A GORENSTEIN IDEAL OF CODIMENSION 4

YONG SU SHIN*

1. Introduction

Let $k$ be an infinite field and let $X = \{P_1, \cdots, P_s\}$ be a set of $s$-distinct points in $\mathbb{P}^n$. We denote by $I(X)$ the defining ideal of $X$ in the polynomial ring $R = k[x_0, \cdots, x_n]$ and by $A$ the homogeneous coordinate ring of $X$, $A = \sum_{t=0}^{\infty} A_t$. The Hilbert function of $X$ (or of $A$) is the function $H : \mathbb{N} \longrightarrow \mathbb{N}$ described by

$$H(X,t) = H(A,t) = \dim_k A_t = \dim_k R_t - \dim_k I_t.$$

The first difference of the Hilbert function of $X$ (or of $A$) is

$$\Delta H(X,t) = \begin{cases} 1, & \text{for } t = 0, \\ H(X,t) - H(X,t-1), & \text{for } t \geq 1. \end{cases}$$

We also denote by $\sigma(X)$ (or $\sigma(A)$) the least integer for which

$$H(X,\sigma) = 0 \text{ and } H(X,\sigma - 1) \neq 0.$$

In [GPS], we obtained the number and the degrees of the generators of an ideal of a $k$-configuration in $\mathbb{P}^2$ and the minimal graded free resolution of the ideal.

In [S], we obtained the number and the degrees of the generators of an ideal of a $k$-configuration in $\mathbb{P}^3$. The aim of this paper is to construct a Gorenstein ideal of codimension 4 from them.


1991 AMS Subject Classification: Primary 13D40; Secondary 14M10.

Key words: $k$-configuration, Hilbert function, Gorenstein ideal.

*This author would like to thank the Mathematics and Statistics Department of Queen's University, Kingston, Ontario, for their kind hospitality during the preparation of this work.
2. \textit{k-configurations in} \( \mathbb{P}^3 \)

Roberts and Roitman [6] introduced the following definition:

\textbf{Definition 2.1.} A \textit{k-configuration} is a finite set \( X \) of points in \( \mathbb{P}^2 \) which satisfies the following conditions:

there exist integers \( 1 \leq d_1 < \cdots < d_m \), and subsets \( X_1, \cdots, X_m \) of \( X \), and distinct lines \( \mathbb{L}_1, \cdots, \mathbb{L}_m \subseteq \mathbb{P}^2 \) such that:

1. \( X = \bigcup_{i=1}^m X_i \);
2. \( |X_i| = d_i \) and \( X_i \subset \mathbb{L}_i \) for each \( i = 1, \cdots, m \), and;
3. \( \mathbb{L}_i \) (\( 1 < i \leq m \)) does not contain any points of \( X_j \) for all \( j < i \).

In this case, the \( k \)-configuration in \( \mathbb{P}^2 \) is said to be of type \( (d_1, \cdots, d_m) \).

\textbf{Theorem 2.2} ([3]). Let \( X \) be a \( k \)-configuration in \( \mathbb{P}^2 \) of type \( (d_1, \cdots, d_m) \) and let \( I \) be the ideal of \( X \). Then \( \nu(I) = m + 1 \) and the minimal graded free resolution of \( I \) as an \( R \)-module is:

\[
0 \longrightarrow R(-(d_1 + m)) \oplus \cdots \oplus R(-(d_i + m - i + 1)) \oplus \cdots \oplus R(-(d_m + 1)) \oplus \cdots \oplus R(-(d_i - m - i)) \oplus \cdots \longrightarrow I \longrightarrow 0
\]

where \( \nu(I) \) is the number of the minimal generators of \( I \).

\textbf{Definition 2.3} ([4]). A \textit{k-configuration} in \( \mathbb{P}^3 \) is a finite set of points which satisfies the following conditions:

there exist subsets \( X_1, \cdots, X_u \) of \( X \) and distinct hyperplanes \( \mathcal{H}_1, \cdots, \mathcal{H}_u \) such that:

1. \( X = \bigcup_{i=1}^u X_i \);
2. \( X_i \subset \mathcal{H}_i \) for any \( i = 1, \cdots, u \);
3. \( \mathcal{H}_i \) (\( 1 < i \leq u \)) does not contain any points of \( X_j \) for any \( j < i \), and;
4. \( X_i \) (\( 1 \leq i \leq u \)) is a \( k \)-configuration in \( \mathcal{H}_i \) of type \( (d_{i1}, \cdots, d_{imi}) \) with \( d_{imi} < m_{i+1} \) for every \( 1 \leq i < u \).

