In this paper, we study the developing maps of the Lie groups with left-invariant affinely flat structures. We make some basic observations on the nature of the developing images and show that the developing map for an incomplete affine structure splits as a product of a covering map of codimension 1 and a diffeomorphism of dimension 1.

1. Introduction

When a Lie group G admits a left invariant linear connection whose torsion and curvature tensor vanish, we say that G has a left invariant *affinely flat* (or *affine* in short) structure. Such structures have been studied in different contexts and purposes by many authors (see for example [1, 2, 3, 6, 12, 15, 18, 19, 21] as samples.) and are classified for the geodesically complete cases on the low dimensional Lie groups [6, 9], sometimes with compatible metrics [5, 11, 14, 17].

If a Lie group G has an affine structure, then the structure naturally induces a so-called developing map into an affine space. Since all the simply connected complete affine spaces are equivalent, we have a developing map into the standard Euclidean space $\mathbb{E}^n$ (i.e., the Euclidean space with its standard flat connection). We intend to investigate this map and its image in this paper.

For the complete case, the developing map becomes a diffeomorphism onto the whole affine space $\mathbb{E}^n$, and hence the image of this map is not interesting even if the different developing maps in general induce the

Received April 17, 1996.

1991 Mathematics Subject Classification: Primary 53C30, 53C15, 53A15.

Key words and phrases: developing map, affinely flat, left-invariant affinely flat structure, left-symmetric algebra.

Research supported by GARC-KOSEF and S. N. U. Research Fund.

different affine structures. But if we consider the general incomplete case, the developing map and image become more interesting already as two dimensional case shows [16]. The three and higher dimensional cases do not seem to be well understood yet. One general result in this direction due to Kozul is that the developing image of a unimodular Lie group with affine structure becomes a cone if the affine structure is convex and hyperbolic [13]. In this paper, we will show that the developing map, in general for incomplete case, becomes a covering map onto an algebraic set and furthermore splits as a product of a covering map of codimension 1 and a diffeomorphism of dimension 1, along with some other observations on the developing map. Unfortunately such splitting is not affine and does not lead us to an induction unless we have a stronger condition on the affine structure.

For the study of left invariant affine structures on Lie groups, it is very convenient and conceptually simple to use a certain type of non-associative algebra, called left symmetric algebra, and this formulation naturally has a lot of algebraic advantages. The comparison of left symmetric algebra with representation view point as well as the interplay between algebra and geometry are explained in [10] and we refer the reader to [10] for more general setup and details. But we will explain all the necessary background in the next section in a somewhat different manner suitable for our purpose.

2. Canonical representation

Let G be an n-dimensional connected Lie groups with its Lie algebra $\mathfrak{g}$. By taking a universal covering group with the pull back structure, we will further assume that G is simply connected in this paper. Suppose G has a left invariant connection ∇ whose torsion and curvature tensor vanish, i.e., if we denote $\nabla_x y$ as a product xy for $x, y \in \mathfrak{g}$, then, we have

$$xy - yx = [xy] \tag{2.1}$$

$$x(yz) - y(xz) - [x, y]z = 0 \tag{2.2}$$

for all $x, y, z \in \mathfrak{g}$. From these conditions, we obtain the identity

$(x,y,z) = (y,x,z)$, where $(x,y,z) = (xy)z - x(yz)$ is the associator of x,y,z.

An algebra which satisfies this identity is called a left symmetric algebra. Hence having a left invariant affine structure on G is the same as having a left symmetric algebra structure on $\mathfrak{g}$ compatible with the Lie structure of $\mathfrak{g}$ in the sense of (2.1).

If we denote the left (right resp.) multiplication by λ (ρ resp.) so that $xy = \lambda_x(y) = \rho_y(x)$, then the flat condition (2.2) is equivalent to that λ is a Lie algebra homomorphism, and the condition (2.1) is just $\mathrm{ad}_x = \lambda_x - \rho_x$ for all $x \in \mathfrak{g}$.

Since G is affinely flat, each point of G has an open neighborhood which is affinely equivalent to an open subset of $\mathbb{E}^n$. The analytic continuation of these local equivalences is well defined since G is simply connected and depends only on the initial data. This analytic continuation is called a developing map, $D : G \to \mathbb{E}^n$, and is rigid in the sense that it is uniquely determined by a local data. Of course, the pull back connection of the standard Euclidean connection under this developing map is the original connection ∇.

The left invariance of ∇ implies that each left translation $l_g : G \to G$, $l_g(h) = gh$, is to be an affine equivalence which then, via D, induces a unique affine map $\phi(g) : \mathbb{E}^n \to \mathbb{E}^n$ such that $D \circ l_g = \phi(g) \circ D$. The unique existence of $\phi(g)$ follows from the rigidity of affine maps. (See [10] for more details.)

