THE DEFORMATION SPACE
OF REAL PROJECTIVE STRUCTURES
ON THE (*n_1n_2n_3n_4)-ORBIFOLD

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ABSTRACT. For positive integers \( n_i \geq 2, i = 1, 2, 3, 4 \), such that \( \sum \frac{1}{n_i} < 2 \), there exists a quadrilateral \( \mathcal{P} = P_1P_2P_3P_4 \) in the hyperbolic plane \( \mathbb{H}^2 \) with the interior angle \( \frac{\pi}{n_i} \) at \( P_i \). Let \( \Gamma \subset Isom(\mathbb{H}^2) \) be the (discrete) group generated by reflections in each side of \( \mathcal{P} \). Then the quotient space \( \mathbb{H}^2/\Gamma \) is a differentiable orbifold of type (*\( n_1n_2n_3n_4 \)). It will be shown that the deformation space of \( \mathbb{R}P^2 \)-structures on this orbifold can be mapped continuously and bijectively onto the cell of dimension \( 4 - |\{i|n_i = 2\}| \).

1. Introduction

Goldman showed that the deformation space of reflection groups in the convex \( k \)-gons of type \( (n_1n_2 \cdots n_k) \) in the projective plane is homeomorphic to the cell of dimension \( 3k - 8 - |\{i|n_i = 2\}| \) for \( k \geq 4, n_i \geq 2, \sum \frac{1}{n_i} < k - 2 \). (See Goldman [2], pp 58-64.) We note that it is similar to the deformation space of real projective structures on the (*\( n_1n_2 \cdots n_k \))-orbifold. (The definition of (*\( n_1n_2 \cdots n_k \))-orbifold will be given in the next section.) We will consider the special case \( k = 4 \): We will define the deformation space of real projective structures on the (*\( n_1n_2n_3n_4 \))-orbifold and show that it can be mapped continuously and bijectively onto the cell of dimension \( 4 - |\{i|n_i = 2\}| \) by concrete matrix calculations, using the deformation theorem in Goldman [3]. But the restriction \( k = 4 \) is actually unnecessary. Our method can be used equally

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well in the cases $k > 4$. We restrict ourselves to the case $k = 4$ only to avoid some complications.

2. Orbifolds

This section will be devoted to some basic definitions and propositions about orbifolds. See Ratcliffe [5], Scott [6], or Thurston [7] for more details. Roughly speaking, an orbifold is a topological space locally modeled on open subsets of $\mathbb{R}^n$ quotient out by some finite groups. More precisely,

**Definition 1.** An $n$-orbifold $\mathcal{O}$ with underlying space $X_\mathcal{O}$ is a Hausdorff topological space $X_\mathcal{O}$ equipped with a covering by open sets $\{U_i\}$ closed under finite intersection such that

- to each $U_i$ is associated a finite group $\Gamma_i$ and an action of $\Gamma_i$ on an open subset $\hat{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\phi_i : \hat{U}_i/\Gamma_i \to U_i$
- whenever $U_i \subset U_j$, there is an inclusion $f_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\hat{\phi}_{ij} : \hat{U}_i \to \hat{U}_j$ equivariant with respect to $f_{ij}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\hat{U}_i & \xrightarrow{\hat{\phi}_{ij}} & \hat{U}_j \\
\downarrow & & \downarrow \\
\hat{U}_i/\Gamma_i & \xrightarrow{\phi_{ij}} & \hat{U}_j/f_{ij}\Gamma_i \\
\downarrow \phi_i & & \downarrow \\
U_i & \subset & U_j
\end{array}
\]

**Definition 2.** The singular set $\Sigma_\mathcal{O}$ of an orbifold $\mathcal{O}$ is the set of all points $x$ in $X_\mathcal{O}$ such that in each local coordinate system $U = \hat{U}/\Gamma$ near $x$, and for each $\hat{x}$ in $\hat{U}$ projecting to $x$, the stabilizer $\Gamma_{\hat{x}}$ of $\hat{x}$ is nontrivial.

