MULTIPLICITY-FREE ACTIONS OF THE ALTERNATING GROUPS

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Abstract. A transitive permutation representation of a group $G$ is said to be multiplicity-free if all of its irreducible constituents are distinct. The character corresponding to the action is called the permutation character, given by $(1_H)^G$, where $H$ is the stabilizer of a point. Multiplicity-free permutation characters are of interest in the study of centralizer algebras and distance-transitive graphs, and all finite simple groups are known to have such characters. In this article, we extend to the alternating groups the result of J. Saxl who determined the multiplicity-free permutation representations of the symmetric groups. We classify all subgroups $H$ for which $(1_H)^{A_n}$, $n > 18$, is multiplicity-free.

1. Introduction

Let $\pi$ be the permutation character associated with the action of a group $G$. If the action is transitive, then $\pi = (1_H)^G$, where $H$ is the stabilizer in $G$ of a point. $\pi$ is said to be multiplicity-free if all of its irreducible constituents are distinct. Such characters arise in the study of centralizer algebras of transitive permutation representations. The multiplicity-free condition is equivalent to the commutativity of the centralizer algebra. It is also well known that if $G$ is the automorphism group of a distance-transitive graph, then the permutation action of $G$ on the vertices is multiplicity-free.

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J. Saxl started the systematic investigation of the multiplicity-free permutation representations of the symmetric groups [10], and the classification program for other groups is well underway. Recently the multiplicity-free permutation characters of the sporadic simple groups have been determined [5].

Saxl’s result was contained in a conference proceedings article where he gives a sketch of the proof. The purpose of this note is to provide an extension of Saxl’s result to the alternating groups. The proof is patterned after that of Saxl, but we give details when necessary. Moreover, extra arguments are needed to investigate some subgroups in the alternating group case. The case where $H$ is a maximal subgroup of $A_n$ was considered in [8]. Most of this work was started and is included in the author’s dissertation [2].

We will prove the following main theorem.

**Theorem 1.** Let $H$ be a subgroup of $A_n$, $n > 18$, and $\Omega$ a set of points on which $H$ acts. Assume that the permutation character $(1_H)^{A_n}$ is multiplicity-free. Then one of the following holds:

1. $n = 2k$ and $A_k \times A_k \subset H \subseteq (S_k \wr S_2) \cap A_{2k}$,
2. $n = 2k$ and $H \subseteq (S_2 \wr S_k) \cap A_{2k}$ of index at most two,
3. $n = 2k + 1$, $H$ fixes a point of $\Omega$, and is one of the two groups in (1) or (2) on the rest of $\Omega$,
4. $A_k \times A_{n-k} \subseteq H \subseteq (S_k \times S_{n-k}) \cap A_n$ for some integer $k$ with $0 \leq k < n/2$,
5. $F_{10} \times A_{n-5} \subseteq H \subseteq (F_{20} \times S_{n-5}) \cap A_n$ where $F_{10}$ and $F_{20}$ are Frobenius groups of orders 10 and 20, respectively,
6. $\text{PSL}(2, 5) \times A_{n-6} \subseteq H \subseteq (\text{PGL}(2, 5) \times S_{n-6}) \cap A_n$, or
7. $\text{PGL}(2, 8) \times A_{n-9} \subseteq H \subseteq (\text{PGL}(2, 8) \times S_{n-9}) \cap A_n$.

**Remark.** The restriction $n > 18$ is not strictly necessary, but is imposed to allow more generality. For small $n$, individual arguments can be used (see for instance [8]). The rest of the paper is devoted to the proof of Theorem 1.

2. Preliminaries

From this point on, we will assume that the permutation character $(1_H)^{A_n}$ is multiplicity-free, and $H \subseteq A_n$, acting on a set $\Omega$ of $n$
elements, where \( n > 18 \). We fix these assumptions, unless otherwise indicated. For convenience, \( H \) will be called a multiplicity-free subgroup. The following lemmas are key tools in the proof.

**Lemma 2.1.** Let \( \Omega_{\{k\}} \) denote the set of \( k \)-element subsets of \( \Omega \), with \( 0 \leq k \leq n/2 \) and \( n > 4 \). Then

1. \( |\text{orb}(H, \Omega_{\{k\}})| \leq k + 1 \), and
2. \( |\text{orb}(H, \Omega_{\{k\}})| \leq k \), if \( H \) is transitive on \( \Omega \).

**Proof.** These are essentially the observations in \([10, \text{p.}341]\) and Praeger \([8, \text{p.}5]\) adapted to the present case.

For any group \( G \), the multiplicity-free condition on a subgroup \( H \) gives an immediate bound on the order of \( H \); for if \( \pi = (1_H)^G \), then:

\[
\pi(1) = [G : H] \leq \sum_{\chi \in \text{Irr}(G)} \chi(1) := d(G).
\]

If \( G = A_n \), we have the following.

