A NOTE ON TIGHT CLOSURE
AND FROBENIUS MAP

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In recent years M.Hochster and C.Huneke introduced the notions of tight closure of an ideal and of the weak $F$-regularity of a ring of positive prime characteristic. Here ‘$F$’ stands for Frobenius. This notion enabled us to play an important role in a commutative ring theory, and other related topics.

In this paper we study the connections between the Frobenius map and the tight closure.

A weakly $F$-regular ring is easily seen to be $F$-pure, but we do not know the converse is true or not in general. We study conditions for an $F$-pure ring to be weakly $F$-regular. And as a corollary we give a proof of R. Fedder’s conjecture in one dimensional case as follows: “$R/xR$ is $F$-pur” implies “$R$ is $F$-pur” whenever $R$ is Cohen-Macaulay ring of dimension one. Finally we study the conditions related to the tight closure that the Cohen-Macaulay ring to be weakly $F$-regular and Gorenstein.

1. Preliminaries

All rings are commutative, Noetherian with identity of prime characteristic $p$. And all modules are finitely generated, unless otherwise specified.

Definition 1.1. [Hochster-Huneke] Let $I \subseteq R$ be an ideal and $R^o$ denote the complement of the union of the minimal primes of $R$ and

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let $I^{[q]}$ denote the ideal $(i^q : i \in I)$. We say that $x \in I^*$, the tight closure of $I$, if there exists $c \in R^*$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$, i.e., for all sufficiently large $q$ of the form $p^e$. If $I = I^*$, we say that $I$ is tightly closed.

**Definition 1.2.** [Hochster-Huneke] A Noetherian ring is called weakly $F$-regular if every ideal is tightly closed. If every localization of $R$ at a multiplicative subset is weakly $F$-regular, then we say that $R$ is $F$-regular.

**Definition 1.3.** [Fedder-Watanabe] A Noetherian local ring of characteristic $p$ is called $F$-rational if every ideal generated by a system of parameter is tightly closed.

Now we introduce the notion of $F$-purity, relying on the special properties of the Frobenius homomorphism. And we discuss the relationship between the $F$-purity and the weak $F$-regularity. Let $R$ be a ring of characteristic $p$. Denote by $^eR$, the ring $R$ viewed as an $R$-module via the $e$-th power of the Frobenius map $F(r) = r^q$, where $q = p^e$. Furthermore, for any $R$-module $M$, $^eM = M \otimes_R {}^eR$ will denote the group $M$ viewed as an $R$-module via $r \cdot m = r^q m$. $R \xrightarrow{F^e} {}^eR$ is therefore an $R$-module homomorphism [9].

**Definition 1.4.** [Hochster-Roberts] A Noetherian ring $R$ of characteristic $p$ is called $F$-pure if for every $R$-module $M$,

$$0 \to M \otimes_R R \to M \otimes_R \overline{1}_R$$

is exact. Equivalently, for some $e > 0$, $0 \to M \to M \otimes_R {}^eR$ is exact.

**Definition 1.5.** [Fedder] We say that a local ring $R$ is $F$-contracted if $R \to \overline{1}_R$ is contracted, which means that every ideal $I$ which is generated by a system of parameter for $R$ satisfies

$$(I \cdot \overline{1}_R) \cap R = I.$$

**Lemma 1.6.** For an $F$-pure or an $F$-contracted ring $R$, the Frobenius map must be injective. Whence $R$ is reduced.
Proof. If $R$ is $F$-pure, then the Frobenius map by tensoring with $R$ is also injective from the definition.

If $R$ is $F$-contracted, and if $F(r) = 0$, then certainly $F(r) \in I \cdot 1 R$ for every ideal $I$ which is generated by a system of parameter for $R$. The contractedness hypothesis then guarantees that $r$ lies in the intersection of all ideals of $R$ which are generated by a system of parameter But this intersection is well known to be 0. Thus, $r = 0$ and the Frobenius map is injective. □

When $R$ is reduced, there is a natural identification of maps:

1. $R \xrightarrow{F} 1 R$.
2. $R \rightarrow R^{1/p}$ where $R^{1/p}$ denotes the ring of the $p$-th roots of elements in $R$.
3. $R^p \rightarrow R$, where $R^p$ denotes the ring of the $p$-th powers of elements in $R$.

Thus, if $I = (a_1, \cdots, a_t)$ is an ideal in $R$, then $1 I$ can be thought of as the ideal $(a_1^{1/p}, \cdots, a_t^{1/p}) \subset R^{1/p}$ under the second identification of maps.