**Example 3.** A manifold without boundary may be regarded as an orbifold whose singular set is empty.

**Definition 4.** Let $\mathcal{O}$ be a 2-orbifold. A point $x \in X_\mathcal{O}$ is a reflector if there is a local coordinate $U \to \mathbb{R}^2/\mathbb{Z}_2$ near $x$ where $\mathbb{Z}_2$ acts as the
reflection in a line through $0 \in \mathbb{R}^2$ and $x$ corresponds to 0. A point $y \in X_\mathcal{O}$ is a corner reflector of order $m$ if there is a local coordinate $V \to \mathbb{R}^2/D_{2m}$ near $y$ where $D_{2m}$ acts as the dihedral group of order $2m$ generated by reflections in two lines through 0 which form an angle of size $\pi/m$ and $y$ corresponds to 0.

**Example 5.** Let $n_1, n_2, \cdots, n_k \geq 2$ be positive integers. The $(n_1 \cdot n_2 \cdots n_k)$-orbifold is a 2-orbifold with the two-dimensional disk as its underlying space and with the boundary of the disk as the singular set such that

- there are $k$ corner reflectors $x_i$ of order $n_i$ on the boundary lying in the (cyclic) order $x_1, x_2, \cdots, x_k$.
- the other boundary points are reflectors.

**Definition 6.** A covering orbifold of an orbifold $\mathcal{O}$ is an orbifold $\tilde{\mathcal{O}}$ with a projection $p : X_{\tilde{\mathcal{O}}} \to X_\mathcal{O}$ such that

- $p$ is a local covering; that is, each $\tilde{x} \in X_{\tilde{\mathcal{O}}}$ has an open neighborhood $\tilde{U}$ homeomorphic to $\tilde{U}/\Gamma$ (in the sense of above definition) such that $p(\tilde{U})$ is an open set $\tilde{U}'$ homeomorphic to $\tilde{U}/\Gamma'$ for some group $\Gamma' \supset \Gamma$ and the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{U}/\Gamma & \to & \tilde{U}/\Gamma' \\
\downarrow & & \downarrow \\
\tilde{U} & \xrightarrow{p} & \tilde{U}'
\end{array}
\]

- $p$ is an even covering, that is, each $x \in X_\mathcal{O}$ has an open neighborhood $V$ homeomorphic to $\hat{V}/\Gamma$ for which each component $\hat{U}_j$ of $p^{-1}(V)$ is isomorphic to $\hat{V}/\Gamma_j$ for some subgroup $\Gamma_j \subset \Gamma$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\hat{V}/\Gamma_j & \to & \hat{V}/\Gamma \\
\downarrow & & \downarrow \\
U_j & \xrightarrow{p} & V
\end{array}
\]

From now on, "covering" will mean orbifold covering.

**Proposition 7.** An orbifold has a universal cover. In other words, if $x \in X_\mathcal{O} - \Sigma_\mathcal{O}$ is a base point for an orbifold $\mathcal{O}$, then there is a (orbifold) covering $p : \tilde{\mathcal{O}} \to \mathcal{O}$ with base point $\tilde{x}$ (with $p(\tilde{x}) = x$) such that for each
other covering \( p' : \tilde{\mathcal{P}} \to \mathcal{O} \) with base point \( \tilde{x}' \) (and \( p'(\tilde{x}') = x \)), there is a unique lifting \( q : \tilde{\mathcal{O}} \to \tilde{\mathcal{P}} \) of \( p \) to a covering of \( \tilde{\mathcal{P}} \) with \( q(\tilde{x}) = \tilde{x}' \).

Note that the universal cover is unique; that is, if \( x \in X_\mathcal{O} - \Sigma_\mathcal{O} \) and \( p_i : \tilde{\mathcal{O}}_i \to \mathcal{O}, i = 1, 2, \) are universal coverings with \( p_i(\tilde{x}_i) = x \) then there is a homeomorphism \( \alpha : \tilde{\mathcal{O}}_1 \to \tilde{\mathcal{O}}_2 \) such that \( \alpha, \alpha^{-1} \) are coverings with \( \alpha(\tilde{x}_1) = \tilde{x}_2 \).

