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SOME MODULES IN CATEGORY $\mathcal{O}$ AND THEIR DECOMPOSITION OVER GENERALIZED KAC-MOODY LIE ALGEBRAS

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ABSTRACT. We extend the notion of equivalence relation $\approx$ for Kac-Moody algebras to generalized Kac-Moody algebras and prove some analogues of results for Kac-Moody algebras.

1. Introduction

In this paper we extend the notion of equivalence relation $\sim$ for Kac-Moody algebras to generalized Kac-Moody algebras(= GKM algebras) and prove some analogues of results for Kac-Moody algebras. Here, GKM algebras are a class of contragredient Lie algebras $G(A)$ over C associated to a real square matrix $A = (a_{ij})_{i,j,\in I}$ indexed by a finite set I which satisfies the conditions:

(C1) either $a_{ii} = 2$ or $a_{ii} \leq 0$;
(C2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in Z$ if $a_{ii} = 2$;
(C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

We also extend the equivalence relation $\approx$ (defined in [2]) from K^g to a larger subset $K^g \subset H^*$ (we use same notation K^g), for GKM algebras. We prove the equivalence classes in K^g for GKM algebras have a property similar to that of the equivalence classes for Kac-Moody algebras. We also prove that their category decomposition theorem for $\mathcal{O}^g$ can be extended.

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2. Decomposition Theorem of Modules in the category $\mathcal{O}$

Let H be a Cartan subalgebra of G, Π the set of simple roots $\{\alpha_i, i \in I\}$. Let W be the Weyl group. Fix an element ρ in H^* such that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$. Here, (,) is a nondegenerate bilinear form on H^*. We denote by P_+ the set of dominant integrals. Let I^{re}(resp. I^{im}) be the subset $\{i \in I | a_{ii} = 2$ (resp. $a_{ii} \leq 0$ $\}$ of the indexing set I. For $\alpha_i \in \Pi$ we call α_i real(resp. imaginary) simple root when $i \in I^{re}$(resp. I^{im}).

First, we recall the definition of the category $\mathcal{O}$ whose objects are G modules M satisfying:

(1) M is $H-$semisimple with finite dimensional weight spaces.

(2) There exist finitely many elements $\mu_1, \mu_2, \cdots \mu_k \in H^*$ such that any weight of M (μ is a weight iff $M_\mu \neq 0$) belongs to some $D(\mu_i)$, where $D_{\mu_i} = \{\mu_i - \gamma | \gamma \in Q_+ = \sum Z_{\geq 0}\alpha_i\}$.

An important class of modules in $\mathcal{O}$ is the class of highest modules, in particular Verma modules.

For $\mu \in H^*$, any $M \in \mathcal{O}$ has a local composition series at μ whose subquotients are irreducible highest weight modules. We call these subquotients components of M. The following Proposition describes the components of Verma modules.

PROPOSITION 2.1. [2, Theorem 3.6] *Let $\lambda, \mu \in H^*$. Then $L(\mu)$ is a component of $M(\lambda)$ iff the ordered pair $\{\lambda, \mu\}$ has the following condition:*

($*$) There exist a sequence $\phi_1, \phi_2, ..., \phi_k$ of positive roots and a sequence $n_1, n_2, ..., n_k$ of positive integers such that

(i) $\lambda - \mu = \sum_{i=1}^{i=k} n_i \phi_i$,

(ii) $2(\lambda + \rho - n_1\phi_1 - \cdots - n_{j-1}\phi_{j-1}, \phi_j) = n_j(\phi_j, \phi_j), \quad \forall\, 1 \leq i \leq k.$

Now fix $\lambda \in H^*$. Define $\mathcal{A}(\lambda)$ to be the subset of H^* of all sums of pairwise perpendicular imaginary simple roots perpendicular to λ. In the set $W \times \mathcal{A}(\lambda)$, we have the extended Bruhat ordering $\geqq$ defined in [4].

The following describes the property (*) in terms of the extended Bruhat ordering.