**Lemma 2.2.** Let \( a \) be an integer with \( 1 \leq a \leq n/2 \). Then we have: \( |H| > (n + 2)^{-1} \min a \cdot 2^a a! (n - 2a)! \).

**Proof.** From the character theory of the alternating groups \([7, \text{Sec.} 2.5]\), we have \( d(A_n) = \frac{1}{2} [d(S_n) + \sum \chi(1)] \), where the sum runs over all self-associated characters of \( S_n \). Hence \( [A_n : H] \leq d(A_n) < d(S_n) \). The character degree sum \( d(S_n) \) can then be computed explicitly, as shown by Saxl in \([10]\).

The next bounds are weaker but allows us to avoid the computations required by Lemma 2.2. They will be useful in later arguments.

**Lemma 2.3.** Let \( n > 18 \), and \([x]\) be the integer part of \( x \). Then

1. \( |H| > [n/2]! \), and
2. \( |H| > 2(k)! \), if \( n = 2k \).

**Proof.** We show that \((n + 2)^{-1} 2^a a! (n - 2a)! > [n/2]!\) for all integers \( a \) with \( 1 \leq a \leq n/2 \), and then apply Lemma 2.2. This can be verified in a straightforward (but tedious) manner. The bound in case (2) is done similarly. We omit the computations.
We now carry out the proof of the main theorem, according as $H$ is primitive (Section 3), transitive but not primitive (Section 4), or intransitive (Section 5). Basic facts on finite groups and their representations can be found in [1], [6] and [7].

3. The primitive case

**Lemma 3.1.** (Praeger-Saxl, [9]) If $G$ is a primitive group of degree $n$ not containing $A_n$, then $|G| < 4^n$.

**Lemma 3.2.** (Liebeck-Praeger-Saxl, [8]) Let $H$ be a multiplicity-free maximal subgroup of $S_n$ or $A_n$ which is primitive on $\Omega$. Then $H$ is doubly transitive on $\Omega$ provided $n > 6$.

We can now prove the result for this case.

**Proposition 3.3.** Let $G$ be primitive on $\Omega$, where $G \leq A_n$, $n > 18$. Then $(1_G)^{A_n}$ is not multiplicity-free.

**Proof.** It is enough to prove the statement for the case where $G$ is maximal in $A_n$; for if $(1_G)^{A_n}$ is not multiplicity-free, then no subgroup of $G$ is multiplicity-free either.

Suppose that $G$ is a multiplicity-free maximal subgroup of $A_n$, primitive on $\Omega$. Then from Lemma 2.2 and Lemma 3.1 we have

$$(n + 2)^{-1}\min a 2^a a!(n - 2a)! \leq |G| < 4^n, \ a \in [1, n/2].$$

As in [9], this forces $n \leq 60$. (A computer check actually shows $n \leq 55$.) Hence by Lemma 3.2, we need to check which doubly-transitive subgroups of $A_n$, $n \leq 55$ are multiplicity-free. For each such subgroup $G$, it is shown in [8] that the inequality $[A_n : G] \leq d(A_n)$ is satisfied only if $n < 12$. Hence for $n > 18$, no primitive subgroup is multiplicity-free. This proves the proposition. \hfill \Box

4. $H$ is transitive but not primitive

In this section we assume that $H$ is a multiplicity-free subgroup of $A_n$ which is transitive but not primitive on $\Omega$, where $|\Omega| = n$, and $n > 18$. Then $\Omega$ decomposes as a disjoint union of $o$ non-trivial blocks
of size $a$ each, where $a$ and $b$ are integers with $n = ab$. Hence $H \subseteq (S_a \wr S_b) \cap A_n$, with $n = ab$. ($S_a \wr S_b$ denotes the ordinary wreath product of the symmetric groups.)

Using Lemma 2.1, it is easy to obtain the next result which shows that the values of $a$ and $b$ are restricted if $H$ is multiplicity-free. This was observed in [8], with a proof given in [10].

**Lemma 4.1.** (Liebeck-Praeger-Saxl, [8]) Let $H$ be transitive but not primitive on $\Omega$. Then $n = 2k$, for some integer $k$, and one of the following holds:

1. $H$ has 2 blocks of size $k$, or
2. $H$ has $k$ blocks of size 2.

We will discuss the possibilities given in Lemma 4.1 separately. Before doing so, we need the following concept. A permutation group $G$ on a set $\Omega$ of $n$ elements, is called $k$-homogeneous if $G$ permutes the set $\Omega_{\{k\}}$ of all $k$-element subsets of $\Omega$ transitively.

We list some results on $k$-homogeneous groups which will be used later. These theorems are all collected in [4, XII§ 6, pp. 366-376].

