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ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF I.I.D. RANDOM VARIABLES

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1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d.) random variables. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants. The purpose of this paper is to find sufficient conditions on $\{a_{ni}\}$ and X such that

$$(1) \qquad \sum_{i=1}^{n} a_{ni}(X_i - EX_i) \to 0 \text{ a.s.}$$

Some convergence theorems for these weighted sums have been obtained by Choi and Sung [2], Chow [4], Chow and Lai [5], Stout [8], Teicher [9], and Thrum [10]. For example, it has been shown in Choi and Sung[2] that conditions $E|X| < \infty$ and $\max_{1\leq i\leq n} |a_{ni}| = O(1/n)$ imply (1). The first major result of this paper is to generalize the result of Choi and Sung[2] for the case of random variables with finite p-th moment($1 \leq p < 2$).

In the case of $a_{ni} = a_i/b_n$ for $1 \leq i \leq n$ and $n \geq 1$, considerations concerning (1) can be found in Adler and Rosalsky [1], Fernoholz and Teicher [6], Jamisons, Orey, and Pruitt [7], and Teicher [9]. Adler and Rosalsky [1] have shown that if $\{X, X_n\}$ is a sequence of i.i.d. random variables with $E|X|^p < \infty$ for some $1 \leq p < 2$, and $\{a_n\}$ and $\{b_n\}$ are

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constants satisfying $0 < b_n \uparrow \infty$,

$$\frac{a_n}{b_n} = O\left(\frac{1}{n^{1/p}}\right), \tag{2}$$

and

$$\sum_{i=1}^{n} |a_i| = O(b_n), \tag{3}$$

then

$$\frac{\sum_{i=1}^{n} a_i(X_i - EX_i)}{b_n} \to 0 \text{ a.s.} \tag{4}$$

Note that Adler and Rosalsky's theorem includes Kolmogorov strong law of large numbers(SLLN) as a special case, where $a_n = 1$ and $b_n = n$ for $n \geq 1$, but not Marcinkiewicz SLLN. The second major result is to improve Adler and Rosalsky's theorem by showing that (i) condition (3) is unnecessary when $1 < p < 2$, and (ii) condition (3) is essential when $p = 1$.

It proves convenient to define $\lg x = \max\{1, \log x\}$, where $\log x$ denotes the natural logarithm. The symbol C denotes a constant which is not necessarily the same one in each appearance.

2. Main Results

The following two lemmas will be used in obtaining our first main result. The proof of Lemma 1 is similar to that of Lemma 2.2 of Choi and Sung [3].

LEMMA 1. *If $E|X|^p < \infty$ for some $0 < p < 2$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2/p}(\lg n)^{1-2/p}} EX^2 I\left(|X| \leq \frac{n^{1/p}}{(\lg n)^{1/p}}\right) < \infty.$$

Proof.

$$\sum_{n=1}^{\infty} \frac{1}{n^{2/p}(\lg n)^{1-2/p}} EX^2 I(|X| \le \frac{n^{1/p}}{(\lg n)^{1/p}})$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{2/p}(\lg n)^{1-2/p}}$$

$$\sum_{i=1}^{n} EX^2 I(\frac{(i-1)^{1/p}}{(\lg(i-1))^{1/p}} < |X| \le \frac{i^{1/p}}{(\lg i)^{1/p}}) \quad (\lg 0 = 1)$$

$$= \sum_{i=1}^{\infty} EX^2 I(\frac{(i-1)^{1/p}}{(\lg(i-1))^{1/p}} < |X| \le \frac{i^{1/p}}{(\lg i)^{1/p}})$$

$$\sum_{n=i}^{\infty} \frac{1}{n^{2/p}(\lg n)^{1-2/p}}$$

$$\le C \sum_{i=1}^{\infty} EX^2 I(\frac{(i-1)^{1/p}}{(\lg(i-1))^{1/p}} < |X| \le \frac{i^{1/p}}{(\lg i)^{1/p}}) \frac{i}{i^{2/p}(\lg i)^{1-2/p}}$$

$$\le C \sum_{i=1}^{\infty} P(\frac{(i-1)^{1/p}}{(\lg(i-1))^{1/p}} < |X| \le \frac{i^{1/p}}{(\lg i)^{1/p}}) \frac{i}{\lg i}$$

$$\le CE|X|^p < \infty,$$

since the first inequality follows from the following.