**Definition 1.7.** [Hochster] The map $R \xrightarrow{\phi} S$ is called **cyclically pure** if for every ideal $I \subset R$, $\{x \in R \mid \phi(x) \in IS\} = I$.

Note that the fact that $\phi$ must be injective follows from the case when $I = 0$. Let $S = 1 R$ and $\phi$ be the Frobenius map. Then, since $I \cdot 1 R = 1 (I[p] R)$, it follows that $R \rightarrow 1 R$ is cyclically $F$-pure if and only if $f^p \in I[p]$ implies $f \in I$. Clearly if $R \rightarrow 1 R$ is $F$-pure, then this map is cyclically $F$-pure. But the converse is true only when $R$ is approximately Gorenstein [6].

**2. Weak $F$-regularity and $F$-purity**

**Proposition 2.1.** A weakly $F$-regular ring $R$ is $F$-pure.

Proof. In fact, the weak $F$-regularity always implies that the map $R \rightarrow 1 R$ is cyclically pure because $f^p \in I[p]$ implies $1 \cdot f^q \in I[q]$ for every $q = p^e$. Whence, $f \in I^* = I$. But if $R$ is approximately Gorenstein, then $R \rightarrow S$ is cyclically pure if and only if it is pure. Since weakly $F$-regular rings are normal, and so approximately Gorenstein. It follows that $R$ is $F$-pure. □
But the converse of Proposition 2.1., that is, the $F$-purity implies the weak $F$-regularity, remains open. However, for the zero dimensional case, we have an affirmative answer.

**Theorem 2.2.** A zero dimensional $F$-pure ring $R$ is weakly $F$-regular.

**Proof.** We may assume that $R$ is local with the maximal ideal $m$. Let $I$ be an ideal of $R$ and let $x \in I^*$. Then there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q = p^e$ because an $F$-pure ring is reduced. But $\bigcup \{P : P \text{ is a minimal prime ideal of } R\} = m$, and $R^o = R - m$ is the units of $R$, we have $x^q \in I^{[q]}$. Thus $x \in I$ by $F$-purity. Hence $I = I^*$ and $R$ is weakly $F$-regular. $\Box$

Now we can prove an one dimensional case of an important conjecture, which is raised by R. Fedder in his paper [2], by using Theorem 2.2.

**Fedder’s Conjecture:** “$R/fR$ is $F$-pure” should imply “$R$ is $F$-pure”, whenever $R$ is Cohen-Macaulay ring and $f \notin Z(R)$.

**Corollary 2.3.** Let $R$ be a one dimensional ring, and let $f \notin Z(R)$. If $R/fR$ is $F$-pure, then $R$ is $F$-pure.

**Proof.** Since $R/fR$ is a zero dimensional $F$-pure ring, $R/fR$ is weakly $F$-regular by Theorem 2.2. Since $\dim R = 1$, $R$ is also weakly $F$-regular[1]. Thus $R$ is $F$-pure. $\Box$

Now we prove that an $F$-pure ring is weakly $F$-regular for the higher dimensional case under additional conditions.

**Definition 2.4.** Let $R$ be a Noetherian reduced ring of characteristic $p$, and let $M$ be an $R$-module. We say that $M$ is $F$-unstable if for every nonzero $x \in M$,

$$\bigcap_{e > 0} \text{Ann}_R(F^e(x)) = (0),$$

where $F^e(x)$ denotes the image of $x = x \otimes 1$ in $F^e(M) = M \otimes_R e^e R$. 
Lemma 2.5. For every ideal $I$ of a domain $R$, $I = I^*$ if and only if $R/I$ is $F$-unstable as an $R$-module.

Proof. Assume that $R/I$ is $F$-unstable and $I$ is not tightly closed. Let $y \in I^* - I$. Then there exists $c \neq 0 \in R$ such that $cy^q \in I^{|q|}$ for all $q = p^e$. Let $x$ be the image of $y$ in $R/I$. Then $F^e(x) \in F^e(R/I) = R/I^{|q|}$, and $cF^e(x) = 0$ in $F^e(R/I)$ for every $e > 0$. Thus $0 \neq c \in \bigcap_{e>0} \text{Ann}_R(F^e(x))$, a contradiction.