**Lemma 4.2.** (Livingston-Wagner [4]) Let $G$ be $k$-homogeneous of degree $n$. Then (i) if $2k \leq n$, then $G$ is $(k - 1)$-homogeneous; (ii) if $4 \leq 2k \leq n$, then $G$ is $(k - 1)$-transitive; and (iii) if $10 \leq 2k \leq n$, then $G$ is $k$-transitive.

**Lemma 4.3.** (Beaumont and Peterson [4]) Suppose that $G$ is a permutation group of degree $n$ and is $k$-homogeneous for all $k = 1, \ldots, n$. Then one of the following holds: (i) $G = A_n$ or $S_n$, (ii) $n = 5$, $G$ is the Frobenius group of order 20, (iii) $n = 6$, $G = PGL(2, 5)$, or (iv) $n = 9$, $G = PGL(2, 8)$ or $PGL(2, 8)$.

**Lemma 4.4.** (Wielandt [4]) Let $k \geq 2$ and $G$ be a $k$-homogeneous group of degree $n$. Suppose that $k + p^a - 1 \leq n$ for any prime power divisor $p^a$ of $k$. Then $G$ is $(k - 1)$-homogeneous.

**Lemma 4.5.** (Kantor [4]) If $G$ is a $k$-homogeneous group which is not 4-transitive, then $G$ is one of $PGL(2, 8)$, $PGL(2, 8)$, or $PGL(2, 32)$ in its natural representation.

We will also use the facts that the only 4-transitive groups are $S_n$ ($n \geq 4$), $A_n$ ($n \geq 6$), and the Mathieu groups $M_{11}$, $M_{12}$, $M_{23}$, and $M_{24}$. If $k \geq 6$, $S_k$ and $A_k$ are the only $k$-transitive groups.
We can now discuss the first case of Lemma 4.1

**Case (1):** $H$ has 2 blocks of size $k$

We assume in this discussion the following. $H$ is a multiplicity-free subgroup of $A_{2k}$, $2k > 18$. The set $\Omega$ decomposes as $\Omega = \Delta \cup \Gamma$ and $\{\Delta, \Gamma\}$ is a complete set of imprimitivity, with $|\Delta| = |\Gamma| = k$.

The action of $H$ on $\Omega$ induces a permutation group on $\Delta$ (as well as on $\Gamma$). The elements either fix $\Delta$ or send $\Delta$ to $\Gamma$. The set $H_0$ of elements which leave $\Delta$ setwise invariant (and hence $\Gamma$ as well) is a normal subgroup of index two in $H$. The pointwise stabilizer $H_\Delta$ of $\Delta$ is a normal subgroup of $H_0$. The factor group $H_0/H_\Delta$ is the permutation group on $\Delta$ induced from $H$. The groups $H_\Gamma$ and $H_0/H_\Gamma$ are defined similarly.

Since $H$ is transitive, $H_\Delta$ and $H_\Gamma$ are conjugate in $H$, i.e., there is an element which interchanges $\Delta$ and $\Gamma$. Hence $|H_\Delta| = |H_\Gamma|$. Next, since $\Omega$ is the disjoint union of $\Delta$ and $\Gamma$, any element which fixes both $\Delta$ and $\Gamma$ pointwise must fix the whole of $\Omega$ pointwise. Thus $H_\Delta \cap H_\Gamma = \{1\}$. From the correspondence theorem, we have that $H_\Delta H_\Gamma / H_\Delta \leq H_0 / H_\Delta$. Also, note that $H_\Gamma \cong H_\Delta H_\Gamma / H_\Delta$. The same remarks hold if we interchange the roles of $\Delta$ and $\Gamma$.

**Lemma 4.6.** Let $k \geq 10$. Then each of $H_0 / H_\Delta$ and $H_0 / H_\Gamma$ is either $S_k$ or $A_k$.

**Proof.** We divide the analysis into several cases, depending on $k$. The notation $\{i^1, 2^j\}$ denotes a subset of $\Omega_{\{k\}}$ with $i$ elements from $\Delta$ and $j$ elements from $\Gamma$.

Case (i): $L. k \geq 20$. Consider $\Omega_{\{10\}}$. Then $\theta_1 = \{1^{10}\}$, $\theta_2 = \{1^9, 2^2\}$, $\theta_3 = \{1^8, 2^2\}$, $\theta_4 = \{1^7, 2^3\}$, $\theta_5 = \{1^6, 2^4\}$ and $\theta_6 = \{1^5, 2^5\}$ are in distinct orbits of $H$. Consider $\theta_1, \theta_2, \theta_3, \theta_4,$ and $\theta_5$. One of these sets must be an $H$-orbit, otherwise if all split, together with $\theta_6$, we get at least 11 orbits on $\Omega_{\{10\}}$, which contradicts Lemma 2.1(2). Hence if one of $\theta_1, \theta_2, \theta_3, \theta_4,$ or $\theta_5$ is an orbit, then the permutation group $H_0 / H_\Delta$ on $\Delta$ is either 10, 9, 8, 7, or 6-homogeneous. By Lemma 4.2(iii), $H_0 / H_\Delta$ is 6-transitive, and must be either $S_k$ or $A_k$.