In this case, the \( k \)-configuration in \( \mathbb{P}^3 \) is said to be of type

\[
(d_{i1}, \cdots, d_{imi}; \cdots; d_{u1}, \cdots, d_{um}).
\]

For simplicity of notation, let \( (d_{ij}) \) denote the tuple of integers \( (d_{i1}, \cdots, d_{imi}; \cdots; d_{u1}, \cdots, d_{um}) \) with \( d_{imi} < m_{i+1} \) for every \( 1 \leq i < u \).
Remark 2.4. (1) All \( k \)-configurations in \( \mathbb{P}^3 \) of type \((d_{ij})\) have the same Hilbert function, which will be denoted by \( H^{(d_{ij})} \).

(2) Let \( H = \{b_t\}_{t \geq 0} \) be a zero-dimensional \( \mathcal{O} \)-sequence with \( b_1 = 4 \). Applying the procedure of Theorem 4.1 in [GMR], we can get integers 
\((d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})\) with \( d_{im_i} < m_{i+1} \) for every \( 1 \leq i < u \) such that
\[ H = H^{(d_{ij})}. \]

Theorem 2.5. Let \( X \) be a \( k \)-configuration in \( \mathbb{P}^3 \) of type \((d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})\) and let \( I \) be the ideal of \( X \). Then \( \nu(I) = \sum_{i=1}^{u} m_i + u + 1 \) and the degrees of the minimal generators of \( I \) are:

\[
\begin{align*}
&u, \ m_1 + u - 1, \ d_{11} + m_1 + u - 2, \cdots, \ d_{1i} + m_1 + u - i - 1, \cdots, \\
&d_{1m_1} + u - 1, \\
&\vdots \\
&m_j + u - j, \ d_{j1} + m_j + u - j - 1, \cdots, \ d_{ji} + m_j + u - i - j, \cdots, \\
&d_{jm_j} + u - j, \\
&\vdots \\
&m_u, \ d_{u1} + m_u - 1, \cdots, \ d_{ui} + m_u - i, \cdots, \ d_{um_u}.
\end{align*}
\]

3. The construction of a Gorenstein ideal of codimension 4

In this section, we shall construct some Gorenstein ideals of codimension 4 using \( k \)-configurations in \( \mathbb{P}^3 \) and find the degrees of the minimal generators of these ideals.

Definition 3.1 ([3]). A weak \( k \)-configuration in \( \mathbb{P}^2 \) is a finite set \( X \) of points in \( \mathbb{P}^2 \) which satisfies the following conditions:

there exist integers \( 1 \leq d_1 \leq \cdots \leq d_m \), subsets \( X_1, \cdots, X_m \) of \( X \) and distinct lines \( L_1, \cdots, L_m \subseteq \mathbb{P}^2 \) such that:

(1) \( i \leq d_i \) for each \( i = 1, \cdots, m \);
(2) \( X = \bigcup_{i=1}^{m} X_i \);
(3) \( |X_i| = d_i \) and \( X_i \subseteq L_i \) for each \( i = 1, \cdots, m \), and;
(4) \( L_i \) (\( 1 < i \leq m \)) does not contain any points of \( X_j \) for all \( j < i \).
In this case, the weak $k$-configuration in $\mathbb{P}^2$ is said to be of type $(d_1, \cdots, d_m)$.

**Theorem 3.2 ([3])**. Let $X$ be a weak $k$-configuration in $\mathbb{P}^2$ of type $(d_1, \cdots, d_m, \cdots, d_{m+\ell})$ where $d_1 < \cdots < d_m = \cdots = d_{m+\ell}$ and $\ell \geq 1$. Let $I$ be the ideal of $X$. If $X$ is a subset of complete intersection in $\mathbb{P}^2$ of type $(m+\ell, d_m)$, then $\nu(I) = m + 1$ and the minimal free resolution of $I$, as an $R$-module, is:

\[
0 \longrightarrow R(-(d_1 + m + \ell)) \oplus \cdots \oplus R(-(d_i + m + \ell - i + 1)) \oplus \cdots \oplus \\
R(-(d_{m-1} + \ell + 2)) \oplus R(-(d_m + \ell + 1)) \\
\longrightarrow R(-(m + \ell)) \oplus R(-(d_1 + m + \ell - 1)) \oplus \cdots \oplus \\
R(-(d_i + m + \ell - i)) \oplus \cdots \oplus R(-(d_{m-1} + \ell + 1)) \oplus R(-d_m) \\
\longrightarrow I \longrightarrow 0.
\]

