CLIFFORD $L^2$-COHOMOLOGY ON THE COMPLETE KÄHLER MANIFOLDS

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0. Introduction

In the study of a manifold $M$, the exterior algebra $\Lambda^*M$ plays an important role. In fact, the de Rham cohomology theory gives many informations of a manifold. Another important object in the study of a manifold is its Clifford algebra $Cl(M)$, generated by the tangent space. It carries an intrinsic first order elliptic operator $D$, the Dirac operator. There is a canonical vector (but not algebra) bundle isomorphism $\Lambda^*(M) \to Cl(M)$. In $\Lambda^*(M)$, the Dirac operator $D$ is $D \cong d + \delta$, where $d$ is the exterior differential and $\delta$ is the adjoint operator of $d$. Therefore many results of the Clifford theory yield the results of the de Rham theory([8]). Moreover the calculus of the pair $Cl(M)$, $D$ carries over formally to bundles of modules over $Cl(M)$. On Kähler manifolds, we obtain operators $D$ and $\bar{D}$ such that $D^2 = \bar{D}^2 = 0$, $D + \bar{D} = \frac{1}{2}D$ and $\bar{D}$ is the formal adjoint of $D$. Using these operators, M. L. Michelsohn([10]) studied the Clifford and spinor cohomology theory and proved some vanishing theorems on compact Kähler manifold. In this paper, we study the Clifford $L^2$-cohomology theory, the decomposition theorem for the $L^2$-Clifford algebra $L^2(Cl^{p,q}(M))$ and prove some vanishing theorems on complete Kähler manifold.

1. Preliminaries

Let $M$ be a $2n$-dimensional Kähler manifold with almost complex
structure $J$ and with connection $\nabla$. Let $\text{Cl}(M)$ be the Clifford bundle generated by the tangent bundle $TM$. Now we define a derivation $\mathcal{J}_0 : \text{Cl}(M) \to \text{Cl}(M)$ induced by $J$ as follows:

\begin{equation}
\mathcal{J}_0(v_1 \cdots v_k) = \sum_{j=1}^{k} v_1 \cdots Jv_j \cdots v_k
\end{equation}

for $v_1, \ldots, v_k \in TM$, where "\cdot" is the Clifford multiplication. If it is clear from the context which multiplication is meant, we omit the Clifford multiplication "\cdot". To study $\mathcal{J}_0$ effectively we consider the complexification $\mathbb{C} \text{l}(M) = \text{Cl}(M) \otimes_{\mathbb{R}} \mathbb{C}$. This algebra has a natural basis given as follows: Let $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ be an orthonormal basis of $T_xM$. Let $T^{1,0}_x$ (resp. $T^{0,1}_x$) be the $i$ eigenspace (resp. $-i$ eigenspace) of $J$ in $T_xM \otimes \mathbb{C}$. Put

$$
\xi_k = \frac{1}{2} \{e_k - iJe_k\}, \quad \bar{\xi}_k = \frac{1}{2} \{e_k + iJe_k\}.
$$

Then $\xi_1, \cdots, \xi_n$ (resp. $\bar{\xi}_1, \cdots, \bar{\xi}_n$) is the basis of $T_x^{1,0}$ (resp. $T_x^{0,1}$). And $\{\xi_k, \bar{\xi}_k\}$ has the following properties:

\begin{equation}
\xi_k \bar{\xi}_\ell + \bar{\xi}_k \xi_\ell = \xi_k \bar{\xi}_\ell + \bar{\xi}_\ell \xi_k = -\delta_{k\ell}, \quad \xi_k \xi_\ell = -\xi_\ell \xi_k, \quad \bar{\xi}_k \bar{\xi}_\ell = -\bar{\xi}_\ell \bar{\xi}_k.
\end{equation}

Denote $\xi_{K\bar{\xi}_I} = \xi_{k_1} \cdots \xi_{k_r} \bar{\xi}_{\ell_1} \cdots \bar{\xi}_{\ell_s}$, where $K$ and $I$ range over all strictly ascending multiindices from $\{1, \cdots, n\}$. For convenience we set $\mathcal{J} = \frac{1}{i} \mathcal{J}_0$. Then by the derivation property, we have

\begin{equation}
\mathcal{J}(\xi_{K\bar{\xi}_I}) = (|K| - |I|)\xi_{K\bar{\xi}_I},
\end{equation}

where $|K|, |I|$ denote the lengths of $K$ and $I$. This gives a decomposition

$$
\text{Cl}(M) = \bigoplus_{p=0}^{n} \text{Cl}^p(M),
$$

where $\text{Cl}^p(M) = \{\phi \in \text{Cl}(M) \mid \mathcal{J}\phi = p\phi\}$.