**Definition 8.** Let \( p : \tilde{\mathcal{O}} \to \mathcal{O} \) be an orbifold covering. A *deck transformation* of the covering is a homeomorphism \( \gamma : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}} \) such that \( p \circ \gamma = p \).

**Definition 9.** The *fundamental group* \( \pi_1(\mathcal{O}) \) of an orbifold \( \mathcal{O} \) is the group of deck transformations of the universal covering.

### 3. The deformation spaces

In this section, we will see the definition of the deformation space of real projective structures on an orbifold.

**Proposition 10.** The quotient space of a connected manifold \( M \) by a group \( \Gamma \) which acts faithfully and properly discontinuously on \( M \) is an orbifold (which we will denote by \( M/\Gamma \)). The quotient map \( M \to M/\Gamma \) is an orbifold covering. If, in addition, \( M \) is simply connected, then it is the universal covering and \( \pi_1(M/\Gamma) \) may be identified with \( \Gamma \).

Henceforth we will consider only orbifolds of the form \( \tilde{M}/\Gamma \), where \( \tilde{M} \) is a simply connected differentiable manifold without boundary and \( \Gamma \) is a group of diffeomorphisms of \( \tilde{M} \) acting faithfully and properly discontinuously on it.

**Definition 11.** Let \( X \) be a real analytic \( n \)-manifold and \( G \) a group of analytic diffeomorphisms of it. Let \( \mathcal{O} \) be an orbifold \( \tilde{M}/\Gamma \). (So \( \Gamma = \pi_1(\mathcal{O}) \).) Then a *development pair* of an \( (X, G) \)-structure on \( \mathcal{O} \) is a pair \( (\text{dev}, H) \) satisfying the following:

- \( \text{dev} : \tilde{M} \to X \) is an immersion.
- \( H \in \text{Hom}(\Gamma, G) \) : the set of all group homomorphisms of \( \Gamma \) into \( G \).
- \( H \) is equivariant with respect to dev, that is, \( \text{dev} \circ \gamma = H(\gamma) \circ \text{dev} : \tilde{M} \to X \) for each \( \gamma \in \Gamma \).
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For a development pair \((dev, H)\), \(dev\) is called a developing map of the structure and \(H\) the holonomy (homomorphism) corresponding to \(dev\).

By the first and third requirements in the above definition, the holonomy is determined by the developing map. That is, if both \((dev, H_1)\) and \((dev, H_2)\) are development pairs then \(H_1 = H_2\).

**Example 12.** Let \(\Gamma\) be a subgroup of Isom(\(H^2\)) acting properly discontinuously on \(H^2\). Since \(H^2\) is simply connected, the quotient map \(H^2 \rightarrow H^2/\Gamma\) is the universal orbifold covering. A development pair of a real projective structure (i.e. (\(RP^2, PGL(3, R)\))-structure or \(RP^2\)-structure) on this orbifold is a pair \((dev, H)\) such that

- \(dev : H^2 \rightarrow RP^2\) is an immersion.
- \(H \in \text{Hom}(\Gamma, PGL(3, R))\).
- \(dev \circ \gamma = H(\gamma) \circ dev : H^2 \rightarrow RP^2\) for each \(\gamma \in \Gamma\).