**Definition 3.3.** A *weak $k$-configuration* in $\mathbb{P}^3$ is a finite set of points which satisfies the following conditions:

there exist subsets $X_1, \cdots, X_u$ of $X$ and distinct hyperplanes $H_1, \cdots, H_u$ such that:

1. $X = \bigcup_{i=1}^{u} X_i$;
2. $X_i \subset H_i$ for any $i = 1, \cdots, u$;
3. $H_i$ ($1 < i \leq u$) does not contain any points of $X_j$ for any $j < i$, and;
4. $X_i$ ($1 \leq i \leq u$) is a weak $k$-configuration in $H_i$ of type $(d_{i1}, \cdots, d_{imi})$.

In this case, the weak $k$-configuration in $\mathbb{P}^3$ is said to be of type

$$(d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u}).$$

From the Theorem 3.2, we obtain the following theorem.

**Theorem 3.4.** Let $X$ be a weak $k$-configuration in $\mathbb{P}^3$ of type $(d_1, \cdots, d_m, \cdots, d_{m+\ell})$ where $d_1 < \cdots < d_m = \cdots = d_{m+\ell}$ and $\ell \geq 1$. Let $I$ be the ideal of $X$. If $X$ is a subset of complete intersection in $\mathbb{P}^3$ of type $(1, m + \ell, d_m)$, then $\nu(I) = m + 2$ and the degrees of the minimal generators of $I$ are

$1, m + \ell, d_1 + m + \ell - 1, \cdots, d_i + m + \ell - i, \cdots, d_m - 1 + \ell + 1, d_m$. 

DEFINITION 3.5 ([4]). A finite set \( Z \) of points in \( \mathbb{P}^n \) is said to be a basic configuration in \( \mathbb{P}^n \) if there exist integers \( r_1, \cdots, r_n \) and distinct hyperplanes \( L_{ij} (1 \leq i \leq n, 1 \leq j \leq r_i) \) such that

\[
Z = H_1 \cap \cdots \cap H_n \text{ as schemes, where } H_i = L_{i1} \cup \cdots \cup L_{ir_i}.
\]

In this case \( Z \) is said to be of type \((r_1, \cdots, r_n)\).

REMARK 3.6. Let \( Z \) be a basic configuration in \( \mathbb{P}^3 \) of type \((u, \alpha, \beta)\) \((u \leq \alpha < \beta)\). Let \( X = \bigcup_{i=1}^{u} X_i \subset Z \) be a \( k \)-configuration in \( \mathbb{P}^3 \) of type \((d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u})\) where \( X_i \) is a \( k \)-configuration in \( \mathbb{P}^2 \) of type \((d_{i1}, \cdots, d_{im_i})\). Let \( m_u < \alpha \) and \( d_{um_u} < \beta \). Assume \( Z_i \subset Z \) is a basic configuration in \( \mathbb{P}^3 \) of type \((1, \alpha, \beta)\) such that \( X_i \subset Z_i \) and \( Y_i = Z_i - X_i \) is a weak \( k \)-configuration in \( \mathbb{P}^3 \) of type \((\beta - d_{im_i}, \cdots, \beta - d_{i1}, \beta, \cdots, \beta)\) for every \( i = 1, \cdots, u \). Let \( Y = \bigcup_{i=1}^{u} Y_i \). Then \( Y \) is a weak \( k \)-configuration in \( \mathbb{P}^3 \).

Moreover,

\[
\Delta H(Z, t) = \Delta H(X, t) + \Delta H(Y, \sigma - 1 - t),
\]

where \( \sigma = \sigma(X) = u + \alpha + \beta - 2 \).