We now introduce two intrinsically defined linear maps $\mathcal{L}, \bar{\mathcal{L}} : \text{Cl}(M) \to \text{Cl}(M)$ as follows; For any $\varphi \in \text{Cl}(M)$, set

\begin{equation}
\mathcal{L}(\varphi) = -\sum_{k=1}^{n} \xi_k \varphi \bar{\xi}_k, \quad \bar{\mathcal{L}}(\varphi) = -\sum_{k=1}^{n} \bar{\xi}_k \varphi \xi_k.
\end{equation}
These operators are independent of the Hermitian basis chosen to define them. We consider the operator \( \mathcal{H} = [\mathcal{L}, \mathcal{L}] \). Then they satisfy the following relations:

\[
[\mathcal{L}, \mathcal{L}] = \mathcal{H}, \quad [\mathcal{H}, \mathcal{L}] = 2\mathcal{L}, \quad [\mathcal{H}, \mathcal{L}] = -2\mathcal{L}.
\]

Hence they define a representation of \( \mathfrak{sl}(2, \mathbb{C}) \), the Lie algebra of \( SL(2, \mathbb{C}) \), on \( \text{Cl}(M) \). Since each of the operators \( \mathcal{L}, \mathcal{L} \) and \( \mathcal{H} \) commutes with \( \mathcal{J} \), we can define the subspaces

\[
\text{Cl}^{p,q}(M) = \{ \varphi \in \text{Cl}(M) \mid \mathcal{H}\varphi = q\varphi, \ \mathcal{J} = p\varphi \}
\]

and obtain a decomposition([10])

\[
\text{Cl}(M) = \bigoplus_{p,q} \text{Cl}^{p,q}(M).
\]

**Proposition 1.1([10]).** For each \( \xi \in T^{1,0}(M) \), one has that \( \xi \cdot \text{Cl}^{p,q} \subseteq \text{Cl}^{p+1,q+1} \) and \( \bar{\xi} \cdot \text{Cl}^{p,q} \subseteq \text{Cl}^{p-1,q-1} \). Furthermore, if \( \xi \neq 0 \), the sequences

\[
\ldots \xrightarrow{\lambda_\xi} \text{Cl}^{p-1,q-1} \xrightarrow{\lambda_\xi} \text{Cl}^{p,q} \xrightarrow{\lambda_\xi} \text{Cl}^{p+1,q+1} \xrightarrow{\lambda_\xi} \ldots
\]

\[
\ldots \xleftarrow{\lambda_\xi} \text{Cl}^{p-1,q-1} \xleftarrow{\lambda_\xi} \text{Cl}^{p,q} \xleftarrow{\lambda_\xi} \text{Cl}^{p+1,q+1} \xleftarrow{\lambda_\xi} \ldots,
\]

where \( \lambda_\xi \) denotes left Clifford multiplication by \( \xi \), are exact.

Moreover, these subspaces \( \text{Cl}^{p,q} \) have the following properties: If \( q - s \neq p + r \), then \( \text{Cl}^{p,q} \cdot \text{Cl}^{r,s} = \{0\} \). Otherwise, \( \text{Cl}^{p,q} \cdot \text{Cl}^{r,q-p-r} \subseteq \text{Cl}^{p+r,q-r} \). In particular, \( \text{Cl}^{p,q} \cdot \text{Cl}^{p,q} \subseteq \text{Cl}^{p,q} \), \( \text{Cl}^{k,k} \cdot \text{Cl}^{k,-k} \subseteq \text{Cl}^{k,k} \) and \( \text{Cl}^{0,0} \cdot \text{Cl}^{0,0} \subseteq \text{Cl}^{0,0} \).