Case (ii): $13 \leq k < 20$: Consider $\Omega_{\{6\}}$. Then $\theta_1 = \{1^6\}$, $\theta_2 = \{1^5, 2^2\}$, $\theta_3 = \{1^4, 2^2\}$, and $\theta_4 = \{1^4, 2^3\}$ are in different orbits. Consider $\theta_1, \theta_2$ and $\theta_3$. As in the previous case, one of these sets must
remain an orbit (otherwise, including \( \theta_4 \), we get at least 7 orbits, which is a contradiction). Thus \( H_0/H_\Delta \) is either 6, 5, or 4-homogeneous. In any case, it is at least 4-homogeneous. A 4-homogeneous group which is not 4-transitive must have degree 9 or 33, by Lemma 4.5 (Kantor’s result). Since \( 13 \leq k < 20 \), we find that \( H_0/H_\Delta \) must be 4-transitive, and hence is either \( S_k \) or \( A_k \).

Case (iii): \( k = 12 \): From the analysis above, we find that \( H_0/H_\Delta \) is 4-homogeneous, and again must be 4-transitive. So \( H_0/H_\Delta \) either contains \( A_{12} \) or \( H_0/H_\Delta = M_{12} \), the Mathieu group on 12 letters. In the former case we are done. In the latter, we get \(|H| \leq 2|M_{12}|^2 = 18,065,520,320\). We show that this gives a contradiction. For if \( H \) were a multiplicity-free subgroup of \( A_{24} \), it should satisfy: \([A_{24}:H] \leq d(A_{24}) = \frac{1}{2}[d(S_{24}) + \sum \chi(1)]\), the sum over all the 11 self-associated irreducible characters of \( S_{24} \). The terms can be computed by hand using the Hook Formula [7, p. 56], and we obtain \([A_{24}:H] \leq 8,836,179,426,416\). Solving for \( H \), we get \( H > 35,108,400,000 \). This rules out the Mathieu group. Note that the bound of Lemma 2.2 does not rule out this possibility.

Case (iv): \( k = 11 \): As in the previous case, we find that \( H_0/H_\Delta \) must be 4-transitive. Hence \( H_0/H_\Delta \) must either contain \( A_{11} \) or \( H_0/H_\Delta = M_{11} \), the Mathieu group on 11 letters. Hence \(|H| \leq 2|M_{11}|^2\). But \( 2|M_{11}|^2 < \frac{1}{24}\min_a 2^a a!(22 - 2a)! \cdot a \in [1,11] \), contradicting Lemma 2.2. (The minimum is attained at \( a = 9 \).)

Case (v): \( k = 10 \): Considering \( \Omega_4(6) \), we find again that \( H_0/H_\Delta \) is 4-homogeneous. Using Kantor’s result, the only possibilities for \( H_0/H_\Delta \) are \( S_{10} \) or \( A_{10} \) as asserted.

The same arguments hold for \( H_0/H_\Gamma \). This completes the proof of the lemma. \( \square \)

**Proposition 4.7.** Suppose that \( H \) has two \( \varphi \)-blocks, each of size \( k \), where \( k \geq 10 \). Then: \( A_k \times A_k \subset H \subset (S_k \wr S_2) \cap A_{2k} \).

**Proof.** The inclusion \( H \subset (S_k \wr S_2) \cap A_{2k} \) is trivial. We show the other. From the previous lemma, we know that \( H_0/H_\Delta \) is either \( S_k \) or \( A_k \). Since \( A_k \) is simple (as \( k \geq 10 \)), we have either \( H_\Gamma \supseteq A_k \) or \( H_\Gamma = \{1\} \). In the former case, we get (since \( H_\Delta \supseteq A_k \) also) that \( H \supseteq H_\Delta \times H_\Gamma \supseteq A_k \times A_k \), and we are done. In the second case, we
have that $H_{\Gamma} = H_{\Delta} = \{1\}$. But then $H_0 / H_{\Delta} \cong H_0$, so that $H_0$ is either $S_k$ or $A_k$. Since $[H : H_0] = 2$, we get that $|H| = 2|H_0| \leq 2(k!)$. This contradicts Lemma 2.3(2).

Finally, it can be shown that the character $1_{A_k \times A_k}^{A_{2k}}$ is not multiplicity-free (see [3]), which gives the strict inclusion in the proposition and completes the proof. 

We now consider the second case of Lemma 4.1

**Case (2): $H$ has $k$ blocks of size 2**

We now consider the case where $H$ is a multiplicity-free subgroup of $A_{2k}$ with $k$ blocks of size 2.