Similarly,

\[
\Delta H(Z_u, t) = \Delta H(X_u, t) + \Delta H(Y_u, \sigma' - 1 - t)
\]

\[
\Delta H(Z', t - 1) = \Delta H(X', t - 1) + \Delta H(Y', \sigma - 1 - t),
\]

where \( Z' = \bigcup_{i=1}^{u-1} Z_i, X' = \bigcup_{i=1}^{u-1} X_i, Y' = \bigcup_{i=1}^{u-1} Y_i \), and \( \sigma' = \alpha + \beta - 1 \).

Hence

\[
\Delta H(Y, \sigma - 1 - t) = \Delta H(Y_u, \sigma' - 1 - t) + \Delta H(Y', \sigma - 1 - t).
\]

Let \( s = \sigma - 1 - t \). Since \( \sigma' - \sigma = -u + 1 \),

\[
\Delta H(Y, s) = \Delta H(Y_u, s - u + 1) + \Delta H(Y', s).
\]

Hence we obtain the following Lemma.
Lemma 3.7. Let $Y$, $Y_u$, and $Y'$ be as in Remark 3.6. Then
\begin{equation}
\triangle H(Y, s) = \triangle H(Y_u, s - (u - 1)) + \triangle H(Y', s), \text{ i.e.,}
\end{equation}
\[H(Y, s) = H(Y_u, s - (u - 1)) + H(Y', s)\]
for every $s \geq 0$.

Remark 3.8. Let $Y$ and $Y_i$ be as in Remark 3.6 and let $Y'' = \bigcup_{i=2}^u Y_i$. Then, from (3.1),
\begin{equation}
H(Y, t) = H(Y_1, t) + H(Y'', t - 1).
\end{equation}

Theorem 3.9. Let $Y$ be as in Remark 3.6. Let $J = I(Y)$. Then $\nu(J) = \sum_{i=1}^u m_i + 3$ and the degrees of the minimal generators of $J$ are:
\[
\beta - d_{1m_1} + \alpha - 1, \quad \cdots, \quad \beta - d_{11} + \alpha - m_1,
\beta - d_{2m_2} + \alpha, \quad \cdots, \quad \beta - d_{21} + \alpha - m_2 + 1,
\vdots
u, \quad \alpha, \quad \beta - d_{um_u} + \alpha + u - 2, \quad \cdots, \quad \beta - d_{u1} + \alpha - m_u + u - 1, \quad \beta.
\]

Proof. Let $Y_i$, $Z$, and $Z_i$ be as in Remark 3.6. Set $H_i$ the hyperplane which contains $Z_i$ and $H_i = I(H_i)$. We shall prove the theorem by induction on $u$. If $u = 1$, then we are done by Theorem 3.4.

Now assume $u > 1$. Let $Y''$ be as in Remark 3.8. Then, by the induction hypothesis, there exist $\sum_{i=2}^u m_i + 3$ minimal generators of $I(Y'')$
\[F_{21}', \quad \cdots, \quad F_{2m_2}', \]
\[\vdots, \quad H_2 \cdots H_u, \quad F_{u0}', \quad F_{u1}', \quad \cdots, \quad F_{um_u}', \quad F_{um_u+1}', \]
with degrees
\[
\beta - d_{2m_2} + \alpha - 1, \quad \cdots, \quad \beta - d_{21} + \alpha - m_2,
\vdots
u - 1, \quad \alpha, \quad \beta - d_{um_u} + \alpha + u - 3, \quad \cdots, \quad \beta - d_{u1} + \alpha - m_u + u - 2, \quad \beta,
\]
respectively where $F_{u0}' = g$ and $F_{um_u+1}' = h$. 
Let \( S = R/(H_1) \) and \( J' = \frac{J + (H_1)}{(H_1)} \). Then

\[
\frac{J}{H_1 \cdot [J : H_1]} = \frac{J}{(H_1) \cap J} \simeq \frac{J + (H_1)}{(H_1)} = J' \subset S.
\]

Thus we have an exact sequence of graded modules

\[
0 \rightarrow [J : H_1](-1) \xrightarrow{\times H_1} J \rightarrow \frac{J + (H_1)}{(H_1)} \rightarrow 0.
\]

(3.3)

Since \([J : H_1] = I(\mathbb{Y}'')\), we can rewrite the exact sequence (3.3) as:

\[
0 \rightarrow I(\mathbb{Y}'')(-1) \xrightarrow{\times H_1} J \rightarrow J' \rightarrow 0.
\]

(3.4)

It follows from (3.4) and (3.2) that

\[
\mathbf{H}(S/J', t) = \begin{cases} 
1, & \text{for } t = 0 \\
\mathbf{H}(R/J, t) - \mathbf{H}(\mathbb{Y}'', t - 1), & \text{for } t \geq 1,
\end{cases} = \mathbf{H}(\mathbb{Y}_1, t),
\]

which implies \( J' \) is a saturated ideal, i.e., \( J + (H_1) = I(\mathbb{Y}_1) \).