Conversely, assume $R/I$ is not $F$-unstable. Then there exist a nonzero $x \in R/I$ and nonzero $c \in \text{Ann}_R(F^e(x))$ for every $e > 0$. Thus $cy^q \in I^{|q|}$ for every $q = p^e$, where $y$ is the representative of $x$ in $R$. Hence $y \in I^*$, but $y \notin I$. That is, $I$ is not tightly closed. \qed

Theorem 2.6. Let $R$ be a complete $F$-pure domain. Then the followings are equivalent:

1. Every ideal of $R$ is tightly closed.
2. Every finite $R$-module is $F$-unstable.
3. If $R$ is a local ring with the unique maximal ideal $m$, then $E_R(R/m)$, the injective hull of $R/m$, is $F$-unstable.

Proof. (1) implies (2); We will prove by induction on the number $n$ of generators of $M$.

(i) $n = 1$; $M$ is a cyclic $R$-module, let $M = Rx, x \in M$. Then $M$ is isomorphic to $R/I$, where $I = \text{Ann}_R(x)$. Since $I$ is tightly closed by the weak $F$-regularity of $R$, $M$ is $F$-unstable by Lemma 2.5.

(ii) $n > 1$; Let $M_1 = \sum_{i=1}^{k-1} Rx_i, M = \sum_{i=1}^k Rx_i$, and $M_2 = M/M_1$, where $x_i \in M$ for every $i = 1, \cdots, k$. Then the induction hypothesis implies that $M_1$ and $M_2$ are $F$-unstable. From the following commutative diagram follows that $M$ is $F$-unstable.

\[
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & \rightarrow & M_1 & \rightarrow & M & \rightarrow & M_2 & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & \rightarrow & M_1 \otimes_R {}^e R & \rightarrow & M \otimes_R {}^e R & \rightarrow & M_2 \otimes_R {}^e R & \rightarrow & 0
\end{array}
\]
(2) implies (3): we can write \( E = E_R(R/m) \) as a direct limit of finite \( R \)-modules. That is, \( E = \lim_{\to} M_i \), where \( M_i \) are finite \( R \)-modules. Here, each \( M_i \) is \( F \)-unstable by the hypothesis. Then,

\[
E \otimes_R e^c R = (\lim_{\to} M_i) \otimes_R e^c R = \lim_{\to} (M_i \otimes_R e^c R).
\]

The second equality follows from the fact that the tensoring, \( \otimes_R e^c R \), commutes with the direct limit. Thus (3) is also true.

(3) implies (1): Assume that \( I^* \supsetneq I \) for any ideal \( I \) of \( R \).

Then \( (0 :_E I^*) \subsetneq (0 :_E I) = \{ r \in E \mid rI = 0 \} \), and \( I^* \cdot (0 :_E I) \neq 0 \).

For, if \( (0 :_E I^*) = (0 :_E I) \), then \( \text{Ann}_R((0 :_E I^*)) = \text{Ann}_R((0 :_E I)) \).

But \( \text{Ann}_R((0 :_E J)) = J \) for any ideal \( J \) of \( R \) \[8\]. This implies that \( I = I^* \), which is a contradiction.

We can therefore choose \( y \in I^* \) and \( x \in (0 :_E I) \) such that \( z = yx \) is a nonzero element of \( E \). Since \( y \in I^* \), there exists \( c \neq 0 \in R \) such that \( cy^q \in I^{[q]} \) for all \( q = p^e \). Since \( x \in (0 :_E I) \), \( F^e(x) \in (0 :_E I^{[q]}) \). Thus, \( 0 = (cy^q)F^e(x) = cF^e(yx) = cF^e(z) \) for every \( e > 0 \). That is, \( c \in \bigcap_{e > 0} (0 :_R F^e(z)) = \bigcap_{e > 0} \text{Ann}_R(F^e(z)) \). Since \( c \neq 0 \), \( E \) is not \( F \)-unstable, which is a contradiction. \( \square \)

3. The Frobenius Map and the Weak \( F \)-regularity

Recall that a ring \( R \) of characteristic \( p \) is \( F \)-contracted if every ideal generated by a system of parameter is contracted with respect to the Frobenius map \( F : R \to R \), that is, \( (I \cdot R) \cap R = I \).

**Proposition 3.1.** Let \( (R, m) \) be a Cohen-Macaulay local ring with the maximal ideal \( m \). Then the followings are equivalent:

1. The map from \( H^i_m(R) \) to \( H^i_m(R) \), induced by the Frobenius map from \( R \) to \( R \), is injective.
2. \( R \) is \( F \)-contracted.
3. There exists a system of parameter which is contracted with respect to the Frobenius map from \( R \) to \( R \).
Proof. See [3, Proposition 1.4].