Let $\{\Delta_1, \ldots, \Delta_k\}$ be the set of blocks, with $|\Delta_i| = 2$, $1 \leq i \leq k$. Let $K = \{k \in H | (\Delta_i)^k = \Delta_i$, for each $i\}$, the subgroup of $H$ leaving each block invariant. The action of $H$ induces a permutation group $H/K$ which acts on the blocks $\Delta_i$. Let $\Delta_i = \{\alpha_i, \beta_i\}$, $1 \leq i \leq k$, and let $K_0 = \langle (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k) \rangle$ be the group generated by the $k$ involutions $(\alpha_i, \beta_i)$. $K_0$ consists of elements which fix each block setwise. Then $|K_0| = 2^k$, $K_0$ is a direct product of $k$ copies of the cyclic group of order 2, and $K \subseteq K_0$.

**Lemma 4.8.** Let $k \geq 10$. Then $H/K$ is either $A_k$ or $S_k$.

**Proof.** As in the proof of Lemma 4.6, we use Lemma 2.1 with care. Consider the set $\Omega_{\{6\}}$. The sets $\theta_1 = \{1^2, 2^2, 3^2\}$, $\theta_2 = \{1^2, 2^2, 3, 4\}$, $\theta_3 = \{1^2, 2, 3, 4, 5\}$, and $\theta_4 = \{1, 2, 3, 4, 5, 6\}$ (with the obvious notation) are in disjoint orbits of $H$. By Lemma 2.1(2), the number of $H$-orbits on $\Omega_{\{6\}}$ is at most 6. Consider $\theta_2, \theta_3, \theta_4$. Then necessarily, at least one of the sets should remain an orbit. If $\theta_2$ is an orbit, then $H/K$ is 4-homogeneous; an element of $H/K$ which sends $\{1^2, 2^2, 3, 4\}$ to another set of the form $\{h^2, i^2, j, k\}$ in the orbit sends $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ to $\{\Delta_h, \Delta_i, \Delta_j, \Delta_k\}$. Similarly, if $\theta_3$ is an orbit, $H/K$ is 5-homogeneous; if $\theta_4$ is an orbit, $H/K$ is 6-homogeneous.

The group $H/K$ is a permutation group of degree $k$. By Wielandt's theorem (Lemma 4.4), if $H/K$ is 6-homogeneous, it is 5-homogeneous. By Livingston-Wagner (Lemma 4.2(i)), if it is 5-homogeneous, then it is 4-homogeneous. In any case, $H/K$ is 4-homogeneous. If the degree $k$ satisfies $10 \leq k \leq 19$, by Kantor's result, all 4-homogeneous groups
are 4-transitive. Hence \( H/K \) is either \( S_k, A_k, M_{11} \) or \( M_{12} \). We show that it is not \( M_{12} \) or \( M_{11} \).

Suppose \( H/K = M_{12} \). Then \( |H| = |M_{12}| |K| \leq |M_{12}| 2^{12} = 11 \cdot 5 \cdot 3^3 \cdot 2^{18} \), since \( |K| \leq 2^{12} \). But by Lemma 2.3(2) \( |H| > 2(12!) = 11 \cdot 7 \cdot 5^2 \cdot 3^5 \cdot 2^{11} \). A comparison shows that this leads to a contradiction. In the same way we find that \( H/K \) cannot equal \( M_{11} \). Hence \( H/K \supseteq A_k \) for \( 10 \leq k \leq 19 \).

Now let \( k \geq 20 \). Considering the orbits of \( H \) on \( \Omega_{10} \) and using similar arguments as before, we see that \( H/K \) is at least 6-homogeneous. By Livingston-Wagner (Lemma 4.2(iii)), \( H/K \) is 6-transitive, and thus is either symmetric or alternating.

Thus for all cases considered, \( H/K \supseteq A_k \) as claimed. \( \Box \)

**Lemma 4.9.** \( |K| \geq 2^{k-1} \).

**Proof.** View \( K_0 \) as the natural permutation module of the group \( A_k \) over the finite field \( GF(2) \). Since \( K \subseteq K_0 \), \( K \) is a submodule of \( K_0 \). From the preceding lemma, \( H/K \subseteq S_k \). Thus \( |H||K|^{-1} \leq k! \). So, \( |K| \geq |H|/k! \). By Lemma 2.3(2), we know \( |H| > 2(k!) \). Hence, \( |K| > 2(k!)/k! = 2 \). This shows that \( K \) is a non-trivial submodule. Hence, using the structure of the natural module [1, p.50 and 74], we have either \( K = K_0 \) or \( K \) is the core. In any case, \( [K_0 : K] \leq 2 \). Since \( |K_0| = 2^k \), we have \( |K| \geq 2^{k-1} \). \( \Box \)

We now have our result for this case.