By Theorem 3.4, there exist \( F_{10}, F_{11}, \ldots, F_{m_1}, F_{1m_1 + 1} \in J \) with degrees

\[
\deg F_{10} = \alpha, \quad \deg F_{11} = \beta - d_{1m_1 + \alpha - 1}, \quad \cdots,
\]

\[
\deg F_{m_1} = \beta - d_{11 + \alpha - m_1}, \quad \deg F_{1m_1 + 1} = \beta
\]

such that \( \overline{F}_{10}, \overline{F}_{11}, \ldots, \overline{F}_{m_1}, \overline{F}_{1m_1 + 1} \) are the minimal generators of \( J' \). Moreover, \( F_{10} = g \) and \( F_{1m_1 + 1} = h \). Let \( \{ F'_{ij} \} \) be the minimal generators of \( I(\mathbb{Y}'') \) and \( \{ F_{ij} \} = \{ F'_{ij}, H_1 \} \cup \{ F_{10}, F_{11}, \ldots, F_{m_1}, F_{1m_1 + 1} \} \).

**Claim:** \( J = \langle \{ F_{ij} \} \rangle \).

**Proof of Claim.** Clearly, \( \langle \{ F_{ij} \} \rangle \subseteq J \). Conversely, for every \( F \in J \), \( \overline{F} \in J' \). Hence

\[
F = F_{10}N_0 + F_{11}N_1 + \cdots + F_{m_1}N_{m_1} + F_{1m_1 + 1}N_{m_1 + 1} + H_1K
\]
for some \( N_0, N_1, \cdots, N_{m_1}, N_{m_1 + 1}, K \in R \). Since \( K \in [J : H_1] = I(\mathcal{X}^m) \),

\[
K = \sum F'_{ij} M_{ij}
\]

for some \( M_{ij} \in R \). Hence

\[
F = F_{10} N_0 + F_{11} N_1 + \cdots + F_{1m_1} N_{m_1} + F_{1m_1 + 1} N_{m_1 + 1} + H_1 K
\]

\[
= F_{10} N_0 + F_{11} N_1 + \cdots + F_{1m_1} N_{m_1} + F_{1m_1 + 1} N_{m_1 + 1} + \sum F'_{ij} M_{ij}
\]

\[
= F_{10} N_0 + F_{11} N_1 + \cdots + F_{1m_1} N_{m_1} + F_{1m_1 + 1} N_{m_1 + 1} + \sum (F'_{ij} H_1) M_{ij}
\]

\[
\in \langle \{F_{ij}\} \rangle.
\]

Since \( F'_{u0} = F_{10} = g, F'_{um_u+1} = F_{1m_1+1} = h, \nu(J) = \sum_{i=1}^u m_i + 3 \)

where the degrees of the minimal generators of \( J \) are

\[
\beta - d_{1m_1} + \alpha - 1, \quad \cdots, \quad \beta - d_{11} + \alpha - m_1,
\]

\[
\beta - d_{2m_2} + \alpha, \quad \cdots, \quad \beta - d_{21} + \alpha - m_2 + 1,
\]

\[\vdots\]

\[u, \quad \alpha, \quad \beta - d_{um_u} + \alpha + u - 2, \quad \cdots, \quad \beta - d_{u1} + \alpha - m_u + u - 1, \quad \beta.\]

Hence we are done.

