ASYMPTOTIC SEQUENCES AND GENERALIZED FRACTIONS

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1. Introduction

Throughout this note, $R$ is a commutative Noetherian ring (with non-zero identity). For an ideal $a$ of $R$, $\bar{a}$ is the integral closure of $a$, so

$$\bar{a} = \{x \in R : x^n + a_1 x^{n-1} + \cdots + a_n = 0, \text{ for } a_i \in a^i\}.$$ 

In [5], Ratliff introduced an asymptotic sequence and a criterion of a locally quasi-unmixed ring. Recall that a locally quasi-unmixed $R$ is for each maximal ideal $m$ the completion $R_m^*$ is equidimensional and if $R$ is local then we call it quasi-unmixed. The elements $a_1, \ldots, a_n$ of $R$ is said to be a poor asymptotic sequence if, for $i = 1, \ldots, n$, $a_i \notin \bigcup_{p \in Q} p$ where $Q = \text{Ass} \left( R/(a_1, \ldots, a_{i-1})^tR \right)$ for all large $t$; it is said to be an asymptotic sequence if, in addition, $(a_1, \ldots, a_n)R \neq R$.

If $a$ is an ideal of $R$, then the asymptotic grade of $a$, denoted $a$-grade$(a)$, is the common length of all maximal asymptotic sequences in $a$ and we interpret $a$-grade$(R) = \infty$.

In [10], Yass showed that the set whose members are poor asymptotic sequences is a triangular subset (see [7]). This triangular subset provides a module of generalized fractions by [7].

In [1], we gave associated prime ideals of modules of generalized fractions defined by some special triangular subsets.

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The purpose of this note is to study associated prime ideals of the module of generalized fractions defined by poor asymptotic sequences and to give an extended criterion of a locally quasi-unmixed ring using this module.

2. Main results

Lemma 1. [10, 2.2.17]. Let \((U_a)_n = \{(a_1, \ldots, a_n) \in R^n : a_1, \ldots, a_n\}

\text{is a poor asymptotic sequence in } R \text{ such that if } a_i = 1 \text{ for some } i = 1, \ldots, n - 1 \text{ then } a_j = 1 \text{ for all } j \geq i\} \). Then \((U_a)_n\) is a triangular subset of \(R^n\).

Proof. We show that \((U_a)_n\) is satisfied the three conditions of triangular subset [7].

Clearly \((U_a)_n \neq \emptyset\), since \((1, \ldots, 1) \in (U_a)_n\).

Next, let \((a_1, \ldots, a_n) \in (U_a)_n\). Then we may assume that \(a\)-grade

\((a_1, \ldots, a_n) R = n\). Hence by 3.16 of [5], \((a_1^{a_1}, \ldots, a_n^{a_n}) \in (U_a)_n\) for all choice \(a_i \in N\).

Finally, let \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in (U_a)_n\). Also we may assume that \(a\)-grade\((a_1, \ldots, a_n) R = n\) and \(a\)-grade\((b_1, \ldots, b_n) R = n\). Hence by 3.10 of [5], there is \((c_1, \ldots, c_n) \in (U_a)_n\) such that for each \(i = 1, \ldots, n,\)

\[(c_1, \ldots, c_i) R \subset (a_1, \ldots, a_i) R \cap (b_1, \ldots, b_i) R.\]

\(\square\)

Theorem 2. Fix a non-negative integer \(n\) such that \(n \leq \sup_{m \in \text{Max}(R)} a\)-grade\((m)\). Put

\[(U_a)_{n+1} = \{(a_1, \ldots, a_{n+1}) \in R^{n+1} : a_1, \ldots, a_{n+1}\}

\text{is a poor asymptotic sequence in } R

\text{such that for some } i(1 \leq i < n)

\text{if } a_i = 1 \text{ then } a_j = 1 \text{ for all } j \geq i\}

\text{and}

\[(U_a)_n = \{(a_1, \ldots, a_n) \in R^n : \text{there is } a_{n+1} \in R

\text{such that } (a_1, \ldots, a_{n+1}) \in (U_a)_{n+1}\}.\]
Consider the following conditions.

\[ A_n = \{ p \in \text{Spec}(R) : \text{a-grade}(p) = \text{ht } p = n \}. \]

\[ A'_n = \{ p \in \text{Spec}(R) : R_p \text{ is quasi-unmixed such that } \text{a-grade}(p) = \text{a-grade}(pR_p) = n \}. \]

\[ B_n = \{ p \in \text{Spec}(R) : a\text{-grade}(p) = a\text{-grade}(pR_p) = n \}. \]

\[ B'_n = \{ p \in \text{Spec}(R) : \text{for some } (a_1, \ldots, a_n) \in (U_a)_n \]

\[ p \in \text{Ass} \left( R/(a_1, \ldots, a_n)^t R \right) \text{ for all large } t \}. \]