**Proposition 4.10.** Suppose that \( H \) has \( k \) blocks, each of size two, where \( k \geq 10 \). Then \( H \) is a subgroup of \( (S_2 \wr S_k) \cap A_{2k} \) of index at most two.

**Proof.** The inclusion is immediate. Since \( |(S_2 \wr S_k) \cap A_{2k}| = 2^{k-1}k! \), we have \( 2^{k-1}k! = [(S_2 \wr S_k) \cap A_{2k} : H][H : K][K : \{1\}] \), from which we get the result, since \( H/K \supseteq A_k \) and \( |K| \geq 2^{k-1} \) from the two previous lemmas. \( \Box \)

### 5. \( H \) is intransitive

In this section we assume that \( H \) is a multiplicity-free subgroup of \( A_n \) acting intransitively on a set \( \Omega \) of \( n \)-elements.
From Lemma 2.1(1) the number of orbits of $H$ on $\Omega_{(1)} \cong \Omega$ is at most two. Since $H$ is intransitive, this number is exactly two, by Lemma 2.1(2). Let $\Gamma$ and $\Delta$ be the two orbits, and take $\Gamma$ to be the smaller orbit with size $|\Gamma| = k \leq n/2$. The action of $H$ induces an action on $\Gamma$. Let $H_\Gamma$ be the pointwise stabilizer of $H$ on $\Gamma$. Then $H_\Gamma = H/H_\Gamma$ is the permutation group on $\Gamma$ induced from $H$. Similarly, $H_\Delta = H/H_\Delta$ is the group induced on $\Delta$. We fix these notations for the rest of the section. The key step in the analysis of this section is the determination of the groups $H_\Gamma, H_\Delta, H_\Gamma, H_\Delta$.

**Lemma 5.1.** For each $t \leq k$, $H^\Gamma$ and $H^\Delta$ are $t$-homogeneous. Furthermore, $H^\Gamma$ is one of the following: $S_k, A_k, PGL(2,5), PGL(2,8)$, or $PGL(2,8)$.

**Proof.** For each integer $t$ with $1 \leq t \leq n/2$, consider the action of $H$ on $\Omega_{(t)}$. We see that the $t + 1$ sets: $\{1^t\}, \{1^t-1, 2\}, \{1^t-2, 2^2\}, \ldots, \{1, 2^t-1\}$, and $\{2^t\}$ are in distinct orbits of $H$ on $\Omega_{(t)}$. (As before $\{1^t, 2^t\}$ means a set with $i$ elements from $\Delta$ and $j$ elements from $\Gamma$.) By Lemma 2.1(1), these are all the orbits of $H$ on $\Omega_{(t)}$. Hence for each $t$, $1 \leq t \leq k$, the groups $H^\Gamma$ and $H^\Delta$ are $t$-homogeneous, since the sets $\{1^t\}$ and $\{2^t\}$ are orbits. This proves the first statement.

Since $H^\Gamma$ is a permutation group of degree $k$ such that $H^\Gamma$ is $t$-homogeneous for all $t, t \leq k$, by Beaumont and Peterson's classification (Lemma 4.3), the possibilities for $H^\Gamma$ are determined. This completes the proof. \[\square\]

**Lemma 5.2.** $H_\Gamma \neq \{1\}$

**Proof.** Suppose that $H_\Gamma = \{1\}$. Then $H^\Gamma = H/H_\Gamma = H/\{1\} \cong H$. Hence $|H| = |H^\Gamma| \leq k! \leq [n/2]!$, since $H^\Gamma \hookrightarrow S_k$. But by Lemma 2.3(1), $|H| > [n/2]!$. This contradiction shows that $H_\Gamma \neq \{1\}$. \[\square\]

We now state and prove our result for the intransitive case.

**Proposition 5.3.** Let $H$ be intransitive on $\Omega$. Then $H$ is one of the groups in cases (3), (4), (5), (6), or (7) of the main theorem.

**Proof.** As before, let $\Gamma$ and $\Delta$ be the two orbits of $H$, with $|\Gamma| = k \leq n/2$. Via the natural embedding, $H = H/(H_\Gamma \cap H_\Delta) \hookrightarrow H/H_\Gamma \times H/H_\Delta = H^\Gamma \times H^\Delta.$
we obtain the inclusion $H \subseteq H^\Gamma \times H^\Delta$. Since $H^\Gamma$ and $H^\Delta$ are permutation groups of degrees $k$ and $n-k$ respectively, we have that $H \subseteq H^\Gamma \times H^\Delta \subseteq S_k \times S_{n-k}$. Since $H \subseteq A_n$, we further have the following inclusions:

$$H \subseteq (H^\Gamma \times H^\Delta)^+ \subseteq (S_k \times H^\Delta)^+ \subseteq (S_k \times S_{n-k})^+,$$

where $G^+$ denotes the subgroup of index two of all even permutations of a group $G$ (i.e. $G^+ = G \cap A_n$).