Now we can define the deformation space of \(RP^2\)-structures on the orbifold \(O = \widetilde{M}/\Gamma\). For convenience, \(PGL(3, R)\) will be identified with \(SL(3, R)\) and \(G\) will be used sometimes in place of them. Let \(D'(O)\) be the set of all developing maps of \(RP^2\)-structures on \(O\). We topologize \(D'(O)\) regarding it as a subspace of \(C^\infty(\widetilde{M}, RP^2)\) with the weak topology. (For the definition of the weak topology, see Hirsch [4].) We also topologize \(\text{Hom}(\Gamma, G)\) by the compact-open topology. Since \(\Gamma\) is countable and discrete, the compact-open topology equals the pointwise convergence topology. Assigning to each developing map the holonomy corresponding to it gives a map \(hol_1 : D'(O) \rightarrow \text{Hom}(\Gamma, G)\), which can be shown to be continuous. Fix a point \(x_0 \in O\) and \(\tilde{x}_0 \in \widetilde{M}\) projecting to it. Let \(Diff_0(O)\) be the identity component in the group \(\{f \in Diff(\widetilde{M}) \mid f(\tilde{x}_0) = \tilde{x}_0, f \circ \gamma = \gamma \circ f \forall \gamma \in \Gamma\}\). \(Diff_0(O)\) acts on \(D'(O)\) by composition to the right. Let \(D(O)\) be the quotient space \(D'(O)/Diff_0(O)\). Since \(hol_1\) is constant on each orbit of the action, it induces \(hol_2 : D(O) \rightarrow \text{Hom}(\Gamma, G)\). Moreover there are actions of \(G\) on both \(D(O)\) and \(\text{Hom}(\Gamma, G) : G\) acts on \(D'(O)\) by compositions to the left. Such an action projects to an action of \(G\) on \(D(O)\). On the other hand, \(G\) acts on \(\text{Hom}(\Gamma, G)\) by conjugations. It can be readily checked that \(hol_2\) induces a well-defined map \(hol : D(O)/G \rightarrow \text{Hom}(\Gamma, G)/G\). We denote \(D(O)/G\) by \(\Sigma(O)\) and call it the *deformation space of \(RP^2\)-structures* on \(O\). We remark that if \(O\) itself is a manifold, then \(hol_2\) is a local homeomorphism. See Goldman [3] for more details.
4. Convex real projective structures

By an affine patch in the projective plane, we mean the complement of a projective line of \( \mathbb{RP}^2 \). An affine patch has a natural structure of an affine plane. Then a convex set in \( \mathbb{RP}^2 \) is an affinely convex subset of an affine patch in \( \mathbb{RP}^2 \). Now, as in the preceding section, let \( \mathcal{O} \) be an orbifold \( \widetilde{M}/\Gamma \). The deformation space \( \mathcal{C}(\mathcal{O}) \) of convex real projective structures on \( \mathcal{O} \) is the subspace of \( \mathcal{T}(\mathcal{O}) \) consisting of equivalence classes of real projective structures each of which has a developing map \( dev : \widetilde{M} \to \mathbb{RP}^2 \) an embedding onto a convex subset of \( \mathbb{RP}^2 \). It is known that if \( \mathcal{O} \) is a closed orientable surface of genus \( > 1 \), then the restriction to \( \mathcal{C}(\mathcal{O}) \) of \( hol : \mathcal{T}(\mathcal{O}) \to \text{Hom}(\Gamma, G)/G \) is an embedding onto a connected component of \( \text{Hom}(\Gamma, G)/G \). See Choi [1] for the proof. However for orbifolds, we do not have the proof.

5. The main part

We turn to our main discussion. Let \( n_i \geq 2, i = 1, 2, 3, 4 \), be integers such that \( \sum (1/n_i) < 2 \). There is a quadrilateral \( \mathcal{P} = P_1P_2P_3P_4 \) in \( \mathbb{H}^2 \) such that the angle at \( P_i \) is \( \pi/n_i \) for each \( i \). Let \( \Gamma \) be the subgroup of \( \text{Isom}(\mathbb{H}^2) \) generated by reflections in each side of \( \mathcal{P} \). The group \( \Gamma \) acts properly discontinuously on \( \mathbb{H}^2 \) and \( \mathcal{P} \) is a fundamental domain for \( \Gamma \). Let us denote the reflection in the side \( P_iP_{i+1} \) by \( r_i \). Then \( \Gamma \) admits a presentation

\[
<r_1, r_2, r_3, r_4 | r_i^{n_i-1} = 1, i = 1, 2, 3, 4>
\]