**Remark 3.10.** Let \( \mathcal{X} = \bigcup_{i=1}^u \mathcal{X}_i \) be a \( k \)-configuration in \( \mathbb{P}^3 \) of type \((d_{11}, \cdots, d_{1m_1}; \cdots; d_{u1}, \cdots, d_{um_u})\) where \( \mathcal{X}_i \) is a \( k \)-configuration in \( \mathbb{P}^3 \) of type \((d_{i1}, \cdots, d_{im_i})\) contained in the hyperplane \( \mathbb{H}_i \). Assume that the hyperplanes \( \mathbb{H}_i \) are parallel to each other. Since \( \mathcal{X}_i \) is a \( k \)-configuration in \( \mathbb{H}_i \) of type \((d_{i1}, \cdots, d_{im_i})\), there exist subsets \( \mathcal{X}_{i1}, \cdots, \mathcal{X}_{im_i} \) and distinct lines \( \mathbb{L}_{i1}, \cdots, \mathbb{L}_{im_i} \) which are contained in \( \mathbb{H}_i \) such that:

1\: \mathcal{X}_i = \bigcup_{k=1}^{m_i} \mathcal{X}_{ik};

2\: |\mathcal{X}_{ik}| = d_{ik} \text{ and } \mathcal{X}_{ik} \subset \mathbb{L}_{ik} \text{ for each } k = 1, \cdots, m_i.

Choose \( \alpha \) and \( \beta \) such that \( m_u < \alpha, d_{um_u} < \beta, \text{ and } \alpha < \beta \). Let \( \mathcal{Y}_i \) be the weak \( k \)-configuration in \( \mathbb{P}^3 \) of type \((\beta - d_{im_i}, \cdots, \beta - d_{i1}, \beta, \cdots, \beta)\) which is obtained by taking the complement of \( \mathcal{X}_i \) in a set \( \mathcal{Z}_i \) where \( \mathcal{Z}_i \) is constructed as follows.

To each line \( \mathbb{L}_{ik} \) of \( \mathbb{H}_i \), add \( \beta - d_{ik} \) distinct new points. Further add \( \alpha - m_i \) new lines each containing \( \beta \) distinct points. (This set will then contain \( \alpha \beta \) distinct points. See Fig. 1)
Let $\mathcal{Y} := \bigcup_{i=1}^{u} \mathcal{Y}_i$ and $J = I(\mathcal{Y})$. Then $\mathcal{Y}$ is a weak $k$-configuration in $\mathbb{P}^3$ of type $(\beta - d_{um}, \ldots, \beta - d_{u1}, \beta, \ldots, \beta, \ldots; \beta - d_{1m}, \ldots, \beta - d_{11}, \beta, \ldots, \beta)$. From the proof of Theorem 3.9, we can see that $\nu(J) \leq \sum_{i=1}^{u} m_i + 2u + 1$. The following example shows that each case of the above inequality can occur.

**Figure 1**

**Example 3.11 (Macaulay [1]).** Consider the following examples.

1. Let $Z$ be a basic configuration in $\mathbb{P}^3$ of type $(2, 3, 5)$ and $\mathcal{Y}_1 \subset Z$ be a weak $k$-configuration in $\mathbb{P}^3$ of type $(3, 4, 5; 4, 5, 5)$. Then the number of minimal generators of the ideal of $\mathcal{Y}_1$ is 6 by Theorem 3.9.

2. Let

$$\mathcal{Y}_2 = \{(1, 2, 1, 1), (2, 4, 1, 1), (3, 6, 1, 1), (0, 1, 1, 1), (1, 3, 1, 1), (2, 5, 1, 1), (3, 7, 1, 1), (0, -1, 1, 1), (1, 1, 1, 1), (2, 3, 1, 1), (3, 5, 1, 1), (0, 1, 0, 1), (0, 2, 0, 1), (0, 3, 0, 1), (1, 2, 0, 1), (1, 3, 0, 1), (2, 0, 0, 1), (2, 1, 0, 1), (2, 2, 0, 1), (2, 3, 0, 1)\}.$$

Then $\mathcal{Y}_2$ is a weak $k$-configuration in $\mathbb{P}^3$ of type $(2, 3, 4; 3, 4, 4)$, and the number of minimal generators of the ideal of $\mathcal{Y}_2$ is 7 from Macaulay [BS].

3. Let

$$\mathcal{Y}'_3 = \{(4, 8, 1, 1), (4, 9, 1, 1), (4, 7, 1, 1), (0, 4, 0, 1), (1, 4, 0, 1), (2, 4, 0, 1)\}.$$
and $\mathcal{Y}_3 = \mathcal{Y}_2 \cup \mathcal{Y}_3$. Then $\mathcal{Y}_3$ is a weak $k$-configuration in $\mathbb{P}^3$ of type $(3, 4, 5; 4, 5, 5)$, and the number of minimal generators of the ideal of $\mathcal{Y}_3$ is 8 from Macaulay [BS].