$$\sum_{n=i}^{\infty} \frac{1}{n^{2/p}(\lg n)^{1-2/p}} \le C \int_i^{\infty} \frac{1}{x^{2/p}(\log x)^{1-2/p}}\, dx \le C \frac{i}{i^{2/p}(\lg i)^{1-2/p}}. \qquad \square$$

LEMMA 2. *If $E|X|^p < \infty$ for some $p > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/p}(\lg n)^{1-1/p}} E|X| I(|X| > \frac{n^{1/p}}{(\lg n)^{1/p}}) < \infty.$$

Proof. The proof is similar to that of Lemma 1 and omitted. □

The next theorem, our first main result, is an extension of Theorem 5 of Choi and Sung [2]. They have proved Theorem 1 when $p = 1$.

THEOREM 1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $E|X|^p < \infty$ for some $1 \leq p < 2$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying*

$$\max_{1 \leq i \leq n} |a_{ni}| = O\left(\frac{1}{n^{1/p}(\lg n)^{1-1/p}}\right).$$

Then (1) holds, i.e., $\sum_{i=1}^{n} a_{ni}(X_i - EX_i) \to 0$ a.s.

Proof. Define

$$X'_n = X_n I\left(|X_n| \leq \frac{n^{1/p}}{(\lg n)^{1/p}}\right) - EXI\left(|X| \leq \frac{n^{1/p}}{(\lg n)^{1/p}}\right),$$
$$X''_n = X_n I\left(|X_n| > \frac{n^{1/p}}{(\lg n)^{1/p}}\right) - EXI\left(|X| > \frac{n^{1/p}}{(\lg n)^{1/p}}\right).$$

Then $X'_n + X''_n = X_n - EX_n$ for $n \geq 1$. To prove

$$\limsup_{n\to\infty} \sum_{i=1}^{n} a_{ni}(X_i - EX_i) \leq 0 \text{ a.s.},$$

it is enough to show that

(5) $$\limsup_{n\to\infty} \sum_{i=1}^{n} a_{ni}X'_i \leq 0 \text{ a.s.}$$

and

(6) $$\lim_{n\to\infty} \sum_{i=1}^{n} a_{ni}X''_i = 0 \text{ a.s.}$$

From the inequality $e^x \le 1 + x + x^2 e^{|x|}/2$ for all $x \in R$, we have for $t > 0$

$$
\begin{aligned}
E[\exp(ta_{ni}X_i')] &\le 1 + \frac{1}{2}t^2 a_{ni}^2 E[X_i'^2 \exp(t|a_{ni}X_i'|)] \\
&\le 1 + C\frac{t^2}{n^{2/p}(\lg n)^{2-2/p}} \exp(C\frac{t}{\lg n}) EX_i'^2 \\
&\le \exp\{C\frac{t^2}{n^{2/p}(\lg n)^{2-2/p}} \exp(C\frac{t}{\lg n}) EX_i'^2\}.
\end{aligned}
$$

By the independence of $\{X_n'\}$

$$
\begin{aligned}
E[\exp(t\sum_{i=1}^n a_{ni}X_i')] &= \prod_{i=1}^n E[\exp(ta_{ni}X_i')] \\
&\le \exp\{C\frac{t^2}{n^{2/p}(\lg n)^{2-2/p}} \exp(C\frac{t}{\lg n}) \sum_{i=1}^n EX_i'^2\}.
\end{aligned}
$$