The quotient space \( \mathcal{O} = \mathbb{H}^2/\Gamma \) is an orbifold of type \((^*n_1n_2n_3n_4)\) and the quotient map \( \mathbb{H}^2 \to \mathcal{O} \) is the universal covering. Throughout this section, \( n_i \)'s are fixed and \( \mathcal{O} \) will always mean the \((^*n_1n_2n_3n_4)\) orbifold \( \mathbb{H}^2/\Gamma \), where \( \Gamma \) is as above. The manifold \( \mathbb{H}^2 \) will be identified with the universal cover of \( \mathcal{O} \) and \( \mathcal{D}', \mathcal{D}, \mathcal{T}, \) and \( \mathcal{C} \) will be used in place of \( \mathcal{D}'(\mathcal{O}), \mathcal{D}(\mathcal{O}), \mathcal{T}(\mathcal{O}), \) and \( \mathcal{C}(\mathcal{O}) \), respectively. Our main purpose is to prove

Theorem 1. \( \mathcal{T} \) can be mapped continuously and bijectively onto the cell of dimension \( 4 - |\{i|n_i = 2\}|. \)

It is known that \( \mathcal{C} = \mathcal{T} \); that is, each developing map \( dev : \mathbb{H}^2 \to \mathbb{RP}^2 \) is an embedding onto a convex set in \( \mathbb{RP}^2 \). Let \( \mathcal{H} \) be the image of
hol. Then $\mathfrak{g}$ is the subspace of $\text{Hom}(\Gamma, G)/G$ consisting of equivalence classes of homomorphisms each of which is the holonomy of an $\mathbb{RP}^2$-structure on $O$. Fix a projective quadrilateral $p = p_1p_2p_3p_4$ in $\mathbb{RP}^2$. (Let $p_1 = [0,0,1], p_2 = [1,0,1], p_3 = [1,1,1], p_4 = [0,1,1]$ in homogeneous coordinates for ease of computations.) Since for any two projective bases of $\mathbb{RP}^2$ there is a unique element in $G$ carrying one to the other, we have a one-to-one correspondence between $\mathfrak{g}$ and $\mathfrak{h}$, where $\mathfrak{h}$ is the subset of $\text{Hom}(\Gamma, G)$ consisting of holonomies corresponding to $dev \in \mathcal{D}'$ such that $dev(P_i) = p_i$ for $i = 1, 2, 3, 4$. Thus the restriction to $\mathfrak{h}$ of the quotient map $\text{Hom}(\Gamma, G) \to \text{Hom}(\Gamma, G)/G$ is one-to-one and onto $\mathfrak{g}$.

**Proof of the Theorem:** We will show that $\mathfrak{h}$ is the cell of dimension $4 - \#\{i | n_i = 2\}$. It will follow easily from Lemmas 1 and 2.

**Lemma 1.** $H \in \text{Hom}(\Gamma, G)$ is in $\mathfrak{h}$ if and only if

\[
H(r_1) = \begin{pmatrix} -1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & -b & -1 \end{pmatrix}, \quad H(r_2) = \begin{pmatrix} -c & 0 & c-1 \\ -d & -1 & d \\ -c-1 & 0 & c \end{pmatrix}, \\
H(r_3) = \begin{pmatrix} -1 & f & -f \\ 0 & e & -e-1 \\ 0 & e-1 & -e \end{pmatrix}, \quad H(r_4) = \begin{pmatrix} -1 & 0 & 0 \\ -g & -1 & 0 \\ -h & 0 & 1 \end{pmatrix}
\]

