INTEGRAL OPERATORS THAT PRESERVE THE SUBORDINATION

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ABSTRACT. Let $H(U)$ be the space of all analytic functions in the unit disk $U$ and let $\mathcal{K} \subset H(U)$. For the operator $A_{\beta, \gamma} : \mathcal{K} \to H(U)$ defined by

$$A_{\beta, \gamma}(f)(z) = \left[ \frac{\beta + \gamma}{z^{\gamma}} \int_0^z f^\beta(t)t^{\gamma-1} \, dt \right]^{1/\beta}$$

and $\beta, \gamma \in \mathbb{C}$, we determined conditions on $g(z), \beta$ and $\gamma$ such that

$$z \left[ \frac{f(z)}{z} \right]^\beta < z \left[ \frac{g(z)}{z} \right]^\beta$$

implies

$$z \left[ \frac{A_{\beta, \gamma}(f)(z)}{z} \right]^\beta < z \left[ \frac{A_{\beta, \gamma}(g)(z)}{z} \right]^\beta$$

and we presented some particular cases of our main result.

1. Introduction

Let $H(U)$ be the space of all analytic functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $f, g \in H(U)$. We say that $f$ is subordinate to $g$, written $f(z) < g(z)$, if $g$ is univalent in $U$, $f(0) = g(0)$ and $f(U) \subset g(U)$.

In [7] the authors determined conditions under which

$$f(z) < g(z) \quad \text{implies} \quad A(f)(z) < A(g)(z)$$

where $A : K \to H(U)$, $K \subset H(U)$ and $A(f)(z) = \left[ \frac{1}{z^{\gamma}} \int_0^z f^\beta(t)t^{\gamma-1} \, dt \right]^{1/\beta}$, $\beta, \gamma \in \mathbb{C}$.

Note that some particular cases of this result were previously obtained in [2], [3] and [9].

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For $h \in A$, $A \subset H(U)$, considering the integral operator $A_h : \tilde{K} \to H(U)$, $\tilde{K} \subset H(U)$ defined by

$$A_h(f)(z) = \left[ \beta \int_0^z f^{\beta}(t)h^{-1}(t)h'(t) \, dt \right]^{1/\beta}, \beta \in \mathbb{C}$$

in [1] the author gives sufficient conditions on $h(z)$ and $g(z)$ such that

$$\left( \frac{zh'(z)}{h(z)} \right)^{1/\beta} f(z) < \left( \frac{zh'(z)}{h(z)} \right)^{1/\beta} g(z) \quad \text{implies} \quad A_h(f)(z) < A_h(g)(z).$$

Let the integral operator $A_{\beta,\gamma} : \mathcal{K} \to H(U)$, $\mathcal{K} \subset H(U)$ defined by

$$A_{\beta,\gamma}(f)(z) = \left[ \frac{\beta + \gamma}{z^\gamma} \int_0^z f^{\beta}(t)t^{\gamma-1}(t) \, dt \right]^{1/\beta}, \quad \beta, \gamma \in \mathbb{C}. \quad (1)$$

In the present paper we determine conditions on $g(z)$, $\beta$ and $\gamma$ so that the next implication holds:

$$z \left[ \frac{f(z)}{z} \right]^{\beta} < z \left[ \frac{g(z)}{z} \right]^{\beta} \implies z \left[ \frac{A_{\beta,\gamma}(f)(z)}{z} \right]^{\beta} < z \left[ \frac{A_{\beta,\gamma}(g)(z)}{z} \right]^{\beta}$$

and in addition some particular cases obtained for different choices of $\beta, \gamma$ and $g(z)$ will be given.

2. Preliminaries

In order to prove our main results, we will need the next definitions and lemmas presented in this section.