**2. Clifford cohomology group**

We recall some facts from [3]: Consider Hilbert spaces \( H_i \) \((0 \leq i \leq N)\), \( H_{N+1} := 0 \) and closed operators \( D_i : H_i \rightarrow H_{i+1} \), with \( D_i^* \) the adjoint operator. Let \( \text{dom}D_i \) be the domain of \( D_i \) and \( \text{ran}D_i \) the range of \( D_i \). We then assume that

\[
\text{ran}D_i \subset \text{dom}D_{i+1} \quad \text{and} \quad D_{i+1} \circ D_i = 0.
\]
Thus we obtain a complex

\[(2.1) \quad 0 \longrightarrow \text{dom} D_0 \xrightarrow{D_0} \text{dom} D_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{N-1}} \text{dom} D_N \longrightarrow 0\]

in the sense of homological algebra with additional functional analytic structure, which is called a Hilbert complex. We will abbreviate the complex (2.1) as \((\text{dom} D, D)\).

**Lemma 2.1([3]).** (The Weak Hodge Decomposition). Let \((\text{dom} D, D)\) be a Hilbert complex. Then for each \(i\), we have an orthogonal decomposition

\[(2.2) \quad H_i = \hat{\mathcal{H}}_i \oplus \overline{\text{im} D_{i-1}} \oplus \overline{\text{im} D_i^*}\]

where \(\hat{\mathcal{H}}_i := \text{Ker} D_i \cap \text{Ker} D_{i-1}^*\).

Put \(\Delta_i := D_i D_i^* + D_i^* D_i\). Then we have

**Lemma 2.2([3]).** \(\hat{\mathcal{H}}_i = \text{Ker} \Delta_i\).

Now, let \(E_i \to M\) \((0 \leq i \leq N)\) be hermitian vector bundles over a Riemannian manifold \(M\) and \(d_i := \Gamma_{\text{cpt}}(E_i) \to \Gamma_{\text{cpt}}(E_{i+1})\) differential operators such that \(d_i \circ d_{i-1} = 0\). Denote the formal adjoint \(d_i^t\) by \(d_i^t\). Then \(d_i\) has a closed extension \(d_{i, \text{max}}\) in the Hilbert space \(H_i := L^2(E_i)\) given by

\[d_{i, \text{max}} := (d_{i, \text{min}})^*,\]

where \(d_{i, \text{min}}\) is the minimal extension or the closure of \(d_i\). Then we have

**Lemma 2.3([3]).** If \((\Gamma_{\text{cpt}}(E_i), d_i)\) is an elliptic complex, then

\[\cdots \longrightarrow \text{dom} d_{i-1, \text{max}} \xrightarrow{d_{i-1, \text{max}}^{-1}} \text{dom} d_{i, \text{max}} \xrightarrow{d_{i, \text{max}}} \text{dom} d_{i+1, \text{max}} \longrightarrow \cdots\]

is a Hilbert complex.

Suppose now that \(M\) is a complete Kähler manifold. We introduce two differential operators \(D, \bar{D} : \Gamma\text{Cl}(M) \to \Gamma\text{Cl}(M)\) by the formulas

\[(2.3) \quad D = \sum_j \xi_j \nabla \tilde{\xi}_j, \quad \bar{D} = \sum_j \tilde{\xi}_j \nabla \xi_j,\]

where \(\nabla\) is the canonical connection. Since \(\nabla\) preserves the subbundles \(\Gamma\text{Cl}^{p,q}(M)\), we have

\[D(\Gamma\text{Cl}^{p,q}) \subset \Gamma\text{Cl}^{p+1,q+1}, \quad \bar{D}(\Gamma\text{Cl}^{p,q}) \subset \Gamma\text{Cl}^{p-1,q-1}\]

for all \(p\) and \(q\). Then we have the following well known fact:
THEOREM 2.4([10]). The operators $D$ and $\bar{D}$ are formal adjoints of one another on $\Gamma_{cpt} \mathcal{C}l(M)$, the set of all sections with the compact support. And they satisfy

$$D^2 = \bar{D}^2 = 0.$$

Furthermore, the complex

$$\ldots \overset{D}{\rightarrow} \Gamma \mathbb{C}^{p-1,q-1} \overset{D}{\rightarrow} \Gamma \mathbb{C}^{p,q} \overset{D}{\rightarrow} \Gamma \mathbb{C}^{p+1,q+1} \overset{D}{\rightarrow} \ldots$$

is elliptic.