for $a, b, c, d, e, f, g, h \in \mathbb{R}$ satisfying

\[
\begin{align*}
d(a-b) &= 2 + 2\cos(2\pi/n_2), \quad d < 0 \quad \text{if } n_2 > 2 \\
d &= 0, \quad a = b \quad \text{if } n_2 = 2 \\
(c - d + 1)(-e + f + 1) &= 2 + 2\cos(2\pi/n_3), \quad c - d + 1 < 0 \quad \text{if } n_3 > 2 \\
c - d + 1 &= 0, \quad -e + f + 1 = 0 \quad \text{if } n_3 = 2 \\
f(-g + h) &= 2 + 2\cos(2\pi/n_4), \quad f > 0 \quad \text{if } n_4 > 2 \\
f &= 0, \quad g = h \quad \text{if } n_4 = 2 \\
ag &= 2 + 2\cos(2\pi/n_1), \quad a > 0 \quad \text{if } n_1 > 2 \\
a &= 0, \quad g = 0 \quad \text{if } n_1 = 2
\end{align*}
\]

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Lemma 2. The set of all \((a, b, c, d, e, f, g, h) \in \mathbb{R}^8\) satisfying \((A), (B), (C)\) and \((D)\) is the cell of dimension \(4 - |\{i | n_i = 2\}|\).

Proof of Lemma 1. We want to find all the holonomies corresponding to \(dev\) such that \(dev(P_i) = p_i\). Let \(dev\) be such a developing map and \(H\) the holonomy corresponding to it. Let \(A_i = H(r_i)\) for \(i = 1, 2, 3, 4\). From the equivariance relation \(dev \circ \gamma = H(\gamma) \circ dev\) for each \(\gamma \in \Gamma\) we see that \(A_i\) fixes the (projective) line \(p_ip_{i+1}\) pointwise since \(r_i\) fixes the (hyperbolic) line \(P_iP_{i+1}\) pointwise. So we get the following equations:

\[
\begin{align*}
(1) & \quad \begin{bmatrix}
A_1 & \begin{pmatrix} s \\ 0 \\ s + t \end{pmatrix}
\end{bmatrix} = \begin{bmatrix}
s \\ 0 \\ s + t
\end{bmatrix} & \forall s, t \in \mathbb{R} \\
(2) & \quad \begin{bmatrix}
A_2 & \begin{pmatrix} s + t \\ t \\ s + t \end{pmatrix}
\end{bmatrix} = \begin{bmatrix}
s + t \\ t \\ s + t
\end{bmatrix} & \forall s, t \in \mathbb{R} \\
(3) & \quad \begin{bmatrix}
A_3 & \begin{pmatrix} s \\ s + t \\ s + t \end{pmatrix}
\end{bmatrix} = \begin{bmatrix}
s \\ s + t \\ s + t
\end{bmatrix} & \forall s, t \in \mathbb{R} \\
(4) & \quad \begin{bmatrix}
A_1 & \begin{pmatrix} 0 \\ s \\ s + t \end{pmatrix}
\end{bmatrix} = \begin{bmatrix}
0 \\ s \\ s + t
\end{bmatrix} & \forall s, t \in \mathbb{R}
\end{align*}
\]

in the homogeneous coordinates.

From the relations \(r_i^2 = 1\), we also have \(A_i^2 = I\).

\[
\begin{align*}
(5) & \quad A_1^2 = I \\
(6) & \quad A_2^2 = I \\
(7) & \quad A_3^2 = I \\
(8) & \quad A_4^2 = I
\end{align*}
\]

Note that \(A_i \neq I\) from the equivariance.
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\[
\begin{array}{c}
\text{in } \mathbb{H}^2 \\
\text{in } \mathbb{RP}^2
\end{array}
\]

\[
\begin{array}{c}
P_1 \\
r_1(P) \\
\mathcal{P} \\
P_2
\end{array}
\quad \xrightarrow{\text{dev}} \quad
\begin{array}{c}
A_1(p) \\
\mathcal{P} \\
p_2
\end{array}
\]