Then we have \( A_n = A'_n, B_n = B'_n \) and

\[ A_n \subset \text{Ass}(U_a)_{n+1}^{-n-1} R \subset B_n. \]

**Proof.** The first and the second assertions follow immediately from 4.5 and 3.8 of [5].

For the third assertion, we show that the first inclusion holds. Let \( p \in A_n \) hence \( \text{ht } p = n \). Let \( \Phi : R \rightarrow R_p \) be the natural map. Then

\[ \Phi(U_a)_{n+1} = \{ (\Phi(a_1), \ldots, \Phi(a_{n+1})) \in R^{n+1}_p : (a_1, \ldots, a_{n+1}) \in (U_a)_{n+1} \} \]

is a triangular subset of \( R^{n+1}_p \) and \( ((U_a)_{n+1}^{-n-1} R)_p = \Phi(U_a)_{n+1}^{-n-1} R_p \) by 2.1 of [3]. Since \( \text{Supp}((U_a)_{n+1}^{-n-1} R) \subset \{ q \in \text{Spec}(R) : \text{ht } q \geq n \} \) by 3.1 of [3], it is enough to show that \( \Phi(U_a)_{n+1}^{-n-1} R_p \neq 0 \). Since \( a\text{-grade}(p) = a\text{-grade}(pR_p) = \dim R_p = n \) and \( R_p \) is quasi-unmixed, we may assume that

\[ \Phi(U_a)_{n+1} \]

\[ = \{ (\Phi(a_1), \ldots, \Phi(a_{n+1})) \in R^{n+1}_p : \text{there exists } j \text{ with } 0 \leq j \leq n \]

\[ \text{such that } \Phi(a_1), \ldots, \Phi(a_j) \text{ form, a subsystem of parameters for } R_p \]

\[ \text{and } \Phi(a_{j+1}) = \cdots = \Phi(a_{n+1}) = 1 \}; \]

for, \( \text{ht}(a_1, \ldots, a_{n+1}) R \geq n + 1 \) and if \( \Phi(a_i) = 1 \) for some \( i < n + 1 \),

then by 3.3 of [7] we have \( \left( \Phi(a_1), \ldots, \Phi(a_{n+1}) \right) = 0. \)
Therefore by 3.5 of [8] we have
\[ \Phi(U_n)_{n+1}^{-n-1}R_p \cong H^n_{pR_p} (R_p) \neq 0. \]

Next for the second inclusion, let \( p \in \text{Ass}((U_n)_{n+1}^{-n-1}R) \). Then by 5.1 of [9] there is \( \frac{r}{(a_1, \ldots, a_n, 1)} \in (U_n)_{n+1}^{-n-1}R \) such that
\[ \left( 0 : \frac{r}{(a_1, \ldots, a_n, 1)} \right) = p. \]

Since \( (a_1, \ldots, a_n)R \subset p \) by 3.3 of [7], we have \( a-grade(p) \geq n \). Suppose that \( a-grade(p) > n \). Let \( Q = \text{Ass} \left(R/(a_1, \ldots, a_n)^tR\right) \) for all large \( t \). Then for all \( q \in Q \), \( a-grade(q) = n \) by 3.7 of [5]. Hence there is \( a_{n+1} \in p \setminus \bigcup_{q \in Q} q \) such that
\[ (a_1, \ldots, a_{n+1}) \in (U_n)_{n+1}. \]

Therefore we have
\[ \left( 0 : \frac{r}{(a_1, \ldots, a_n, 1)} \right) = \left( 0 : \frac{r}{(a_1, \ldots, a_{n+1})} \right) = p \]
by 5.1 of [9] again. Hence we have the following contradiction.
\[ \frac{a_{n+1} r}{(a_1, \ldots, a_{n+1})} = \frac{r}{(a_1, \ldots, a_n, 1)} := 0. \]

On the other hand, by Corollary (p. 38) of [4]
\[ p \in \text{Ass}((U_n)_{n+1}^{-n-1}R) \iff pR_p \in \text{Ass}(\Phi(U_n)_{n+1}^{-n-1}R_p). \]

Note that, for all \( (a_1, \ldots, a_{n+1}) \in (U_n)_{n+1} \), \( a-grade((\Phi(a_1), \ldots, \Phi(a_i)) R) \geq i \) for \( i = 1, \ldots, n \) by 2.9 of [5]. Hence replacing \( R \) with \( R_p \), we have \( a-grade(pR_p) = n \), using the same argument as above. \( \Box \)
COROLLARY 3. The following statements are equivalent.