Now we set

(2.4) \[ \Delta := D\bar{D} + \bar{D}D. \]

Then $\Delta$ is a formally self-adjoint elliptic operator. To understand $\Delta$ we introduce two “real” operators on $\mathcal{C}l(M)$:

(2.6) \[ D = \sum_j \{ e_j \nabla e_j + (Je_j) \nabla Je_j \}, \quad D^c = \sum_j \{ e_j \nabla Je_j - (Je_j) \nabla e_j \}. \]

The first operator is called the Dirac operator. Then we can easily see that

(2.7) \[ D = \frac{1}{4} (D + iD^c), \quad \bar{D} = \frac{1}{4} (D - iD^c). \]

Since $D^2 = 0$, we have that $D^2 = (D^c)^2$ and $DD^c + D^cD = 0$. It follows that

(2.7) \[ \Delta = \frac{1}{4} D^2. \]

Since $D$ is essentially self-adjoint, we have

(2.8) \[ \text{Ker} D = \text{Ker} D^2 = \text{Ker} \Delta. \]

Now, we consider the usual inner product

(2.9) \[ \langle \varphi_1, \varphi_2 \rangle := \int_M < \varphi_1, \varphi_2 > \]
for any $\varphi_1, \varphi_2 \in \Gamma_{cpt} Cl(M)$. Let $L^2(Cl^{p,q}(M))$ be the completion of $\Gamma_{cpt} Cl^{p,q}$ with respect to $\ll , \gg$. We recall that the operators $\mathcal{D}$ and $\bar{\mathcal{D}}$ are formal adjoint to one another with respect to $\ll , \gg$. Then $\mathcal{D}$ and $\bar{\mathcal{D}}$ have closed extensions in $L^2(Cl^{p,q}(M))$ defined by

$$
(2.10) \quad \mathcal{D}_{\text{max}} := (\bar{\mathcal{D}}_{\text{min}})^*, \quad \bar{\mathcal{D}}_{\text{max}} := (\mathcal{D}_{\text{min}})^*.
$$

where $\bar{\mathcal{D}}_{\text{min}}$ (resp. $\mathcal{D}_{\text{min}}$) is a minimal extension of $\bar{\mathcal{D}}$ (resp. $\mathcal{D}$) and $(\ )^*$ is the adjoint operator of $(\ )$ with respect to $\ll , \gg$. Since $\Delta$ and $\mathcal{D}$ are essentially self-adjoint, we have $\mathcal{D}_{\text{max}} = \mathcal{D}_{\text{min}}$ and $\bar{\mathcal{D}}_{\text{max}} = \bar{\mathcal{D}}_{\text{min}}([3])$. And hence we denote the closed extensions as the same symbols. Consequently, from Lemma 2.3 and Theorem 2.4, we obtain the Hilbert complexes

$$
(2.11) \quad \ldots \xrightarrow{\mathcal{D}} L^2(Cl^{p-1,q-1}(M)) \xrightarrow{\mathcal{D}} L^2(Cl^{p,q}(M)) \xrightarrow{\mathcal{D}} L^2(Cl^{p+1,q+1}(M)) \xrightarrow{\mathcal{D}} \ldots ,
$$

$$
\ldots \xleftarrow{\bar{\mathcal{D}}} L^2(Cl^{p-1,q-1}(M)) \xleftarrow{\bar{\mathcal{D}}} L^2(Cl^{p,q}(M)) \xleftarrow{\bar{\mathcal{D}}} L^2(Cl^{p+1,q+1}(M)) \xleftarrow{\bar{\mathcal{D}}} \ldots .
$$

Now, we put

$$
(2.12) \quad L^2H^{p,q} := \text{Ker}\mathcal{D}/\text{Im}\bar{\mathcal{D}} \cap L^2(Cl^{p,q}(M)),
$$

$$
(2.13) \quad L^2\mathcal{H}^{p,q} := \text{Ker}\mathcal{D} \cap \text{Ker}\bar{\mathcal{D}} \cap L^2(Cl^{p,q}(M)),
$$

$$
(2.14) \quad L^2H^{p,q} := \text{Ker}\Delta \cap L^2(Cl^{p,q}(M)).
$$

Here $L^2H^{p,q}$ and $L^2H^{p,q}$ are called the Clifford $L^2$-cohomology group and $L^2$-harmonic space, respectively. Then we have