PROPOSITION 2.2. [3, PROPOSITION 2.9]. *Let* $\lambda \in P_+$ *and* $(w_i, \beta_i) \in W \times \mathcal{A}(\lambda)$, *for* $i = 1, 2$. *Then* $\{w_2(\lambda + \rho - \beta_2) - \rho, w_2(\lambda + \rho - \beta_1) - \rho\}$ *has the property (*) if and only if* $(w_1, \beta_1) \geqq (w_2, \beta_2)$.

We note that the relation (*) is not symmetric. We extend the equivalence relation $\sim$ to GKM algebras in the same way as in Kac-Moody algebras.

DEFINITION 2.3. For $\lambda, \mu \in H^*$ we define $\lambda \sim \mu$ if there exists a sequence $\lambda = \lambda_0, \lambda_1, ..., \lambda_k = \mu$ in H^* such that for every $0 \leq i \leq k$, either $\{\lambda_i, \lambda_{i+1}\}$ or $\{\lambda_{i+1}, \lambda_i\}$ has the property (*).

DEFINITION 2.4. Let Λ be an equivalence class of H^* under $\sim$. A modules $M \in \mathcal{O}$ is said to be of type Λ iff all the components of M have highest weights belonging to Λ.

The following Lemma 2.5 and Proposition 2.6 are proved for a contragradient Lie algebra which is more general than a generalized Kac-Moody algebra.

LEMMA 2.5. [2, Proposition 4.4] *Let* $M(\lambda)$ *and* $M(\mu)$ *be Verma modules with highest weights* λ *and* μ, *respectively. Then* $\mathrm{Ext}_G(M(\lambda), M(\mu)) = 0$ *if* λ *and* μ *are inequivalent.*

PROPOSITION 2.6. [2, Theorem 4.2] *Let* G *be any GKM algebras. Let* M, N *be two* G *modules in* $\mathcal{O}$, *such that* $M(resp. N)$ *is of type* $\Lambda_M(resp.\ \Lambda_N)$. *Then*

(1) *If* $\Lambda_M \neq \Lambda_N$ *then* $\mathrm{Ext}_G(M, N) = 0$.
(2) *There exists a unique set of* $\{M_\Lambda\}_\Lambda$ *of submodules of* M *such that*
   (i) M_Λ *is of type* Λ *and*
   (ii) $M = \oplus_\Lambda M_\Lambda$.

3. Subcategory $\mathcal{O}^g$ and equivalence classes in K^g

In this section we extend the notion of category $\mathcal{O}^g$ for Kac-Moody algebras and generalize their category decomposition theorem to arbitrary symmetrizable GKM algebras.

DEFINITION 3.1. Let $C = \{\lambda \in H^* \mid (\lambda, \alpha_i) \geq 0,\ \forall$ simple roots $\alpha_i\}$. Put $K = \cup_{\lambda,\beta} W(\lambda + \rho - \beta)$, where λ runs in C and $\beta \in \mathcal{A}(\lambda)$.

Now we set $K^g = -\rho + K, C^g = -\rho + C$.

PROPOSITION 3.2. (1) *K is $W-$invariant.*
(2) *Every orbit of W in K contains a unique element of C.*

Proof. By definition (1) is clear. For (2) one modifies the proof of Proposition 1.3 in [5] slightly. □

LEMMA 3.3. *Let $w(\lambda + \rho - \beta) - \rho \in K^g$ and $\{w(\lambda + \rho - \beta) - \rho, \mu\}$ satisfy condition $(*)$. Then $\mu = \sigma(\lambda + \rho - \beta_0) - \rho$ for some $\sigma \in W$ and some $\beta_0 \in \mathcal{A}(\lambda)$. In particular $\mu \in K^g$.*

Proof. By definition of K^g, λ is in C. Then for any β in $\mathcal{A}(\lambda)$ and simple root α_i we have $(\lambda - \beta, \alpha_i) \geq 0$. Thus $\lambda - \beta \in C$. In the proof of Proposition 4.2 in [3], this Lemma was proved assuming that $\lambda - \beta \in P_+$. However, if one looks at the proof carefully one can find that his proof carries over to the case $\lambda - \beta \in C$. □

We denote by $\mathcal{O}^g$ the full subcategory of the category $\mathcal{O}$ consisting of those modules $M \in \mathcal{O}$ such that all the irreducible subquotients have highest weights in K^g.