Define the map $f : (S_k \times S_{n-k})^+ \to S_{n-k}$ which sends $(\sigma, \tau) \mapsto \tau$. Then $f$ is a homomorphism of $(S_k \times S_{n-k})^+$ onto $S_{n-k}$ with kernel $A_k \times \{1\}$. Similarly, the map $g : (S_k \times H^\Delta)^+ \to H^\Delta$ which maps $(\sigma, \tau) \mapsto \tau$ is a surjective homomorphism with kernel $A_k \times \{1\}$ also.

By transitivity of induction, we have

$$(1_H)^{A_n} = \left\{ (1_H)^{(S_k \times H^\Delta)^+} \right\}^{(S_k \times S_{n-k})^+} \subseteq A_n$$

$$= \left\{ 1_{(S_k \times H^\Delta)^+} + \cdots \right\}^{(S_k \times S_{n-k})^+} \subseteq A_n$$

$$= \left\{ 1_{(S_k \times H^\Delta)^+} \right\}^{(S_k \times S_{n-k})^+} \subseteq A_n.$$

Since $(1_H)^{A_n}$ is multiplicity-free by assumption, the permutation character $\{1_{(S_k \times H^\Delta)^+} \}^{(S_k \times S_{n-k})^+}$ must be multiplicity-free also.

Let $T = \{t_i\}$ be a complete set of representatives of the cosets of $H^\Delta$ in $S_{n-k}$. We can identify $T$ with the set of coset representatives of $(S_k \times H^\Delta)^+$ in $(S_k \times S_{n-k})^+$ via $t_i \mapsto (1, t_i)$.

The group $(S_k \times S_{n-k})^+$ acts on the cosets of $(S_k \times H^\Delta)^+$ with trivial action on the first factor, i.e., if $(\rho, \sigma) \in (S_k \times S_{n-k})^+$, then

$$(S_k \times H^\Delta)^+(1, t_i)(\rho, \sigma) = S_k \rho \times H^\Delta t_i \sigma = S_k \times H^\Delta t_i \sigma.$$ 

Hence the $(S_k \times S_{n-k})^+$-module affording the permutation character $\{1_{(S_k \times H^\Delta)^+} \}^{(S_k \times S_{n-k})^+}$ is inflated via the earlier defined epimorphism $f$ from the $S_{n-k}$-module affording the character $(1_{H^\Delta})^{S_{n-k}}$. Consequently, since $\{1_{(S_k \times H^\Delta)^+} \}^{(S_k \times S_{n-k})^+}$ is multiplicity-free, then $(1_{H^\Delta})^{S_{n-k}}$ is also multiplicity-free. Hence $H^\Delta$ is a multiplicity-free subgroup of $S_{n-k}$ and we can apply Saxl’s theorem ([10, p. 340]) to
classify $H^\Delta$. Let $|\Delta| = n - k = m$. Then we have the following possibilities for $H^\Delta$:

(i) $A_{m-t} \times A_t \subseteq H^\Delta \subseteq S_{m-t} \times S_t$, $0 \leq t < m/2$,

(ii) $m = 2t$ and $A_t \times A_t \subseteq H^\Delta \subseteq S_t \wr S_2$,

(iii) $m = 2t$ and $H^\Delta \subseteq S_2 \wr S_t$ of index at most four,

(iv) $m$ is odd and $H^\Delta$ fixes a point of $\Delta$ and is one of the groups in (ii) and (iii) on the rest of $\Delta$,

(v) $A_{m-t} \times G_t \subseteq H^\Delta \subseteq S_{m-t} \times G_t$, where $t$ is 5, 6, or 9, and $G_t$ is $F_{20}$, $PGL(2,5)$, and $PGL(2,8)$ respectively.

Since $H^\Delta$ is transitive on $\Delta$, the intransitive cases (iv) and (v) are immediately excluded. Case (i) is possible only if $t = 0$; in this case, we have $A_{n-k} \subseteq H^\Delta \subseteq S_{n-k}$. We discuss this case and the remaining possibilities, i.e., (ii) and (iii) in the next step of the proof.

Case I: $A_{n-k} \subseteq H^\Delta \subseteq S_{n-k}$:

Recall that $H_\Delta$ and $H_\Gamma$ are normal subgroups of $H$ with $H_\Delta \cap H_\Gamma = \{1\}$. Hence $H_\Delta H_\Gamma \leq H$, and $H_\Delta H_\Gamma / H_\Delta \leq H / H_\Delta = H^\Delta$ by the correspondence theorem. From Lemma 5.2, we know that $H_\Gamma \neq \{1\}$. Hence $H_\Delta H_\Gamma$ strictly contains $H_\Delta$. Since $H^\Delta$ is a subgroup of $S_{n-k}$ containing $A_{n-k}$, $H_\Delta H_\Gamma / H_\Delta$ is either $S_{n-k}$ or $A_{n-k}$ (since $n - k \geq n/2 \geq 10$). In any case, we get $H_\Delta H_\Gamma / H_\Delta \supseteq A_{n-k}$. From the 2nd Isomorphism Theorem, we have $H_\Gamma \cong H_\Gamma / \{1\} \cong H_\Delta H_\Gamma / H_\Delta \supseteq A_{n-k}$. Thus, $H_\Gamma \supseteq A_{n-k}$.