Let $c \in \mathbb{C}$ with $\Re c > 0$ and let $N = N(c) = \frac{|c| \sqrt{1 + 2\Re c} + \Im c}{\Re c}$. If $k$ is the univalent function $k(z) = \frac{2Nz}{1 - z^2}$ then we define the "open door" function $R_c$ by

$$R_c(z) = k \left( \frac{z + b}{1 + b^2} \right), \quad z \in U. \quad (2)$$

Note that $R_c$ is univalent in $U$, $R_c(0) = c$ and $R_c(U) = k(U)$ is the complex plane slit along the half-lines $\Re w = 0$, $\Im w \geq N$ and $\Re w = 0$, $\Im w \leq -N$. 628
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Let $A$ be the set of functions $f(z) = z + a_2 z^2 + \cdots$ that are analytic in the unit disk $U$ and we denote by $D = \{ \phi \in H(U) : \phi(z) \neq 0 \text{ for } z \in U, \phi(0) = 1 \}$.

**Lemma 2.1.** [6] Let $\phi, \Phi \in D$ and let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\text{Re} (\alpha + \delta) > 0$. If $f \in A$ satisfies

$$\frac{\alpha f'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta < R_{\alpha+\delta}(z),$$

where $R_c$ is defined by (2) and if the function $F$ is defined by

(3) \hspace{1cm} F = A_{\beta, \gamma}(f)

then

$$F \in A, \frac{F(z)}{z} \neq 0, z \in U \text{ and } \text{Re} \left[ \beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right] > 0, z \in U.$$  

(All powers in (1) are principal ones.)

A function $L(z; t), z \in U, t \geq 0$ is called to be a subordination (or a Loewner) chain if $L(\cdot; t)$ is analytic and univalent in $U$ for all $t \geq 0$, $L(z; \cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in U$ and $L(z; s) < L(z; t)$ when $0 \leq s \leq t$.

**Lemma 2.2.** [8, p. 159] The function $L(z; t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t \to +\infty} |a_1(t)| = +\infty$ is a subordination chain if and only if

$$\text{Re} \left[ z \frac{\partial L}{\partial z} \right] > 0, z \in U, t \geq 0.$$

A function $f \in A$ is called to be a convex (and univalent) function in $U$ if $\text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > 0, z \in U$ and we represent the class of such functions by $K$. We denote by $K(\gamma), \gamma \leq 1$ the class of convex functions of order $\gamma$, i.e.

$$K(\gamma) = \left\{ f \in A : \text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > \gamma, z \in U \right\}.$$  

**Lemma 2.3.** [5] Let $F$ be analytic in $U$ and let $G$ be analytic and univalent in $\bar{U}$ with $F(0) = G(0)$. If $F$ is not subordinate to $G$, the there exist points $z_0 \in U, \zeta_0 \in \partial U \text{ an } m \geq 1$ for which $F(|z| < |z_0|) \subset G(U), F(z_0) = G(z_0)$ and $z_0 F'(z_0) = m \zeta_0 G'(\zeta_0)$.  

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Lemma 2.4. Suppose that the function \( \psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C} \) satisfies the condition \( \text{Re } \psi(is, t; z) \leq 0 \) for all \( s \in \mathbb{R}, \ t \leq - \frac{1 + s^2}{2} \) and all \( z \in U \). If \( p \in H(U) \) with \( p(0) = 1 \) then
\[
\text{Re } \psi(p(z), zp'(z); z) > 0, \ z \in U \text{ implies } \text{Re } p(z) > 0, \ z \in U.
\]

More general forms of this lemma may be found in [5].

Lemma 2.5. [5] Let \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and let \( h \in H(U) \) with \( h(0) = c \). If \( \text{Re } [\beta h(z) + \gamma] > 0, \ z \in U \) then the solution of the differential equation
\[
q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)
\]
with \( q(0) = c \) is regular in \( U \) and satisfies \( \text{Re } [\beta q(z) + \gamma] > 0, \ z \in U \).