The following is an immediate consequence of Lemma 3.3.

PROPOSITION 3.4. *If $\lambda \in K^g$, then $M(\lambda) \in \mathcal{O}^g$*

Next we define an equivalence relation $\approx$ in K^g (resp. $\mathcal{A}(\lambda_0)$) by using K^g (resp. $\mathcal{A}(\lambda_0)$) in place of H^* in the definition of $\sim$.

For $\lambda_0 \in C^g$, and β_0 in $\mathcal{A}(\lambda_0)$. We denote by $[\beta_0]$ the equivalence class under $\approx$ containing β_0 and $[\mathcal{A}(\lambda_0)]$ be the set of all equivalence classes $[\beta_0]$ in $\mathcal{A}(\lambda_0)$. Let $W(\lambda_0)$ be the subgroup of W generated by $\{s_\phi \mid \phi$ a real root and $2(\lambda_0 + \rho,\ \phi)/(\phi, \phi) \in Z\}$. Consider disjoint union $\dot{\cup}(W/W(\lambda_0) \times [\mathcal{A}(\lambda_0)])$.

PROPOSITION 3.5. *The set of equivalence classes under $\approx$ is in bijective correspondence with the set $\dot{\cup}(W/W(\lambda_0) \times [\mathcal{A}(\lambda_0)])$.*

Proof. Let $\lambda_0 \in C^g$ and $\beta_0 \in \mathcal{A}(\lambda_0)$. Consider a set $A = \{\sigma w(\lambda_0 + \rho - \beta) - \rho\}_{w \in W(\lambda_0), \beta \in [\beta_0]}$. First note that $A \in K^g$. We show that A is an equivalence class under $\approx$. For any element $w \in W(\lambda_0)$ we write $w = s_{\phi_1} s_{\phi_2} \cdots s_{\phi_k}$ where each ϕ_i is real and $2(\lambda_0 + \rho, \phi_i)/(\phi_i, \phi_i) \in Z$. Since $\beta_0 \in A(\lambda_0)$ and the Weyl group is generated by real simple reflections, we have $2(\beta_0, \phi_i)/(\phi_i, \phi_i) \in Z$ for $\phi_i, 1 \leq i \leq k$. Therefore for each ϕ_i we have $2(\lambda_0 + \rho - \beta_0, \phi_i)/(\phi_i, \phi_i) \in Z$. Using this with the fact $w = s_{\phi_1} s_{\phi_2} \cdots s_{\phi_k}$ one can show that $\sigma(\lambda_0 + \rho - \beta_0) - \rho \approx \sigma w(\lambda_0 + \rho - \beta_0) - \rho$.

We now consider $\beta \in [\beta_0]$. By definition of $[\beta_0]$ there exist $\beta_0, \beta_1, \ldots, \beta_k = \beta$ in $\mathcal{A}(\lambda_0)$ such that either $\{\beta_i, \beta_{i+1}\}$ or $\{\beta_{i+1}, \beta_i\}$ has property (*). By Proposition 2.2 either $\beta_i \geqq \beta_{i+1}$ or $\beta_{i+1} \geqq \beta_i$, which implies either $(\sigma w, \beta_i) \geqq (\sigma w, \beta_{i+1})$ or $(\sigma w, \beta_{i+1}) \geqq (\sigma w, \beta_i)$. Hence $\sigma w(\lambda_0 + \rho - \beta_{i+1}) - \rho \approx \sigma w(\lambda_0 + \rho - \beta_i) - \rho$.