For a Gorenstein local ring $R$ of dimension $n$, it is a well-known fact from local duality theory that $H^n_m (R)$ is isomorphic to $E$, the injective hull of $R/m$, where $m$ is the unique maximal ideal of $R$. Hochster and Roberts proved that $R$ is $F$-pure if and only if $E \to E \otimes^1 R$ is injective [7]. Hence we have the following:

**PROPOSITION 3.2.** Let $R$ be a local Gorenstein ring with the maximal ideal $m$. Then the followings are equivalent:

1. $R$ is $F$-pure.
2. $R$ is $F$-contracted.
3. There exists a system of parameter which is contracted with respect to the Frobenius map.
4. $H^n_m (R) \to H^n_m (1R)$ is injective, where $\dim R = n$.

Proof. (2), (3), and (4) are equivalent by Proposition 3.1. And the implication of (1) to (2) is clear. Now it remains only to prove that (4) implies (1). But $H^n_m (R) \cong E$, the injective hull of $R/m$, implies that $E \to E \otimes^1 R$ is injective. Thus, $R$ is $F$-pure. \hfill \Box

Now we discuss the relationship between the $F$-contractedness and the weak $F$-regularity, and characterize the Gorenstein ring of dimension zero.

**PROPOSITION 3.3.** Let $R$ be a local Gorenstein ring and let $x_1, \cdots, x_d$ be a system of parameter. If the image of $I$ in $R/I$ is contracted with respect to the Frobenius map

$$F : R/I \to (R/I),$$

where $I = (x_1, \cdots, x_d)R$, then $R$ is weakly $F$-regular.

Proof. $R/I$ is a zero-dimensional Gorenstein $F$-pure ring by the hypothesis and Proposition 3.2. Thus $R/I$ is weakly $F$-regular by Theorem 2. Since $x_1, \cdots, x_d$ is a regular sequence in $R$, $R$ is also Gorenstein. Thus $R$ is weakly $F$-regular. \hfill \Box

In Proposition 3.3, the condition that $R$ is Gorenstein can be replaced by the condition that $R$ is Cohen-Macaulay.
LEMMA 3.4. Let $R$ be a reduced ring of dimension zero. Then $R$ is Gorenstein, and weakly $F$-regular.

Proof. We may assume that $R$ is local. Since $R$ is a direct product of finite number of fields, $R$ is normal. But we know that any normal local ring is approximately Gorenstein. Since $R$ is zero-dimensional local, $R$ is Gorenstein. Now we need only to show that $R$ is weakly $F$-regular. It is enough to show that $(0)$, a system of parameterideal of $R$, is tightly closed. Let $r \in (0)^*$. Then there exists $c \in R^o$ such that $cr^q = 0$ for all $q = p^e$. But $R^o = R \setminus Z(R)$, since $R$ is Noetherian reduced. We have $r^q = 0$ and $r = 0$. Thus, $(0) = (0)^*$, as required. \hfill \Box

THEOREM 3.5. Let $R$ be a Cohen-Macaulay local ring of dimension $d$ and let $I$ be an ideal of $R$ which is generated by a system of parameter If $R/I$ is reduced, then $R$ is Gorenstein and $R$ is $F$-regular.

Proof. Since $R/I$ is zero-dimensional and reduced, $R/I$ is Gorenstein and (weakly) $F$-regular by Lemma 3.4. And since a system of parameter for $R$ is a regular sequence in $R$, $R$ is also Gorenstein. We know that if $x_1, \cdots, x_d$ form a regular sequence in a Gorenstein local ring and $R/(x_1, \cdots, x_d)R$ is weakly $F$-regular, then $R$ is weakly $F$-regular [1]. \hfill \Box

Now we prove that Proposition 3.3. is still true when $R$ is Cohen-Macaulay.

PROPOSITION 3.6. Let $R$ be a Cohen-Macaulay local ring of dimension $d$, and let $x_1, \cdots, x_d$ be a system of parameter If the image of $I = (x_1, \cdots, x_d)R$ in $R/I$ is contracted with respect to the Frobenius map

$$F : R/I \to 1(R/I),$$

then $R$ is weakly $F$-regular and Gorenstein.

Proof. The condition that the image of $I = (x_1, \cdots, x_d)R$ in $R/I$ is contracted with respect to the Frobenius map implies that $R/I$ is $F$-contracted, and $R/I$ is reduced. Thus $R$ is Gorenstein and $F$-regular by Theorem 3.5. \hfill \Box
Corollary 3.7. Let $R$ be a Cohen-Macaulay local ring of dimension $d$. If $R/I$ is $F$-pure and $I$ is an ideal generated by an s.o.p., then $R$ is $F$-regular.

References


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