Finally we denote by \( \mathcal{F}_{\beta, \gamma} \) the class of functions \( f \in A \) that satisfy
\[
\beta \frac{zf'(z)}{f(z)} + \gamma < R_{\beta + \gamma}(z).
\]

3. Main Results

First we will determine the subset, \( \mathcal{K} \subset H(U) \) such that the integral operator given by (1) will be well defined, considering a more general form of this operator.

Lemma 3.1. Let \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \), \( \text{Re } (\beta + \gamma) > 0 \) and let \( h \in A \) with \( h(z)h'(z)/z \neq 0, \ z \in U \). If \( f \in A \) and
\[
\beta \frac{zf'(z)}{f(z)} + (\gamma - 1) \frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)} < R_{\beta + \gamma}(z)
\]
then
\[
F \in A, \ \frac{F(z)}{z} \neq 0, \ z \in U \text{ and } \text{Re } \left[ \beta \frac{zF'(z)}{F(z)} + \gamma \frac{zh'(z)}{z} \right] > 0, \ z \in U
\]
where
\[
F(z) = I_h(z) = \left[ \frac{\beta + \gamma}{h^\gamma(z)} \int_0^z f^\beta(t)h^{\gamma - 1}(t)h'(t) \, dt \right]^{1/\beta}.
\]
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Proof. In order to prove the above result we will use Lemma 2.1 for $\alpha = \beta$, $\Phi(z) = [h(z)/z]^\gamma$ and $\phi(z) = [h(z)/z]^{\gamma-1}h'(z)$. From the assumptions we have $\Phi, \phi \in D$, $A_{\beta, \gamma} = I_h$ and a simple calculus shows that the conditions of Lemma 2.1 are satisfied, hence we obtain our result. □

Remark. Taking $h(z) = z$ in Lemma 3.1 and using the fact that $I_h = A_{\beta, \gamma}$, for the case $h(z) = z$ we have the next implication:

Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $\text{Re} (\beta + \gamma) > 0$. Then $f \in F_{\beta, \gamma}$ implies $F \in A$, $\frac{F(z)}{z} \neq 0$, $z \in U$ and $\text{Re} \left[ \beta \frac{zF'(z)}{F(z)} + \gamma \right] > 0$, $z \in U$ where $F(z) = A_{\beta, \gamma}(f)(z)$.

Theorem 1. Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $0 < \beta + \gamma \leq 1$. Let $f, g \in F_{\beta, \gamma}$ and for $\beta \neq 1$ suppose in addition that $f(z)/z \neq 0$, $g(z)/z \neq 0$, $z \in U$.

If $\text{Re} \left[ 1 + \frac{z\phi''(z)}{\phi'(z)} \right] > 1 - (\beta + \gamma)$, where $\phi(z) = z \left[ \frac{g(z)}{z} \right]^\beta$ then

$z \left[ \frac{f(z)}{z} \right]^\beta < z \left[ \frac{g(z)}{z} \right]^\beta$ implies $z \left[ \frac{A_{\beta, \gamma}(f)(z)}{z} \right]^\beta < z \left[ \frac{A_{\beta, \gamma}(g)(z)}{z} \right]^\beta$.

Proof. Denoting $F = A_{\beta, \gamma}(f)$, $G = A_{\beta, \gamma}(g)$, $\psi(z) = z [f(z)/z]^\beta$, $\phi(z) = z [g(z)/z]^\beta$, $\Psi(z) = z [F(z)/z]^\beta$, $\Phi(z) = z [G(z)/z]^\beta$, we need to prove that $\psi(z) < \phi(z)$ implies $\Psi(z) < \Phi(z)$. Then $\psi, \phi \in A$ and by the above remark we have $F(z)/z \neq 0$ and $G(z)/z \neq 0$ hence $\psi, \phi \in H(U)$ and moreover $\psi, \phi \in A$.