**Corollary 2.5.** Let $M$ be a complete Kähler manifold. Then we have

$$
L^2(Cl^{p,q}(M)) = L^2\mathcal{H}^{p,q} \oplus \text{Im}\bar{\mathcal{D}} \oplus \text{Im}\mathcal{D},
$$

and

$$
L^2H^{p,q} \simeq L^2\mathcal{H}^{p,q} \simeq L^2H^{p,q}.
$$

**Proof.** The first follows from Lemma 2.3 and Lemma 2.4. The second is obvious from [3, Lemma 3.2].
Remark ([10]). We study the relationship between Dolbeault cohomology and Clifford cohomology. First, we prepare the some facts: Let $\Lambda^{r,s}(M)$ be the standard Dolbeault decomposition of $\Lambda^*(M) \otimes \mathbb{C}$. Then there are operators

$$\partial : \Gamma \Lambda^{r,s} \longrightarrow \Gamma \Lambda^{r+1,s}, \quad \bar{\partial} : \Gamma \Lambda^{r,s} \longrightarrow \Gamma \Lambda^{r,s+1}$$

given by the formulas;

$$\partial = \sum_j \bar{\xi}_j \wedge \nabla \xi_j, \quad \bar{\partial} = \sum_j \xi_j \wedge \nabla \bar{\xi}_j,$$

where $\nabla$ is the Kähler connection and $\{\xi_j, \bar{\xi}_j\}$ is as before. The formal adjoints of $\partial$ and $\bar{\partial}$ are given respectively by

$$\partial^* = -\sum_j i(\xi_j) \nabla \bar{\xi}_j, \quad \bar{\partial}^* = -\sum_j i(\bar{\xi}_j) \nabla \xi_j,$$

where $i(\cdot)$ denotes the interior product. It is well known that under the isomorphism $\mathbb{C}l(M) \cong \Lambda^*(M) \otimes \mathbb{C}$, we have $\mathcal{D} \cong \bar{\partial} + \partial^*$ and $\bar{\mathcal{D}} \cong \partial + \bar{\partial}^*$. Note that the $(p,q)$-decomposition of $\mathbb{C}l(M)$ constructed above does not directly correspond to the Dolbeault decomposition. In fact,

$$\mathbb{C}l^{p,*}(M) \cong \bigoplus_{s-r=p} \Lambda^{r,s}(M),$$

where $\mathbb{C}l^{p,*}(M) = \bigoplus_q \mathbb{C}l^{p,q}(M)$. Moreover,

$$H^{s-r,n-r-s}(M) \cong H^{r,s}_{Dol}(M),$$

where $H^{r,s}_{Dol}(M) = H \cap \Lambda^{r,s}(M)$, $H$ is the harmonic space. The relations (2.17) and (2.18) hold for the space of $L^2$ sections.

3. Vanishing theorems

In this section, we shall prove some vanishing theorems under various curvature conditions. Let $M$ be a Kähler manifold and consider a hermitian vector bundle $S \rightarrow M$ of left modules over $\mathbb{C}l(M)$ with a hermitian metric $<\cdot,\cdot>$ such that:
(1) Module multiplication by unit tangent vectors is unitary, i.e.,
\begin{equation}
\langle \xi \cdot \phi, \psi \rangle + \langle \phi, \bar{\xi} \cdot \psi \rangle = 0,
\end{equation}
for any $\phi, \psi \in \Gamma(S)$ and $\xi \in \Gamma(TM) \otimes \mathbb{C}$.

(2) With respect to the canonical hermitian connection, covariant differentiation is a derivation of module multiplication. That is, for $\phi \in \Gamma(\mathcal{Cl}(M))$ and $s \in \Gamma(S)$, we have
\begin{equation}
\nabla (\phi \cdot s) = (\nabla \phi) \cdot s + \phi \cdot (\nabla s).
\end{equation}

Now, we recall some basic results from [10]. For each $j$, we set $\omega_j = -\xi_j \bar{\xi}_j$, $\bar{\omega}_j = -\bar{\xi}_j \xi_j$. To each (possibly empty) subset $I = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, n\}$ with complementary subset $\{j_1, \ldots, j_{n-p}\}$ we set $\omega_I = \omega_{i_1} \cdots \omega_{i_p} \bar{\omega}_{j_1} \cdots \bar{\omega}_{j_{n-p}}$ and we denote $|I| = p$. Then we have
\begin{equation}
1 = \prod_{j=1}^{n} (\omega_j + \bar{\omega}_j) = \sum_{r=1}^{n} \pi_r,
\end{equation}
where $\pi_r = \sum_{|I|=r} \omega_I$. Moreover, we have an orthogonal decomposition of the bundle
\begin{equation}
S = \bigoplus_{r=0}^{n} S^r, \quad S^r = \pi_r \cdot S.
\end{equation}