(1) $R$ is locally quasi-unmixed.

(2) $a$-grade$(I) = \text{ht } I$ for all ideals $I$ in $R$.

(3) $a$-grade$(m) = \text{ht } m$ for all maximal ideals $m$ in $R$.

(4) If $I$ is an ideal of the principal class in $R$, then $a$-grade$(I) = \text{ht } I$.

(5) $\text{Ass}((U_a)_{n+1}^{-n-1}R) = \{p \in \text{Spec}(R) : \text{ht } p = n\}$ for all $n = 0, 1, 2, \ldots$.

(6) The following complex defined in [6, p. 52]

$$0 \rightarrow R \rightarrow (U_a)_{1}^{-1}R \rightarrow \cdots \rightarrow (U_a)_{i}^{-i}R \rightarrow \cdots$$

is of Cousin type for $R$ with respect to the height filtration $\mathcal{F} = (F_i)_{i \geq 0}$ where $F_i = \{p \in \text{Spec}(R) : \text{ht } p \geq i\}$ (see [6, 3.1]).

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$ See 4.1 of [5].

$(2) \Rightarrow (5)$ By the hypothesis we have $a$-grade$(p) = \text{ht } p$ for all $p \in \text{Spec}(R)$. Therefore Theorem 2 completes the proof.

$(5) \Rightarrow (3)$ Suppose that $\text{ht } m = n$ for some $m \in \text{Max}(R)$. Then $m \in \text{Ass}((U_a)_{n+1}^{-n-1}R)$ by the hypothesis. Hence by Theorem 2 we have $a$-grade$(m) = n$.

$(1) \Rightarrow (6)$ Using $(5)$ and the following Corollary 4, the assertion follows from 3.2 of [2].

$(6) \Rightarrow (5)$ This follows immediately from 3.4 of [2]. \qed

COROLLARY 4. If $R$ is locally quasi-unmixed, then there is an isomorphic $R$-homomorphism

$$\Theta : (U_a)_{n+1}^{-n-1}R \rightarrow \bigoplus_{\text{ht } p = n} ((U_a)_{n+1}^{-n-1}R)_p \quad \text{for all } n = 0, 1, 2, \ldots$$

where, for all $x \in (U_a)_{n+1}^{-n-1}R$ and $p$ of height $n$, the component of $\Theta(x)$ in the summand $((U_a)_{n+1}^{-n-1}R)_p$ is $x/p$.

Proof. By Corollary 3(5) and 3.2 of [1] it is sufficient to show that, for some $(a_1, \ldots, a_n) \in (U_a)_n$ and an ideal $a$ of $R$ containing $(a_1, \ldots, a_n)$ $R$ and not contained in any $p \in \text{Spec}(R)$ such that $\text{ht } p = n$, there is $a_{n+1} \in a$ such that $(a_1, \ldots, a_{n+1}) \in (U_a)_{n+1}$. But this is clear by the definition of a poor asymptotic sequence. \qed
Remark 5. If $\text{Ass}((U_a)_{n+1}^{-n-1} R) = B_n$ for all $n = 0, 1, 2, \ldots$, then the converse of Corollary 4 holds.

Proof. We show that Corollary 3(3) is satisfied. Assume that $a$-grade $(m) = n$ for some $m \in \text{Max}(R)$. Then $a$-grade$(mR_m) = n$ by 3.5 of [5]. Hence we have $m \in \text{Ass}((U_a)_{n+1}^{-n-1} R)$ by the hypothesis.

On the other hand, by the assumption we have

$$\text{Ass}((U_a)_{n+1}^{-n-1} R) = \text{Ass}(\bigoplus_{htp = n} ((U_a)_{n+1}^{-n-1} R)_p) \subset \{p \in \text{Spec}(R) : ht \ p = n\}$$

by 3.1 of [3]. Hence we have $ht \ m = n$. \hfill \Box

Example 6. Let $R = k[X, Y, Z]/(X) \cap (Y, Z) = k[x, y, z]$ and $m = (x, y, z)$. Then $R$ is not quasi-unmixed and $a$-grade$(x, y) = a$-grade$(m) = a$-grade$(mR_m) = 1$. But $ht(x, y) = 1$ and $ht(m) = 2$. Hence we have $A_n \neq B_n$ in Theorem 2.

References

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