Let $\epsilon > 0$ be given. By putting $t = 2\lg n/\epsilon$, we obtain

$$
\begin{aligned}
P(\sum_{i=1}^n a_{ni}X_i' > \epsilon) &\le e^{-t\epsilon} E[\exp(t\sum_{i=1}^n a_{ni}X_i')] \\
&\le e^{-t\epsilon} \exp\{C\frac{t^2}{n^{2/p}(\lg n)^{2-2/p}} \exp(C\frac{t}{\lg n}) \sum_{i=1}^n EX_i'^2\} \qquad (7) \\
&\le \frac{1}{n^2} \exp\{C\frac{\lg n}{n^{2/p}(\lg n)^{1-2/p}} \sum_{i=1}^n EX_i'^2\}.
\end{aligned}
$$

On the other hand, Lemma 1 and Kronecker lemma entail that

$$
\frac{1}{n^{2/p}(\lg n)^{1-2/p}} \sum_{i=1}^n EX_i'^2 \to 0.
$$

Hence, the power of exp in the last expression of (7) is bounded by $\alpha \lg n (\alpha > 0)$ for all sufficiently large n. Thus, choosing $0 < \alpha < 1$, we have

$$
\sum_{n=1}^\infty P(\sum_{i=1}^n a_{ni}X_i' > \epsilon) \le C\sum_{n=1}^\infty \frac{1}{n^{2-\alpha}} < \infty,
$$

which implies (5) by Borel-Cantelli lemma.
Now we show that (6) holds. For the case $p=1$, note that

$$
\begin{aligned}
&|\sum_{i=1}^{n} a_{ni}X_i''| \le C\frac{1}{n}\sum_{i=1}^{n}|X_i''| \\
\le & C\frac{1}{n}\sum_{i=1}^{n}|X_i|I(|X_i|>\frac{i}{\lg i}) + C\frac{1}{n}\sum_{i=1}^{n}E|X|I(|X|>\frac{i}{\lg i}) \\
\le & C\frac{1}{n}\sum_{i=1}^{n}|X_i|I(|X_i|>N) \\
&+ C\frac{1}{n}\sum_{i=1}^{n}|X_i|I(\frac{i}{\lg i}<|X_i|\le N) + C\frac{1}{n}\sum_{i=1}^{n}E|X|I(|X|>\frac{i}{\lg i}).
\end{aligned}
$$

The first term on the last expression converges to $CE|X|I(|X|>N)$ a.s. by Kolmogorov SLLN. The second term clearly converges to 0. The third term converges to 0 since $E|X|I(|X|>i/\lg i)\to 0$ as $i\to\infty$. Thus

$$
\limsup_{n\to\infty}|\sum_{i=1}^{n}a_{ni}X_i''| \le \lim_{N\to\infty} CE|X|I(|X|>N)=0 \text{ a.s.},
$$

and so (6) holds when $p=1$. Next we assume that $1<p<2$. By observing that

$$
\begin{aligned}
\max_{2^k\le n<2^{k+1}}|\sum_{i=1}^{n}a_{ni}X_i''| &\le C\max_{2^k\le n<2^{k+1}}\frac{1}{n^{1/p}(\lg n)^{1-1/p}}\sum_{i=1}^{n}|X_i''| \\
&\le C\frac{1}{(2^{k+1})^{1/p}(\lg 2^{k+1})^{1-1/p}}\sum_{i=1}^{2^{k+1}}|X_i''|,
\end{aligned}
$$

we will obtain (6) if we show that

$$
\frac{1}{(2^k)^{1/p}(\lg 2^k)^{1-1/p}}\sum_{i=1}^{2^k}|X_i''|\to 0 \text{ a.s.} \tag{8}
$$