From the fact that $D \circ l_g = \phi(g) \circ D$ along with the rigidity, we can immediately deduce that ϕ is a homomorphism from G to $\mathrm{Aff}(n)$, the group of affine transformetions on $\mathbb{E}^n$.

Now if we denote the canonical evaluation map at $e \in \mathbb{E}^n$ by Ev_x : $\mathrm{Aff}(n) \to \mathbb{E}^n$, $Ev_x(a) = a \cdot x := a(x)$, and let $ev_x = Ev_x \circ \phi$, then the developing map D is the same as ev_x with $x = De$, $e =$ identity of G. Since D is an evaluation map as well as a local diffeomorphism, $\Omega = D(G)$ is an open orbit of x and $D = ev_x$ becomes a covering map onto its image. Take $x = De$ as our origin so that the affine space $\mathbb{E}^n$ becomes a vector space $V = \mathbb{R}^n$. We can then write $\mathrm{Aff}(n)$ as a semi-direct product $V \rtimes Gl(V)$ and its Lie algebra $\mathfrak{aff}(n)$ as a sum $V + \mathfrak{gl}(V)$. Hence ϕ has two components $\phi = (q, L) : G \to V \rtimes Gl(V)$ and correspondingly, $d\phi = (t, h) : \mathfrak{g} \to V + \mathfrak{gl}(V)$. Note that for each

$x \in \mathfrak{g}$,

$$t(x) = \frac{d}{dt}\Big|_0 q(\exp tx) = \frac{d}{dt}\Big|_0 (\exp tx)\cdot 0 = d(ev_0)\big|_e(x). \tag{2.3}$$

Now identify the vector space $V = T_0V = T_x\mathbb{E}^n$ with $\mathfrak{g} = T_eG$ by $d(ev)|_e = dD|_e$, then we obtain a developing map of G into $\mathfrak{g}$ which, as well, is the evaluation map at the origin of $\mathfrak{g}$. With these identifications, the homomorphism $\phi : G \to \mathrm{Aff}(\mathfrak{g})$, will be called the *canonical representation*, where $\mathrm{Aff}(\mathfrak{g})$ is the group of affine transformations of the vector space $\mathfrak{g}$. Note that from (2.3), the translation part of $d\phi$ is identity and we obtain the following diagram:

$$\begin{array}{ccc} \mathfrak{g} & \stackrel{d\phi=(id,\lambda)}{\longrightarrow} & \mathfrak{aff}(\mathfrak{g}) = \mathfrak{g} + \mathfrak{gl}(\mathfrak{g}) \\ \big\downarrow \exp & & \big\downarrow \exp \\ G & \stackrel{\phi=(q,L)}{\longrightarrow} & \mathrm{Aff}(\mathfrak{g}) = \mathfrak{g} \rtimes \mathrm{Gl}(\mathfrak{g}) \end{array}$$

Here we use the notation λ elaborately for the linear part of $d\phi$ anticipating λ is the same as left multiplication.

Let $g = \exp a$, $a \in \mathfrak{g}$ and $g \in G$. Then

$$\begin{aligned} \exp(a, \lambda_a) &= 1 + (a, \lambda_a) + \frac{1}{2!}(a, \lambda_a)^2 + \cdots \\ &= (a + \frac{1}{2!}\lambda_a(a) + \frac{1}{3!}\lambda_a^2(a) + \cdots, 1 + \lambda_a + \frac{1}{2!}\lambda_a^2 + \cdots) \\ &= (\text{“}e^a - 1\text{”}, e^{\lambda_a}) \end{aligned}$$

where "$e^a - 1$" $= a + \frac{1}{2!}a\cdot a + \frac{1}{3!}a\cdot(a\cdot a) + \cdots$. Note that $(a, \lambda_a)^2, \cdots$ is calculated using $\begin{pmatrix} \lambda_a & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda_a & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_a^2 & \lambda_a(a) \\ 0 & 0 \end{pmatrix}, \cdots$, etc.

Observe that for each $y \in \mathfrak{g}$, y is a left invariant vector field and since left translation corresponds to $\phi(g) = (L_g, q_g)$, the derivative of y in the direction of x can be calculated as follows:

$$\nabla_x y = \frac{d}{dt}\Big|_0 L_{\exp(tx)}(y) = \frac{d}{dt}\Big|_0 e^{\lambda_{tx}}(y) = \frac{d}{dt}\Big|_0 e^{t\lambda_x}(y) = \lambda_x(y).$$

Therefore λ is really the left multiplication as claimed above.