Then the complex
\begin{equation}
0 \to \Gamma_{cpt}(S^0) \xrightarrow{\mathcal{D}} \Gamma_{cpt}(S^1) \xrightarrow{\mathcal{D}} \cdots \xrightarrow{\mathcal{D}} \Gamma_{cpt}(S^n) \to 0
\end{equation}
is elliptic. By Lemma 2.3, its completion becomes a Hilbert complex. Similarly with Corollary 2.5, we have
\begin{equation}
L^2 \mathcal{H}^r(M, S') \cong L^2 \mathcal{H}^r(M, S) \cong L^{2r} \mathcal{H}^r(M, S).
\end{equation}

Now, we define invariant operators on $\Gamma(S)$ by
\begin{equation}
\nabla^* \nabla = -\sum_j \nabla_{\xi_j, \xi_j}, \quad \bar{\nabla}^* \bar{\nabla} = -\sum_j \nabla_{\bar{\xi}_j, \bar{\xi}_j},
\end{equation}
\begin{equation}
\mathcal{R} = \sum_{j, k} \xi_j \bar{\xi}_k R_{\xi_j, \xi_k}, \quad \bar{\mathcal{R}} = \sum_{j, k} \bar{\xi}_j \xi_k R_{\bar{\xi}_j, \bar{\xi}_k},
\end{equation}
where $R_{V, W} = \nabla_{V, W} - \nabla_{W, V}$ is the curvature tensor and where $\nabla_{V, W} = \nabla_V \nabla_W - \nabla_{\nabla_V W}$ is the invariant second covariant derivative. Then we have the following result([10]):
PROPOSITION 3.1. For any two sections \( s_1, s_2 \in \Gamma(S) \), at least one of which has compact support, the following holds:

\[
\int_M \angle \nabla^* \nabla s_1, s_2 \rangle = \int_M \langle \nabla s_1, \nabla s_2 \rangle,
\]

where \( \angle \nabla s_1, \nabla s_2 \rangle = \langle \nabla_{\xi}, s_1, \nabla_{\xi}, s_2 \rangle \). Hence \( \nabla^* \nabla \) is a formally self adjoint, nonnegative operator. Similarly, this holds for \( \bar{\nabla}^* \bar{\nabla} \). Moreover, the zero order operators \( R \) and \( \bar{R} \) are self-adjoint.

Moreover, by the straight calculation, we obtain the Bochner-Weitzenböck type formula([10]);

(3.8) \[ \bar{D}D + \bar{D}D = \nabla^* \nabla + R = \bar{\nabla}^* \bar{\nabla} + \bar{R}. \]

From this formula, we obtain the first important consequence

THEOREM 3.2. For any \( s \in \text{dom} \bar{D}D \cap \text{dom} \bar{D}D \), we have

(3.9) \[ \|D\| s \|^2 + \|\bar{D}s\|^2 = \|
abla s\|^2 + \angle R, s \rangle = \|
abla s\|^2 + \angle \bar{R}, s \rangle, \]

where \( \|
abla s\|^2 = \angle \nabla_{\xi}, s, \nabla_{\bar{\xi}}, s \rangle \) and \( \|
abla s\|^2 = \angle \nabla_{\xi}, s, \nabla_{\bar{\xi}}, s \rangle \).

Proof. First we consider a function \( \omega_\ell \) such that \( 0 \leq \omega_\ell(x) \leq 1 \) for any \( x \in M, \text{supp} \ \omega_\ell \subset B(x_0, 2\ell) \), \( \omega_\ell(x) = 1 \) for any \( x \in B(x_0, \ell) \), \( \lim_{\ell \to \infty} \omega_\ell = 1 \) and \( |d\omega_\ell| \leq C/\ell \) almost everywhere on \( M \), where \( C \) is a positive constant independent of \( \ell \in \mathbb{R}_+ \), \( x_0 \in M \) and \( B(x_0, r) \) is the Riemannian open ball with radius \( r \) and center \( x_0 \).