Figure 1: A figure illustrating the equivariance

So (1) and (5) mean that \( A_1 \) is a reflection in the line \( p_1p_2 \). Solving them, we get

\[
A_1 = \begin{pmatrix}
-1 & -a & 0 \\
0 & 1 & 0 \\
0 & -b & -1
\end{pmatrix} \quad a, b \in \mathbb{R} \tag{9}
\]

Solving (2) and (6), we get

\[
A_2 = \begin{pmatrix}
-c & 0 & c - 1 \\
-d & -1 & d \\
-c - 1 & 0 & c
\end{pmatrix} \quad c, d \in \mathbb{R} \tag{10}
\]

Solving (3) and (7), we get

\[
A_3 = \begin{pmatrix}
-1 & f & -f \\
0 & e & -e - 1 \\
0 & e - 1 & -e
\end{pmatrix} \quad e, f \in \mathbb{R} \tag{11}
\]

Solving (4) and (8), we get

\[
A_4 = \begin{pmatrix}
-1 & 0 & 0 \\
-g & -1 & 0 \\
-h & 0 & 1
\end{pmatrix} \quad g, h \in \mathbb{R} \tag{12}
\]

Finally, we consider the relations \((r_i r_{i+1})^{n_i+1} = 1\) together with the equivariance. These induce the relations \((A_i A_{i+1})^{n_i+1} = I\). We will only
consider the relation \((A_1A_2)^{n_2} = I\). Here we must take the two cases (a) \(n_2 = 2\) and (b) \(n_2 > 2\) separately.

(a) \(n_2 = 2\): We have \((A_1A_2)^2 = I\) or \(A_1A_2 = A_2A_1\). Substituting (9) and (10) into this equation, we get \(d = 0, a = b\) from the following.

\[
\begin{pmatrix}
  c + ad & a & -c - ad + 1 \\
  -d & -1 & -d \\
  bd + c + 1 & b & -bd - c
\end{pmatrix}
= 
\begin{pmatrix}
  c & ac - bc + b & -c + 1 \\
  d & ad - bd - 1 & -d \\
  c + 1 & ac - bc + a & -c
\end{pmatrix}
\]

(b) \(n_2 > 2\): Since \(A_1A_2\) has 1 as an eigenvalue and \((A_1A_2)^{n_2} = I\), \(A_1A_2\) has \(e^{2\pi ki/n_2}\) and \(e^{-2\pi ki/n_2}\) as the other two complex eigenvalues for a \(k \in \{1, 2, \ldots, n_2 - 1\}\). Since \(r_1r_2\) is the rotation around \(P_2\) by the angle \(2\pi/n_2\), \(A_1A_2\) has \(e^{2\pi i/n_2}\) and \(e^{-2\pi i/n_2}\) as eigenvalues other than 1, by the equivariance. So the trace of \(A_1A_2\) equals \(1 + 2 \cos(2\pi/n_2)\). Thus we get \(d(a - b) = 2 + 2 \cos(2\pi/n_2) \geq 0\). Note that for small positive values \(r\),

\[
\begin{pmatrix}
  1 \\
  r \\
  1
\end{pmatrix}
= 
\begin{pmatrix}
  -1 - ar \\
  r \\
  -1 - br
\end{pmatrix}
= 
\begin{pmatrix}
  \frac{1 + ar}{1 + br} \\
  \frac{-r}{1 + br} \\
  1
\end{pmatrix}
\]

in the homogeneous coordinates.

We must have \(d < 0\). Suppose on the contrary that \(d > 0\) or, equivalently, \(a > b\). Then the interiors of \(A_1(p)\) and \(A_2A_1(p)\) overlap. To see this, note that \((1 + ar)/(1 + br) > 1\) for small positive values \(r\). So the intersection of \(A_1(p)\) with an open neighborhood of \(p_2\) looks like Fig. 2.

\[p_1\]

\[\frac{1 + ar}{1 + br}, \frac{-r}{1 + br}, 1\]

\[\frac{[1, r, 1]}{1, r, 1}\]

\[A_1(p)\]

\[p\]

\[p_2\]

\[p_3\]

Figure 2

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Let $U$ be a small open neighborhood of $p_2$ such that $\text{dev}$ restricted to an open neighborhood $\mathcal{U}$ of $P_2$ is a diffeomorphism onto $U$ and let $W$ be one of the four regions of $U$ formed by the projective lines $p_2p_1$ and $p_2p_3$ in Fig. 3.