**Corollary 3.12.** Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, and $J$ be as in Remark 3.6 and let $I = I(\mathcal{X})$. Then $I + J$ is a Gorenstein ideal of codimension 4 and

$$\nu(I + J) = 2 \sum_{i=1}^{u} m_i + u + 1.$$ 

**Proof.** By Remark 1.4 in [PS], $I + J$ is a Gorenstein ideal of codimension 4. Let $H$, be as in the proof of Theorem 3.9 and $H = \prod_{i=1}^{u} H_i$. Let \{\(H, F_{10}, F_{11}, \ldots, F_{1m_1}, \ldots; F_{u0}, F_{u1}, \ldots, F_{um_u}\)\} be the set of the minimal generators of $I$ and let \{\(H, G_{10}, \ldots, G_{1m_1}, G_{1m_1+1}; G_{21}, \ldots, G_{2m_2}, \ldots; G_{u1}, \ldots, G_{um_u}\)\} be the set of the minimal generators of $J$ where $F_{u0}|G_{10}$ and $F_{um_u}|G_{1m_1+1}$. (This is always possible.) So we have that

\[
H, F_{10}, F_{11}, \ldots, F_{1m_1}, \ldots, F_{u0}, F_{u1}, \ldots, F_{um_u},
\]
\[
G_{11}, \ldots, G_{1m_1}, G_{1m_1+1}, G_{21}, \ldots, G_{2m_2}, \ldots, G_{u1}, \ldots, G_{um_u}
\]
certainly generate $I + J$.

We first show that no other $F_{ij}$ can be eliminated from the set. If $F_{ij}\langle H, \ldots, \hat{F}_{ij}, \ldots, F_{um_u}; G_{11}, \ldots, G_{um_u}\rangle$ (where $\hat{\ast}$ means that $\ast$ is omitted), then

\[
F_{ij} = \alpha H + \alpha_{10} F_{10} + \cdots + \alpha_{ij} F_{ij} + \cdots + \alpha_{um_u} F_{um_u} + \beta_{11} G_{11} + \cdots + \beta_{um_u} G_{um_u}
\]

for some $\alpha, \alpha_{10}, \ldots, \hat{\alpha}_{ij}, \ldots, \alpha_{um_u}, \beta_{11}, \ldots, \beta_{um_u} \in R$. Thus

\[
\alpha_{10} F_{10} + \cdots - F_{ij} + \cdots + \alpha_{um_u} F_{um_u} = - (\alpha H + \beta_{11} G_{11} + \cdots + \beta_{um_u} G_{um_u})
\]

in $I \cap J = \langle H, G_{10}, G_{1m_1+1}\rangle$.

Hence there exist $\alpha', \alpha'', \alpha''' \in R$ such that

\[
\alpha_{10} F_{10} + \cdots - F_{ij} + \cdots + \alpha_{um_u} F_{um_u} = - (\alpha' H + \alpha'' G_{10} + \alpha''' G_{1m_1+1}),
\]
\[ F_{ij} = \alpha' H + \alpha_{10} F_{10} + \cdots + \alpha_{ij} F_{ij} + \cdots + \alpha_{um_u} F_{um_u} + \alpha'' G_{10} + \alpha''' G_{1m_1+1} \in \langle H, F_{10}, \cdots, F_{um_u}, \rangle, \]

a contradiction.

Hence \( F_{ij} \notin \langle H, F_{10}, \cdots, F_{um_u}, G_{11}, \cdots, G_{um_u} \rangle \).