For any \( s \in L^2(S) \), we calculate \( \angle D\bar{D}s + \bar{D}Ds, \omega_\ell^2 s \rangle \) on \( B(2\ell) \). We choose \( \{\xi, \bar{\xi}\} \) such that \( \langle \nabla \xi \rangle_x = \langle \nabla \bar{\xi} \rangle_x = 0 \). By the definition of \( \bar{D} \) and (2.2), we get

\[ \angle D\bar{D}s, \omega_\ell^2 s \rangle = 2 \langle \omega_\ell Ds, \bar{\xi}(\nabla \xi, \omega_\ell)s \rangle + \|\omega_\ell \bar{D}s\|^2. \]

Using (3.1) and \( \bar{\xi} \xi + \xi \bar{\xi} = -\|\xi\|^2 \), we obtain

\[ \|\xi \cdot s\|^2 + \|\bar{\xi} \cdot s\|^2 = \|\xi\|^2 \|s\|^2. \]

Hence we get \( \|\xi \cdot s\| \leq \|\xi\| \|s\| \) for any \( \xi \in TM \otimes \mathbb{C} \). Therefore, by this inequality and Schwarz inequality, we have

\[ |\langle \bar{D}s, \bar{\xi}(\nabla \xi, \omega_\ell)s \rangle| \leq \|\bar{D}s\| \|\bar{\xi}(\nabla \xi, \omega_\ell)s\| \leq \|\bar{D}s\| \|\nabla \xi, \omega_\ell\| \|s\| \leq \frac{C}{\ell} \|\bar{D}s\| \|s\|. \]
Since \( \|s\| \) and \( \|\mathcal{D}s\| \) are finite, letting \( \ell \to \infty \), we have \( <\mathcal{D}s, \varepsilon_j (\nabla \varepsilon_j \omega_{\ell})s > \to 0 \). This implies that \( <\mathcal{D}\mathcal{D}s, s > = \|\mathcal{D}s\|^2 \). Similarly, we get \( <\mathcal{D}\mathcal{D}s, s > = \|\mathcal{D}s\|^2 \). On the other hand, by Proposition 3.1 and (3.2), we have

\[
<\nabla^* \nabla s, \omega_{\ell}^2 s > = 2 <\omega_{\ell} \nabla s, \nabla \omega_{\ell} \cdot s > + \|\omega_{\ell} \nabla s\|^2.
\]

By similar method, we have \( |<\omega_{\ell} \nabla s, \nabla \omega_{\ell} \cdot s > | \to 0 \) as \( \ell \to \infty \). Hence we have \( <\nabla^* \nabla s, s > = \|\nabla s\|^2 \). Hence we complete the proof of the first equation of (3.9). For the second part, the proof is similar. \( \square \)

From Theorem 3.2, we have

\[
2(\|\mathcal{D}s\|^2 + \|\mathcal{D}_{\bar{s}}s\|^2) = \|\nabla s\|^2 + \|\mathcal{D}_{\bar{s}}s\|^2 + <(\mathcal{R} + \mathcal{\bar{R}})s, s >.
\]

Hence for any \( s \in \text{Ker} \mathcal{D} \cap \text{Ker} \mathcal{D}_{\bar{s}} \), if \( R = \mathcal{R} + \mathcal{\bar{R}} \) is non-negative, then we have \( \|\nabla s\| = \|\mathcal{D}_{\bar{s}}s\| = 0 \). This implies that \( s \) is a parallel section. In addition, if \( R \) is positive at some point, then \( s = 0 \). Hence we have

**Theorem 3.3.** Let \( M \) be a complete Kähler manifold and let \( S \) be any hermitian vector bundle of modules over \( \text{Cl}(M) \). If \( R \) is non-negative and positive at some point of \( M \), then the Clifford \( L^2 \)-cohomology group is trivial. This is,

\[
L^2 \mathcal{H}^r(M, S) = \{0\}, \quad \text{for any } r = 0, 1, \cdots, n.
\]

Moreover, on \( TM \subset \text{Cl}(M) \), we have ([8])

\[
\mathcal{R} + \mathcal{\bar{R}} = \frac{1}{2} \text{Ric}.
\]

Thus, from (3.6) and Theorem 3.3, we have

**Corollary 3.4.** On the complete Kähler manifold, if the Ricci curvature is non-negative and positive at some point, then every \( L^2 \)-harmonic 1-form is necessary zero.