By Lemma 2, we have for any $\epsilon > 0$

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\Big(\frac{1}{(2^k)^{1/p}(\lg 2^k)^{1-1/p}} \sum_{i=1}^{2^k} |X_i''| > \epsilon\Big) \\
&\le \frac{1}{\epsilon} \sum_{k=1}^{\infty} \frac{1}{(2^k)^{1/p}(\lg 2^k)^{1-1/p}} \sum_{i=1}^{2^k} E|X_i''| \\
&= \frac{1}{\epsilon} \sum_{i=1}^{\infty} E|X_i''| \sum_{k:2^k \ge i} \frac{1}{(2^k)^{1/p}(\lg 2^k)^{1-1/p}} \\
&\le \frac{1}{\epsilon} \sum_{i=1}^{\infty} E|X_i''| \frac{1}{(\lg i)^{1-1/p}} \sum_{k:2^k \ge i} \frac{1}{(2^k)^{1/p}} \\
&\le C \sum_{i=1}^{\infty} \frac{E|X_i''|}{i^{1/p}(\lg i)^{1-1/p}} < \infty,
\end{aligned}$$

where $\sum_{k:2^k \ge i}$ means that the summation is taken over all k such that $2^k \ge i$. Hence (8) follows by Borel-Cantelli lemma.

By replacing X_i by $-X_i$ in the above argument we obtain that

$$\liminf_{n\to\infty} \sum_{i=1}^{n} a_{ni}(X_i - EX_i) \ge 0 \text{ a.s.}$$

Thus the conclusion follows. □

The following theorem is an extension of Theorem 2 of Adler and Rosalsky [1]. It has less stringent condition than Theorem 1 when $a_{ni} = a_i/b_n$ for $1 \le i \le n$ and $1 < p < 2$.

THEOREM 2. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $E|X|^p < \infty$ for some $1 \le p < 2$. Let $\{a_n\}$ and $\{b_n\}$ be constants satisfying $0 < b_n \uparrow \infty$. Assume that condition (2) holds. Then*

(i) $\sum_{i=1}^{n} a_i(X_i - EX_i)/b_n \to 0$ a.s. *if* $1 < p < 2$.
(ii) $\sum_{i=1}^{n} a_i(X_i - EX_i)/b_n \to 0$ a.s. *if* $p = 1$ *and condition (3) holds.*

Proof. We need only to prove (i), since (ii) follows from Theorem 2 of Adler and Rosalsky [1]. Assume that $1 < p < 2$. Define $Y_n = X_n I(|X_n| \le n^{1/p})$ for $n \ge 1$. From the proof of Theorem 2 of Adler and Rosalsky [1],

$$\frac{\sum_{i=1}^{n} a_i(X_i - EY_i)}{b_n} \to 0 \text{ a.s.}$$

The proof will be completed by showing that

$$\frac{\sum_{i=1}^{n} a_i E(X_i - Y_i)}{b_n} \to 0.$$

By Kronecker lemma, it is enough to show that

$$\sum_{n=1}^{\infty} \frac{a_n E(X_n - Y_n)}{b_n} \text{ converges.} \tag{9}$$

It follows from condition (2) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|\frac{a_n E(X_n - Y_n)}{b_n}\right| &\le \sum_{n=1}^{\infty} \frac{|a_n|}{|b_n|} E|X| I(|X| > n^{1/p}) \\
&= \sum_{n=1}^{\infty} \frac{|a_n|}{|b_n|} \sum_{i=n}^{\infty} E|X| I(i^{1/p} < |X| \le (i+1)^{1/p}) \\
&= \sum_{i=1}^{\infty} E|X| I(i^{1/p} < |X| \le (i+1)^{1/p}) \sum_{n=1}^{i} \frac{|a_n|}{|b_n|} \\
&\le C \sum_{i=1}^{\infty} E|X| I(i^{1/p} < |X| \le (i+1)^{1/p}) \sum_{n=1}^{i} \frac{1}{n^{1/p}} \\
&\le C \sum_{i=1}^{\infty} i^{(p-1)/p} E|X| I(i^{1/p} < |X| \le (i+1)^{1/p}) \\
&\le C \sum_{i=1}^{\infty} E|X|^p I(i^{1/p} < |X| \le (i+1)^{1/p}) \\
&\le C E|X|^p < \infty,
\end{aligned}$$

which implies (9) and the proof is complete. □

COROLLARY 1. (Marcinkiewicz SLLN). *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $E|X|^p < \infty$ for some $1 \leq p < 2$. Then*

$$\frac{\sum_{i=1}^n (X_i - EX_i)}{n^{1/p}} \to 0 \text{ a.s.}$$

Proof. Let $a_n = 1$ and $b_n = n^{1/p}$ for $n \geq 1$. Then condition (2) holds true. So Corollary 1 follows from Theorem 2. □

The following corollary has been proved by Teicher [9].