Differentiating the equality $G(z) = A_{\beta, \gamma}(g)(z)$ we have

$$G^\beta(z) \left[ \beta \frac{zG'(z)}{G(z)} + \gamma \right] \frac{1}{\beta + \gamma} = g^\beta(z).$$

Since $\Phi(z) = z \left[ \frac{G(z)}{z} \right]^\beta$, by differentiating this relation we obtain

$$\beta \frac{zG'(z)}{G(z)} + \gamma = \beta + \gamma - 1 + \frac{z\Phi'(z)}{\Phi(z)}$$

and replacing this in (4) we deduce that

$$\phi(z) = \left( 1 - \frac{1}{\beta + \gamma} \right) \Phi(z) + \frac{1}{\beta + \gamma} z\Phi'(z).$$

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Letting \( L(z; t) = \left(1 - \frac{1}{\beta + \gamma}\right) \Phi(z) + \frac{1 + t}{\beta + \gamma} z\Phi'(z) \), then \( L(z; 0) = \phi(0) \).

If \( L(z; t) = a_1(t)z + \cdots \) then
\[
a_1(t) = \frac{\partial L(0; t)}{\partial z} = \left(1 + \frac{t}{\beta + \gamma}\right) \Phi'(0) = 1 + \frac{t}{\beta + \gamma}
\]

hence \( \lim_{t \to +\infty} |a_1(t)| = +\infty \) and since \( \Re(\beta + \gamma) > 0 \) we obtain \( a_1(t) \neq 0 \) for all \( t \geq 0 \).

In order to prove that \( L(z; t) \) is a subordination chain we will use Lemma 2.2.

A simple computation shows that
\[
(6) \quad \Re \left[ z \frac{\partial L/\partial z}{\partial L/\partial t} \right] = \Re \left[ \beta + \gamma + \frac{z\Phi''(z)}{\Phi'(z)} \right] + t \Re \left[ 1 + \frac{z\Phi''(z)}{\Phi'(z)} \right]
\]

and we need to show that
\[
(7) \quad \Re \left[ 1 + \frac{z\Phi''(z)}{\Phi'(z)} \right] > 0, \quad z \in U
\]

and
\[
(8) \quad \Re \left[ \beta + \gamma + \frac{z\Phi''(z)}{\Phi'(z)} \right] > 0, \quad z \in U.
\]

Letting \( q(z) = 1 + \frac{z\Phi''(z)}{\Phi'(z)} \) and by differentiating (5) we have
\[
\phi'(z) = \left(1 - \frac{1}{\beta + \gamma}\right) \Phi'(z) + \frac{1}{\beta + \gamma} \left(\Phi'(z) + z\Phi''(z)\right),
\]

then by computing the logarithmical derivative of the above equality we deduce
\[
(9) \quad q(z) + \frac{zq'(z)}{q(z) + \beta + \gamma - 1} = 1 + \frac{zq''(z)}{q'(z)} \equiv h(z).
\]

Taking in Lemma 2.5 \( \beta \equiv 1, \gamma \equiv \beta + \gamma - 1 \) since \( h(0) = 1 = c \) then the condition \( \Re[\beta h(z) + \gamma] > 0 \) is equivalent to the assumption of the Theorem. It follows from Lemma 2.5 that the differential equation (9) has a solution \( q \in H(U) \) with \( q(0) = 1 \) and this solution verify \( \Re q(z) > 1 - \beta + \gamma, \quad z \in U \). From \( \beta + \gamma \leq 1 \) we have \( \Re q(z) > 1 - \beta - \gamma \geq 0, \quad z \in U \) hence inequality (7) is proved.
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Using (7) and the fact that $\Phi \in A$, then $\Phi$ is univalent in $U$ and from the above inequality we have

$$Re \left[ \frac{z\Phi''(z)}{\Phi'(z)} + \beta + \gamma \right] = Re q(z) + \beta + \gamma - 1 > 0, \ z \in U,$$

then relations (6) and (7) are proved and by using Lemma 2.2 we conclude that $L(z; t)$ is a subordination chain.