![Figure 3](image)

Taking sufficiently small $U$, we may assume that $W \subset A_1(p)$. Since $A_2$ is a reflection in $p_2p_3$, $A_2(W) \cap A_1(p)$ has nonempty interior. So $U \cap A_2A_1(p) \cap A_1(p)$ has nonempty interior. Since $\mathcal{U} \cap r_2r_1(\mathcal{P}) \cap r_1(\mathcal{P})$ has the empty interior, $\text{dev}$ is not a local diffeomorphism. Hence a contradiction. The other relations $(A_iA_{i+1})^{n_{i+1}} = I, i = 2, 3, 4$ can be treated in the same way.

Conversely, suppose $H \in \text{Hom}(\Gamma, G)$ satisfies the conditions of Lemma 1. Then $H$ induces a tessellation of a convex set $\Omega_H = \bigcup_{\gamma \in \Gamma} H(\gamma)(p)$. See Goldman [2] for the proof of the fact that $\Omega_H$ is convex. So it is evident that there is a $\text{dev} \in \mathcal{D}'$ satisfying the equivariance relation: $\text{dev} \circ \gamma = H(\gamma) \circ \text{dev} \quad \forall \gamma \in \Gamma$.

**Proof of Lemma 2.** We only consider the two cases (I) $n_1, n_2, n_3, n_4 > 2$ and (II) $n_1 = 2, n_2, n_3, n_4 > 2$. The other cases can be treated similarly.

(I) Let all $n_i$'s be greater than 2. Define $p : \mathbb{R}^8 \rightarrow \mathbb{R}^4$ by

$$p(a, b, c, d, e, f, g, h) = (d, c - d + 1, f, a)$$

Let $S$ be the set of all $(a, b, c, d, e, f, g, h) \in \mathbb{R}^8$ satisfying (A),(B),(C),(D). Then $p|_S$ is one-to-one and onto the subset $J = \{(x, y, z, w) \in \mathbb{R}^4 | x <$
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0, y < 0, z > 0, w > 0} of \( \mathbb{R}^4 \): Define \( j : J \to S \) by

\[
j(x, y, z, w) = (w, w - (2 + 2 \cos(2\pi/n_2))/x, x + y - 1, x,
\quad z + 1 - (2 + 2 \cos(2\pi/n_3))/y, (2 + 2 \cos(2\pi/n_1))/w,
\quad (2 + 2 \cos(2\pi/n_4))/w + (2 + 2 \cos(2\pi/n_4))/z).
\]

This map was obtained just by solving (A), (B), (C), (D), letting \( d = x, c - d + 1 = y, f = z, a = w \). So it is trivial that \( p|_S : S \to J \) and \( j \) are inverses. Since \( J \) is a 4-cell, we are done in this case.

(II) Let \( n_1 = 2, n_2, n_3, n_4 > 2 \). Define \( p : \mathbb{R}^8 \to \mathbb{R}^3 \) by

\[
p(a, b, c, d, e, f, g, h) = (d, c - d + 1, f)
\]

Let \( S \) be as in Case (I) and \( J = \{(x, y, z) \in \mathbb{R}^3 | x < 0, y < 0, z > 0 \} \).

Define \( j : J \to S \) by

\[
j(x, y, z) = (0, -(2 + 2 \cos(2\pi/n_2))/x, x + y - 1, x, z + 1,
\quad -(2 + 2 \cos(2\pi/n_3))/y, z, 0, (2 + 2 \cos(2\pi/n_4))/z).
\]

Then \( p|_S : S \to J \) and \( j \) are inverses. Since \( J \) is a 3-cell, we are done in this case, too.

\[\square\]

References


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