Now, we shall consider some special cases of Theorem 3.3. To begin, we suppose that \( M \) is a Kähler spin manifold, i.e., we assume that
there exists a principal Spin-bundle, \( P_{\text{Spin}}(M) \rightarrow M \), with a \( \text{Spin}_{2n} \)-equivalent map \( \tau : P_{\text{Spin}}(M) \rightarrow P_{\text{SO}}(M) \), to the bundle of real oriented orthonormal frame on \( M \). The bundle of spinors, \( S \), is then defined to be vector bundle associated to the unitary representation \( \Delta \) of \( \text{Spin}_{2n} \) given by the unique irreducible complex representation of \( Cl_{2n} \), i.e., \( S = P_{\text{Spin}} \times_{\Delta} \mathbb{C}^{2^n} \). This bundle is naturally a bundle of modules over \( \mathbb{C}l(M) \) and carries a canonical connection induced from the lift of the riemannian connection on \( P_{\text{SO}}(M) \). Since \( M \) is Kähler, this bundle \( S \) is naturally holomorphic and its connection is hermitian. On this bundle \( S \), the curvature tensor \( R^S \) is given by

\[
R^S_{V,W} = \frac{1}{4} \sum_{\alpha, \beta = 1}^{2n} < R_{V,W} X_\alpha, X_\beta > X_\alpha X_\beta,
\]

where \( X_1, \ldots, X_{2n} \) is any real orthonormal basis of the tangent space. Choosing a basis \( e_1, \ldots, J e_n \), we can write \( R^S \) as

\[
R^S_{V,W} = 2 \sum_{j,k=1}^{n} < R_{V,W} \xi_j, \bar{\xi}_k > \xi_j \bar{\xi}_k + \sum_{j=1}^{n} < R_{V,W} \xi_j, \bar{\xi}_j > .
\]

Hence we have

\[
\mathcal{R}^S = \sum_{j,k=1}^{n} \xi_j \bar{\xi}_k R^S_{\xi_j, \xi_k}
= \sum_{i,j,k=1}^{n} < \mathcal{R}_{\xi_i, \bar{\xi}_i, \xi_j, \xi_k} > \xi_j \bar{\xi}_k
= -\frac{1}{2} \sum_{j,k=1}^{n} \text{Ric}(\xi_j, \xi_k) \xi_j \bar{\xi}_k,
\]

where \( \text{Ric} \) is Ricci tensor on \( M([10]) \). Since \( \text{Ric} \) is hermitian symmetric, we may choose our basis so that \( \text{Ric}(\xi_j, \xi_k) = 1/2 \lambda_j \delta_{jk} \), where \( \lambda_j = \text{Ric}(e_j, e_j) = \text{Ric}(Je_j, Je_j) \), for \( j = 1, \ldots, n \), are the eigenvalues. Then we have

\[
\mathcal{D} \mathcal{D} + \mathcal{D} \mathcal{D} = \nabla^* \nabla + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \omega_j = \bar{\nabla}^* \bar{\nabla} + \frac{1}{4} \sum_{j=1}^{n} \lambda_j \bar{\omega}_j.
\]
We note that $\tilde{\nabla}^* \nabla + \tilde{\nabla}^* \tilde{\nabla} = \frac{1}{2} \tilde{\nabla}^* \tilde{\nabla}$ where

\begin{equation}
\tilde{\nabla}^* \tilde{\nabla} = - \sum_j (\nabla_{e_j, e_j} + \nabla_{J e_j, J e_j})
\end{equation}

is a self-adjoint, elliptic operator whose kernel is the space of parallel sections. We note that the scalar curvature $\kappa$ of $M$ is given by

\begin{equation}
\kappa = \text{trace}_H(\text{Ric}) = 2 \sum_j \lambda_j.
\end{equation}

Hence we get

**Theorem 3.5([10]).** On the spinor bundle $S$, we have

$$4(\mathcal{D} \mathcal{D} + \mathcal{D} \mathcal{D}) = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4} \kappa,$$

where $\kappa$ is the scalar curvature of $M$.

Summing up Theorem 3.3 and Theorem 3.5, we have

**Theorem 3.6.** Let $M$ be a complete Kähler spin manifold. If $\kappa \geq 0$ for all $x \in M$ and $\kappa > 0$ for some point $x_0 \in M$, then there are no non-trivial $L^2$-harmonic spinors.

**References**


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