Next consider $\nu \in K^g$ such that $\nu \approx \sigma(\lambda_0 + \rho - \beta_0) - \rho$. By definition of $\approx$ there exist $\nu = \lambda_1, \lambda_2, \ldots, \lambda_n = \sigma(\lambda_0 + \rho - \beta_0) - \rho$ in K^g such that either $\{\lambda_i, \lambda_{i+1}\}$ or $\{\lambda_{i+1}, \lambda_i\}$ has property (*). Without loss of generality, we may assume $\lambda_{n-1} = \nu$. In case $\{\sigma(\lambda_0 + \rho - \beta_0) - \rho, \nu\}$ has property (*), by Lemma 3.3 $\nu = w\sigma(\lambda_0 + \rho - \beta) - \rho$ for some $w \in W(\lambda_0)$ and $\beta \in [\beta_0]$. Using induction on the length of w one can show that $\sigma^{-1} w \sigma \in W(\lambda_0)$. This proves $\nu \in A$. Consider the case $\{\nu, \sigma(\lambda_0 + \rho - \beta_0) - \rho\}$ has property (*). Since $\nu \in K^g$ by Proposition 2.2 $\sigma(\lambda_0 + \rho - \beta_0) - \rho = w(\nu + \rho - \beta) - \rho$ for some $w \in W$ and $\beta \in [\beta_0]$. Therefore $w^{-1}\sigma(\lambda_0 + \rho - \beta_0) - \rho + \beta = \nu$ and $\sigma\sigma^{-1}w^{-1}\sigma(\lambda_0 + \rho - \beta_0 + \sigma^{-1}w\beta) - \rho = \nu$. Moreover, if one look at the proof of Proposition 2.2 one can conclude that $\sigma^{-1}w^{-1}\sigma \in W(\lambda_0)$ and $\sigma^{-1}w\beta$ is a sum of simple imaginary root with integer coefficients, which implies $\beta_0 - \sigma^{-1}w\beta \approx \beta_0$ in $\mathcal{A}(\lambda)$. This proves $\nu \in A$.

Now we define a map from $K^g/\approx$ to $(\dot{\cup} W/W(\lambda_0) \times [\mathcal{A}(\lambda_0)])$ as follows: Given any equivalence class Λ^g any element μ in Λ^g is of the form $\mu = \tau(\lambda_0 + \rho - \beta_0) - \rho$ for some $\tau \in W$ and $\beta_0 \in A(\lambda_0)$. We corresponds Λ^g to the set $\tau W(\lambda_0) \times [\beta_0]$ in $\dot{\cup}(W/W(\lambda_0) \times [\mathcal{A}(\lambda_0)])$. We show this map is well defined. Suppose $\mu = \tau(\lambda_0 + \rho - \beta_0) - \rho = \theta(\lambda_\theta + \rho - \beta) - \rho$ for some $\lambda_\theta \in C$ and $\beta \in \mathcal{A}(\lambda_\theta)$. By Proposition 3.2 $\lambda_0 + \rho - \beta_0 = \lambda_\theta + \rho - \beta$ and $\tau = \theta$. Hence $\lambda_0 - \lambda_\theta = \beta_0 - \beta$. Since $\beta_0 - \beta$ is the sum of simple imaginary roots with integer coefficients $W(\lambda_0) = W(\lambda_\theta)$ and $\beta_0 \approx \beta$. This proves the Proposition. $\square$

For an equivalence class Λ^g, we let $M \in \mathcal{O}^g$ is said to be of type Λ^g iff all the components of M have highest weights belonging to Λ^g.

The following is an immediate consequence of Lemma 3.3 and Proposition 2.6.

PROPOSITION 3.6. (1) *Let $M \in \mathcal{O}^g$. Then there exist a unique family M_{Λ^g} of submodules of M such that*

(i) $M_{\Lambda^g} \in \mathcal{O}^g_{\Lambda^g}$ *and*

(ii) $M = \oplus_{\Lambda^g} M_{\Lambda^g}$.

(2) *Let M be any module in the category $\mathcal{O}$(resp. $\mathcal{O}^g$) such that it has no irreducible subquotients $L(\nu)$ with $\nu \approx 0$(resp.$\nu \approx 0$), then $Ext(G, M) = 0$.*

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