From the above calculation, we have for $g = \exp a$, $a \in \mathfrak{g}$, $g \in G$, and $x \in \mathfrak{g}$,

$$ev_x(g) = g \cdot x = q_g + L_g(x) = \text{“}e^a - 1\text{”} + e^{\lambda_a}(x),$$

and hence we can deduce that $d(ev_x)|_e = 1 + \rho_x$ since

$$\begin{aligned} d(ev_x)|_e(v) &= \frac{d}{dt}\Big|_0 ev_x(\exp tv) = \frac{d}{dt}\Big|_0 \text{“}e^{tv} - 1\text{”} + e^{t\lambda_v}(x) \\ &= v + \lambda_v(x) = v + v \cdot x = (1 + \rho_x)(v). \end{aligned}$$

The function $1 + \rho_x$ has a fundamental importance in understanding the geometry of left symmetric algebra, i.e., that of the canonical representation. If G has a complete left invariant affine structure, then $D = ev_0$ is a diffeomorphism for each $x \in \mathfrak{g}$, and $x = g \cdot 0$ for some $g \in G$. Therefore $ev_x = ev_{g \cdot 0} = ev_0 \circ r_g$ becomes a diffeomorphism also and hence $d(ev_x)|_e$ is an isomorphism for all $x \in \mathfrak{g}$. Conversely, if $d(ev_x)|_e$ is an isomorphism for all x, then all the orbits are open and hence by connectedness argument, there is only one orbit and the covering map $D = ev_x$ is a diffeomorphism since $\mathfrak{g}$ is simply connected, which implies that the affine structure of G is complete. Hence the affine structure is complete if and only if $1 + \rho_x$ is non-singular for all $x \in \mathfrak{g}$, as is well observed in [8, 9, 18].

As an extreme opposite, if $1 + \rho_x = 0$ for some x, then ev_x becomes a constant map and x is a fixed point of G-action on $\mathfrak{g}$. Such an affine structure is called radiant. (See [7] for more information.) While the developing image Ω of G is the whole affine space $\mathfrak{g}$ when the affine structure is complete, Ω is a cone for the radiant case as the following proposition shows.

PROPOSITION 2.1. *Let G be a simply connected Lie group with a left invariant radiant affine structure. Then the developing image $\Omega = D(G)$ is a cone.*

Proof. Let $1 + \rho_{x_0} = d(ev_{x_0})|_e = 0$. By choosing x_0 as our new origin, we may assume $\phi(g)$ is a linear transformation for all $g \in G$ by conjugation with a translation. Suppose $x \in \Omega$. Then since Ω is open, $\alpha x \in \Omega$ for all $\alpha \in (1 - \varepsilon, 1 + \varepsilon)$, and hence $\alpha x = g \cdot x$ for some

$g \in G$. Now $\Omega = ev_x(G) = ev_x \circ r_g(G) = ev_{g \cdot x}(G) = ev_{\alpha x}(G) = \alpha ev_x(G) = \alpha\Omega$. Note that $ev_{\alpha x}(G) = \alpha ev_x(G)$ since $\phi(g)$ is linear and $\phi(g)(\alpha x) = \alpha(\phi(g)(x))$ for all $g \in G$. This shows that Ω is invariant under expansion by all factors $\alpha \in (1-\varepsilon, 1+\varepsilon)$ and hence by all $\alpha \in \mathbb{R}$.□

3. Developing map for incomplete affine structure

In this section, we will investigate the developing image of the general incomplete affine structure, and start with some basic properties of a polynomial function $p(x) = \det(1 + \rho_x) : \mathfrak{g} \to \mathbb{R}$ which is called the *characteristic polynomial* in [8].

Let $x = g \cdot o$, $g \in G$, be a point of $\mathfrak{g}$ and let $g = \exp a$, $a \in \mathfrak{g}$. Then we have

$$ev_x = ev_{g \cdot o} = ev_o \circ r_g = ev_o \circ l_g \circ A_{g^{-1}}$$

where $r_g : G \to G$ is the right translation and $A_g : G \to G$ is the adjoint map given by $A_g(h) = ghg^{-1}$. Differentiating both sides of this equation, we obtain

$$\begin{aligned} dev_x\big|_e &= dev_o\big|_g \circ dl_g\big|_e \circ dA_{g^{-1}}\big|_e \\ &= L_g \circ Ad_{g^{-1}} \\ &= e^{\lambda_a} \circ e^{-ad_a}, \end{aligned}$$

where the second equality follows from the identity $ev_0 \circ l_g = \phi(g) \circ ev_0$ noting that $d(\phi(g)) = L_g$. Taking determinants of both sides, we get

$$\begin{aligned} \det(1 + \rho_x) &= \det(dev_x\big|_e) = \det e^{\lambda_a} \det e^{-ad_a} \\ &= e^{\operatorname{tr}\, \lambda_a} \cdot e^{-\operatorname{tr} ad_a} = e^{\operatorname{tr} \rho_a} \end{aligned}$$