COROLLARY 2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $E|X|^p < \infty$ for some $1 \leq p < 2$. If $\{a_n\}$ and $\{v_n\}$ are positive constants satisfying $0 < v_n \uparrow$, $\sum_{i=1}^n a_i^p \to \infty$ and*

$$\frac{a_n}{(\sum_{i=1}^n a_i^p)^{1/p}} = O(\frac{v_n}{n^{1/p}}), \tag{10}$$

then

$$\frac{\sum_{i=1}^n a_i(X_i - EX_i)}{v_n(\sum_{i=1}^n a_i^p)^{1/p}} \to 0 \text{ a.s.}$$

Proof. Let $b_n = v_n(\sum_{i=1}^n a_i^p)^{1/p}$ for $n \geq 1$. Under condition (10), (2) holds since

$$\frac{a_n}{b_n} = \frac{a_n}{v_n(\sum_{i=1}^n a_i^p)^{1/p}} = O(\frac{1}{n^{1/p}}).$$

Also, condition (3) holds when $p = 1$ since

$$\sum_{i=1}^n |a_i| = \frac{b_n}{v_n} \leq \frac{b_n}{v_1} = O(b_n).$$

Thus, the conclusion follows from Theorem 2. □

The next example shows that Theorem 2(ii) can fail if condition (3) is not assumed.

EXAMPLE. Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with probability density function

$$f(x) = \frac{c}{x^2(\log x)^2} I_{[2,\infty)}(x), -\infty < x < \infty,$$

where c is taken such that

$$\int_2^\infty \frac{c}{x^2(\log x)^2}\, dx = 1.$$

Then $E[X] = c/\log 2$.

Let $a_n = 1/n, 0 < b_n \uparrow \infty$ and $b_n = o(\log(\log n))$ for all $n \geq 1$(for example, $b_n = \log(\log(\log n)))$. Let Y_n be as in the proof of Theorem 2, i.e., $Y_n = X_n I(|X_n| \leq n)$ for $n \geq 1$. Since $0 < b_n \uparrow \infty$, we have

$$\frac{a_n}{b_n} = \frac{1}{nb_n} = O(\frac{1}{n}),$$

hence (2) holds with $p = 1$ and it follows from the proof of Theorem 2 of Adler and Rosalsky [1] that

$$\frac{\sum_{i=1}^n a_i(X_i - EY_i)}{b_n} \to 0 \text{ a.s.} \tag{11}$$

But,

$$\sum_{i=1}^n |a_i| = \sum_{i=1}^n \frac{1}{i} \sim \log n,$$

which shows that condition (3) fails. Note that

$$\begin{aligned}
&\frac{1}{b_n}\sum_{i=1}^n a_i E(X_i - Y_i) \\
&= \frac{1}{b_n}\sum_{i=1}^n a_i EXI(|X| > i) \sim \frac{1}{b_n}\sum_{i=1}^n \frac{1}{i}\int_i^\infty xf(x)\, dx \\
&\quad \sim \frac{1}{b_n}\sum_{i=2}^n \frac{1}{i}\int_i^\infty \frac{c}{x(\log x)^2}\, dx \\
&= \frac{c}{b_n}\sum_{i=2}^n \frac{1}{i\log i} \sim \frac{c}{b_n}\int_{i=2}^n \frac{1}{x\log x}\, dx \sim \frac{c\log(\log n)}{b_n}.
\end{aligned}$$

Thus, noting that $b_n = o(\log(\log n))$, it follows that

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i E(X_i - Y_i) \to \infty \tag{12}$$

Combining (11) and (12), we have

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i (X_i - EX_i) \to -\infty \text{ a.s.}$$

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