Now, by using Lemma 2.3 we will show that $\Psi(z) \prec \Phi(z)$. Without loss of generality we can assume that $\Phi(z)$ is regular and univalent in $\overline{U}$. If not, let $\psi_r(z) = \psi(rz)$, $\phi_r(z) = \phi(rz)$, $\Psi_r(z) = \Psi(rz)$ and $\Phi_r(z) = \Phi(rz)$, where $0 < r < 1$. Then $\Phi_r$ is regular and univalent in $\overline{U}$ and we need to prove that

$$\psi_r(z) \prec \phi_r(z) \implies \Psi_r(z) \prec \Phi_r(z), \text{ for all } 0 < r < 1$$

and by letting $r \to 1^-$ we obtain $\Psi(z) \prec \Phi(z)$.

Suppose that $\Psi(z) \not\prec \Phi(z)$. Then by Lemma 2.3 there exist $z_0 \in U$, $t_0 \geq 0$ and $\zeta_0 \in \partial U$ such that $\Psi(z_0) = \Phi(\zeta_0)$. $z_0\Psi'(z_0) = (1 + t_0)\zeta_0\Phi'(\zeta_0)$. We deduce that

$$L(\zeta_0; t_0) = \left( 1 - \frac{1}{\beta + \gamma} \right) \Phi(\zeta_0) + \frac{1 + t_0}{\beta + \gamma} \zeta_0\Phi'(\zeta_0) =$$

$$= \left( 1 - \frac{1}{\beta + \gamma} \right) \Psi(z_0) + \frac{1}{\beta + \gamma} z_0\Psi'(z_0) = \psi(z_0),$$

and since $L(z; t)$ is a subordination chain and $\phi(z) = L(z; 0)$ it follows that $\psi(z_0) = L(\zeta_0; t_0) \notin \phi(U)$ and this contradicts the assumption of the Theorem. \hfill $\Box$

Next we will presents a few particular cases of this Theorem obtained for appropriate choices of $\beta$, $\gamma$ and $g(z)$.

**COROLLARY 3.2.** Let $f \in \mathcal{F}_{1, \gamma}$ and $g \in K(-\gamma)$ where $-1 < \gamma \leq 0$. Then

$$f(z) \prec g(z) \implies A_{1, \gamma}(f)(z) \prec A_{1, \gamma}(g)(z).$$

**Proof.** In order to use our Theorem for $\gamma = 0$ we need to prove that

$$g \in K(-\gamma) \implies Re \frac{zg'(z)}{g(z)} > -\gamma, \ z \in U.$$
Letting \( p(z) = \frac{1}{1+\gamma} \left[ \frac{zg'(z)}{g(z)} + \gamma \right] \), since \( g \) is univalent then \( g(z)/z \neq 0, z \in U \), hence \( p \in H(U) \) and \( p(0) = 1 \). Twice differentiating the previous equality and using the fact that \( g \in K(-\gamma) \) we have

\[
\text{Re} \left[ p(z) + \frac{zp'(z)}{(1+\gamma)p(z) - \gamma} \right] > 0, z \in U.
\]

Denoting by \( \psi(w_1, w_2) = \frac{w_1 + w_2}{(1+\gamma)w_1 - \gamma} \) then

\[
\text{Re} \psi(is, t) = \text{Re} \frac{-\gamma t}{\gamma^2 + (1+\gamma)^2 s^2} \leq 0 \text{ for all } s \in R \text{ and } t \leq -\frac{1}{2}(1 + s^2).
\]

From Lemma 2.4 we conclude that (10) implies \( \text{Re} \ p(z) > 0, z \in U \), i.e.

\[
\text{Re} \ \frac{zg'(z)}{g(z)} > -\gamma, z \in U \quad \text{so} \quad \frac{zg'(z)}{g(z)} + \gamma < R_{1+\gamma}, \text{ or } g \in \mathcal{F}_{1,\gamma}.
\]

For the case \( \gamma = 0 \), this result was obtained in [3] and later improved in [7] by the condition \( g \in K(-1/2) \).