We now show that no \( G_{kl} \) can be eliminated from this set. Assume \( G_{kl} \in \langle H, F_{10}, \cdots, F_{um_u}, G_{11}, \cdots, G_{um_u} \rangle \). Then

\[ G_{kl} = \alpha H + \alpha_{10} F_{10} + \cdots + \alpha_{um_u} F_{um_u} + \beta_{11} G_{11} + \cdots + \beta_{kl} G_{kl} + \cdots + \beta_{um_u} G_{um_u} \]

for some \( \alpha, \alpha_{10}, \cdots, \alpha_{um_u}, \beta_{11}, \cdots, \beta_{kl}, \cdots, \beta_{um_u} \in R \). Thus

\[ - (\alpha H + \alpha_{10} F_{10} + \cdots + \alpha_{um_u} F_{um_u}) = \beta_{11} G_{11} + \cdots - G_{kl} + \cdots + \beta_{um_u} G_{um_u} \in I \cap J = \langle H, G_{10}, G_{1m_1+1} \rangle. \]

Hence

\[ \beta_{11} G_{11} + \cdots - G_{kl} + \cdots + \beta_{um_u} G_{um_u} = -(\beta H + \beta' G_{10} + \beta'' G_{1m_1+1}) \]

for some \( \beta, \beta', \beta'' \in R \). It follows that

\[ G_{kl} = \beta H + \beta' G_{10} + \beta_{11} G_{11} + \cdots + \beta_{kl} G_{kl} + \cdots + \beta_{um_u} G_{um_u} + \beta'' G_{1m_1+1} \in \langle H, G_{10}, \cdots, G_{um_u} \rangle. \]

a contradiction. Thus \( G_{kl} \notin \langle H, F_{10}, \cdots, F_{um_u}, G_{11}, \cdots, G_{um_u} \rangle \); we are done.
Remark 3.13. Let \( Z = \bigcup_{i=1}^u Z_i \) be a basic configuration in \( \mathbb{P}^3 \) of type \((u, m_u, d_{um_u}) \) \((u \geq 2)\) where \( Z_i \) is a basic configuration in \( \mathbb{P}^3 \) of type \((1, m_u, d_{um_u}) \) and \( X \subset Z \) be a \( k \)-configuration in \( \mathbb{P}^3 \) of type \((d_{11}, \ldots, d_{1m_1}, \ldots, d_{u1}, \ldots, d_{um_u}) \). Let \( X_i = Z_i \cap X \). Then \( X = \bigcup_{i=1}^u X_i \). Let \( Y_i = Z_i - X_i \). Then \( Y = : Z - X = \bigcup_{i=1}^u Y_i \), and \( Y \) is a weak \( k \)-configuration in \( \mathbb{P}^3 \). Assume that \( Y_i \) is a weak \( k \)-configuration in \( \mathbb{P}^2 \) of type \((d_{um_i} - d_{im_i}, \ldots, d_{um_u} - d_{i1}, d_{um_u}, \ldots, d_{um_u}) \).

The proof of the following theorem is the same as that of Theorem 3.9, so we shall omit it.

Theorem 3.14. Let \( Y \) be as in Remark 3.13. Let \( J = I(Y) \). Then \( \nu(J) = \sum_{i=1}^u m_i + 3 \) and the degrees of the minimal generators of \( J \) are:

\[
\begin{align*}
&u, \quad d_{um_u} - d_{1m_1} + m_u - 1, \quad \cdots, \quad d_{um_u} - d_{11} + m_u - m_1, \\
&\vdots \\
&m_u, \quad d_{um_u} - d_{u1m_u-1} + m_u - u - 3, \quad \cdots, \\
&d_{um_u} - d_{u11} + m_u - m_u-1 + u - 2, \quad d_{um_u}, \\
&m_u + u - 2, \quad d_{um_u} - d_{um_u-1} + m_u + u - 3, \quad \cdots, \quad d_{um_u} - d_{u1} + u - 1.
\end{align*}
\]

We also get the following corollary by the same method as in the proof of Corollary 3.12.

Corollary 3.15. Let \( X \) and \( Y \) be as in Remark 3.13. Let \( I = I(X) \) and \( J = I(Y) \). Then \( I + J \) is a Gorenstein ideal of codimension 4 and

\[
\nu(I + J) = 2 \sum_{i=1}^u m_i + u + 1.
\]

References

1. D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra, Source and object code available for Unix and Macintosh computers. Contact the authors, or download from zariski.harvard.edu via anonymous ftp.


Department of Mathematics  
Sung Shin Women’s University  
Dong Sun Dong 3Ka, Sung Book Ku  
Seoul 136-742, Korea,

**E-mail:** ysshin@cc.sungshin.ac.kr  
ysshin@edgett.mast.queensu.ca