Recall that we have $\mathrm{ad}_a = \lambda_a - \rho_a$ from the compatibility of our left symmetric product with Lie structure, whence $\operatorname{tr} ad_a = \operatorname{tr} \lambda_a - \operatorname{tr} \rho_a$.

The following proposition seems to be first observed by Helmsletter in complex affine case [18] and also proved for real case in [10]. We will give another conceptual proof here.

PROPOSITION 3.1. *Let* $p(x) = \det(1 + \rho_x) : \mathfrak{g} \to \mathbb{R}$. *Then* $p(g \cdot x) = \Delta(g)p(x)$, $g \in G$, *where* $\Delta : G \to \mathbb{R}_+$ *is a group homomorphism given by* $\Delta(g) = \det(1 + \rho_{g \cdot o})$.

Proof. Observe that $\operatorname{tr} \rho_{[a,b]} = \operatorname{tr} \lambda_{[a,b]} - \operatorname{tr} ad_{[a,b]} = 0$ since λ and ad are Lie algebra homomorphisms. Hence $\operatorname{tr} \rho : \mathfrak{g} \to \mathbb{R}$ is a Lie algebra homomorphism and we get corresponding Lie group homomorphism $\Delta : G \to \mathbb{R}_+$ whose differential is $\operatorname{tr} \rho$ so that $e^{\operatorname{tr} \rho} = \Delta \circ \exp$. (Note that G is assumed to be simply connected.)

The above discussion shows that $p \circ ev_o \circ \exp = e^{\operatorname{tr} \rho}$ and hence $\Delta = p \circ ev_o$ at least on a neighborhood of $e \in G$, i.e., $\Delta(g) = p(g \cdot o)$ for g near e, and if we let $x = h \cdot o$, $h \in G$, then

$$p(g \cdot x) = p(gh \cdot o) = \Delta(gh) = \Delta(g)\Delta(h) = \Delta(g)p(h \cdot o) = \Delta(g)p(x).$$

This holds for all small x in a neighborhood of $o \in \mathfrak{g}$. Since both sides are polynomials, they agree for all $x \in \mathfrak{g}$. If the identity holds for g near e, so does for all $g \in G$ since any element of G can be written as a product of elements near e. □

The following proposition is well known [7, 9], but we give a proof since it is simple using the observations made so far.

PROPOSITION 3.2. *Let us denote* $\Omega = ev_o(G)$ *as before. Then* Ω *is the connected component of* $\{x \in \mathfrak{g} | p(x) = \det(1 + \rho_x) \neq 0\}$ *containing* $o \in \mathfrak{g}$.

Proof. Let Ω_0 be the component of $\{x \in \mathfrak{g} | p(x) \neq 0\}$ which contains o. G acts on Ω_0 since $\det(1 + \rho_{g \cdot x}) = \Delta(g) \det(1 + \rho_x) \neq 0$ if $x \in \Omega_0$. Ω is the orbit of $o \in \Omega_0$ and hence $\Omega \subset \Omega_0$. Now for any $y \in \Omega$, since $dev_y\big|_e = 1 + \rho_y$ is an isomorphism, ev_y is a covering map and the orbit of y, $G \cdot y = ev_y(G)$ is an open set. Therefore $\Omega = ev_o(G)$ is open, and hence dosed in Ω_0, being the complement of other orbits in Ω_0. Since Ω_0 is connected, $\Omega = \Omega_0$. □

As we saw in the proof of 3.1, we have a following commutative diagram.

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\operatorname{tr}\rho} & \mathbb{R} \\
{\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\
G & \xrightarrow{\Delta} & \mathbb{R}_+ \\
{\scriptstyle ev_o}\downarrow & & \| \\
\mathfrak{g} & \xrightarrow{p(x)=\det(1+\rho_x)} & \mathbb{R}_+
\end{array}$$

Now suppose that $\operatorname{tr}\rho = 0$. Then Δ becomes a trivial homomorphism and $p(x) = \det(1+\rho_x) = 1$ identically, which shows that the affine structure is complete. The converse is asked by Helmstetter and Perea [8, 18], and proved by Goldman and Hirsch [7]. (See also [20], [4] for algebraic proofs.) Hence we are naturally interested in the incomplete case, i.e., when $\operatorname{tr}\rho \neq 0$. In this case, we obtain the following theorem.