Taking \( \beta + \gamma = 1 \) in our Theorem we have:

**COROLLARY 3.3.** Let \( \beta \in \mathbb{C}^* \), let \( f, g \in \mathcal{F}_{\beta,1-\beta} \), and for \( \beta \neq 1 \) suppose in addition that \( f(z)/z \neq 0, g(z)/z \neq 0 \), \( z \in U \). If \( \phi(z) = z \left[ \frac{g(z)}{z} \right]^\beta \in K \), then

\[
z \left[ \frac{f(z)}{z} \right]^\beta < z \left[ \frac{g(z)}{z} \right]^\beta \quad \text{implies} \quad z \left[ \frac{F(z)}{z} \right]^\beta < z \left[ \frac{G(z)}{z} \right]^\beta
\]

where \( F = A_{\beta,1-\beta}(f) \), \( G = A_{\beta,1-\beta}(g) \).

For the case \( g(z) = ze^{\lambda z} \) we may easily prove the next Corollary:

**COROLLARY 3.4.** Let \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \), \( 0 < \beta + \gamma \leq 1 \), and let

\[
\lambda \in \mathbb{C} \text{ with } |\lambda| \leq \frac{2 + \beta + \gamma - \sqrt{(\beta + \gamma)^2 + 4}}{2|\beta|}.
\]
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Let $f \in \mathcal{F}_{\beta, \gamma}$ and for $\beta \neq 1$ suppose in addition that $\frac{f(z)}{z} \neq 0$, $z \in U$. Then

$$z \left[ \frac{f(z)}{z} \right]^\beta < ze^{\lambda \beta z} \implies z \left[ \frac{F(z)}{z} \right]^\beta < \frac{\beta + \gamma}{z^{\beta + \gamma - 1}} \int_0^z t^{\beta + \gamma - 1} e^{\lambda \beta t} dt$$

where $F = A_{\beta, \gamma}(f)$.

Proof. For $g(z) = ze^{\lambda z}$, $\lambda \in \mathbb{C}$, we have $g \in \mathcal{F}_{\beta, \gamma}$ if and only if $h(z) = \lambda \beta z + \beta + \gamma < R_{\beta + \gamma}(z)$. But

$$|\lambda \beta| \leq \beta + \gamma + 1$$

is equivalent to $|h(z) - (\beta + \gamma)| < \beta + \gamma + 1$, $z \in U$, and this last condition is sufficient for $g \in \mathcal{F}_{\beta, \gamma}$. A simple calculus shows that $\phi(z) = ze^{\lambda \beta z}$ and $1 + z \frac{\phi''(z)}{\phi'(z)} = 1 + \lambda \beta z + \frac{\lambda \beta z}{1 + \lambda \beta z}$ and in order to use our Theorem we must to determine the largest $r = |\lambda \beta|$ such that

$$Re \Phi(\zeta) > 1 - (\beta + \gamma), |\zeta| < r \text{ where } \Phi(\zeta) = 1 + \zeta + \frac{\zeta}{1 + \zeta}.$$ 

Since $r \leq 1$ then $|\lambda \beta| \leq 1$ which implies (11). If $\zeta = re^{i\theta}, \theta \in [0, 2\pi]$ then

$$Re \Phi(re^{i\theta}) = 2 + r \cos \theta - \frac{1 + r \cos \theta}{r^2 + 2r \cos \theta + 1}.$$ 

It is easy to show that $Re \Phi(re^{i\theta}) \geq 2 - r - \frac{1}{1 - r} = t(r), \theta \in [0, 2\pi]$ and

$$t(r) \geq 1 - (\beta + \gamma) \text{ if and only if } r \leq \frac{2 + \beta + \gamma - \sqrt{(\beta + \gamma)^2 + 4}}{2} = r_* \in (0, 1)$$

or $|\lambda \beta| \leq r_*$. \hfill \Box

References


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