THEOREM 3.3. *Suppose we have a left invariant affine structure on a simply connected Lie group G with nontrivial $\operatorname{tr}\rho : \mathfrak{g} \to \mathbb{R}$. Then the followings hold.*

(i) *The affine structure is incomplete.*

(ii) *G can be written as a semi-direct product $G = R \cdot K$, where $K = \ker\Delta$, $\Delta : G \to \mathbb{R}_+$ with $d\Delta = \operatorname{tr}\rho$, and R is the 1-parameter subgroup generated by v with $\operatorname{tr}\rho_v = 1$.*

(iii) *There is a equivariant diffeomorphism $g : \Omega = ev_o(G) \to \mathbb{R} \times M$, where $M = K \cdot o$ and $G = R \cdot K$ action on $\mathbb{R} \times M$ is given by (**) below.*

(iv) *The covering map $ev_o : G \to \Omega$ is the product of a covering map $ev_o : K \to M = K \cdot o$ and a diffeomorphism $ev_o : R \to R \cdot o = g^{-1}(\mathbb{R} \times \{0\})$.*

Proof. We already discussed (i) above. Let $s = \operatorname{tr}\rho$ and choose any $v \in \mathfrak{g}$, with $s(v) = 1$ and let $R = \{\exp tv\}$ be the 1-parameter subgroup generated by v. Then $\Delta(\exp tv) = e^{s(tv)} = e^{s(v)t}$. and $\Delta : R \to \mathbb{R}_+$ is an isomorphism. Let $\mathfrak{k} = \ker s$ and $K = \ker\Delta$. Since $G/K \cong \mathbb{R}_+$ is contractible, $\pi_i(K) \cong \pi_i(G)$ for all $i \geq 0$. In particular, K is connected normal subgroup of G whose Lie algebra is $\mathfrak{k}$. Since $\Delta| : R \to \mathbb{R}_+$

is an isomorphism, $\Delta|^{-1}$ gives a splitting for the short exact sequence $1 \to K \to G \to \mathbb{R}_+ \to 1$ and $G = K \cdot R$ as a semidirect product.

Note that $\Delta(\exp tv) = e^t$ and hence

$$(*) \qquad \begin{cases} p(K \cdot x) = \Delta(K)p(x) = p(x) \\ p(\exp tv \cdot x) = e^t p(x) \end{cases}$$

Let M^{n-1} be the K-orbit of o. Then since K action is locally simply transitive, K is a covering space of M. Now define a map $f : \mathbb{R}\times M \to \Omega$ by $f(t,y) = (\exp tv) \cdot y = x$. By $(*)$, p plays the role of a Morse function on Ω and f becomes a diffeomorphism whose inverse is given by $f^{-1}(x) = (t,y)$, where $t = \log p(x)$ and $y = \exp p(-tv)\cdot x$. Now R action on Ω is given by $\exp(sv) \cdot x = \exp(sv) \cdot ((\exp(tv) \cdot y) = (\exp(s+t)v) \cdot y$ and K action on Ω is given by $k \cdot x = k \cdot (\exp(tv) \cdot y) = \exp(tv) \cdot (k^t \cdot y)$ where $k^t = \exp(-tv)k\exp(tv)$, which is the action of R on the normal subgroup K giving the semi-direct product structure of $G \cong K \rtimes R$. Hence the associated $G = RK$ action on $\mathbb{R} \times M$ induced by f will be

$$(**) \qquad \begin{cases} \exp(rv) \cdot (t,y) = (r+t, y) \\ k \cdot (t,y) = (t, k^t \cdot y), \ k^t = \exp(-tv)k\exp(tv). \end{cases}$$

Note also that if $\gamma\cdot o = o$ for $\gamma \in G$, then $\Delta(\gamma) = \Delta(\gamma)\cdot p(o) = p(\gamma\cdot o) = p(o) = 1$ and $\gamma \in K$. Hence we see that the isotropy subgroups of K and G at o agree, i.e., $K_o = G_o$. This shows that the covering map $ev_o : G \to \Omega$ is, in fact, a product of two covering maps $ev_o : K \to M = K \cdot o$ and $ev_o : R \to R \cdot o$ ($\cong \mathbb{R}$) and the latter is bijective. This proves the theorem. □

We remark that in the above theorem M is a homogeneous algebraic manifold given by the equation $p(x) = \det(1 + \rho_x) = 1$ as well as all the level manifolds given by $p^{-1}(t)$, $t \in \mathbb{R}$.

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