We next determine $H_\Delta$. Since $H / H_\Delta \subseteq S_{n-k}$ and $H_\Delta H_\Gamma / H_\Delta \supseteq A_{n-k}$, we see that $[H / H_\Delta : H_\Delta H_\Gamma / H_\Delta] \leq 2$. By the 3rd Isomorphism Theorem, we have $(H / H_\Delta) / (H_\Delta H_\Gamma / H_\Delta) \cong H / H_\Delta H_\Gamma$. Similarly, $(H / H_\Gamma) / (H_\Delta H_\Gamma / H_\Gamma) \cong H / H_\Delta H_\Gamma$. This shows that $[H / H_\Gamma : H_\Delta H_\Gamma / H_\Gamma] \leq 2$ also. Since $H_\Delta \cong H_\Delta H_\Gamma / H_\Gamma$ and $H_\Gamma = H / H_\Gamma$, we have $[H_\Gamma : H_\Delta] \leq 2$ (abusing notation). Hence to determine $H_\Delta$, we only need to determine the normal subgroups of $H_\Gamma$ of index at most two.

From Lemma 5.1, $H_\Gamma$ can be one of the following groups: $S_k$, $A_k$, $F_{20}$, $PGL(2,5)$, or $PGL(2,8)$.

Suppose that $H_\Gamma = S_k$ or $A_k$. Then $H_\Delta$ is either $S_k$ or $A_k$. In any case, $H_\Delta \supseteq A_k$.

Let $H_\Gamma = F_{20}$. In this case the degree $k = 5$. The Frobenius group of order 5 has a normal subgroup of index two which is the product of
a Frobenius kernel of order 5 and a cyclic group of order 2. We denote this by $F_{10}$. Hence $H_\Delta \supseteq F_{10}$.

Next, let $H^\Gamma = PGL(2, 5)$. Here $k = 6$. $PSL(2, 5)$ is a normal subgroup of index two. Hence $H_\Delta \supseteq PSL(2, 5)$.

Finally, $P\Gamma L(2, 8)$, the projective semi-linear group of degree 9, has no subgroup of index two. Hence $H_\Delta = P\Gamma L(2, 8)$.

Since $H_\Delta$ and $H_\Gamma$ commute ($H_\Delta \cap H_\Gamma = \{1\}$), we have the following inclusions:

$$H_\Delta \times H_\Gamma \subseteq H \subseteq (H^\Gamma \times H^\Delta)_+.$$

Substituting the information on $H^\Gamma$ and $H_\Delta$ we obtained above, and the fact that for this case $H^\Delta \subseteq S_{n-k}$ and $H_\Gamma \supseteq A_{n-k}$, we obtain cases (4), (5), (6), and (7) of the main theorem. Note that the statement of Case (4) includes $k = 0$ since $(1_{A_n})^{S_n}$ is clearly multiplicity-free.

We now consider the remaining two possibilities for $H^\Delta$ (cases (ii) and (iii) in our earlier discussion). We will discuss them in a single case below.

Case II. $H^\Delta \subseteq S_t \wr S_2$ or $S_2 \wr S_t$.

Suppose first that the other orbit $\Gamma$ has more than one element. Then $|\Gamma| > 1$ and so $|\Delta| = n - k > 1$ also, since $k \leq n/2$. By Lemma 2.1(1), the number of orbits of $H$ on $\Omega_{\{2\}}$ is at most three. The sets $\{1^2\}$ and $\{2^2\}$, consisting of pairs of unordered points of $\Delta$ and $\Gamma$, respectively, as well the sets $\{1, 2\}$ consisting of a point each from $\Delta$ and $\Gamma$ are in disjoint orbits. Hence these are exactly the three orbits of $H$ on $\Omega_{\{2\}}$. In particular, since $\{1^2\}$ is an orbit, the group $H^\Delta$ is 2-homogeneous on $\Delta$. It is well known that this implies that $H^\Delta$ is primitive on $\Delta$. However, this contradicts the assumption that $H^\Delta$ is contained in one of the imprimitive groups $S_t \wr S_2$ or $S_2 \wr S_t$. The contradiction arises from our assumption that $\Gamma$ contains more than one element. Hence $|\Gamma| = 1$.

Thus for Case II, the group $H$ is the stabilizer of one point, the degree of $\Omega$ is odd, and on the rest of $\Omega$, $H$ is one of the groups obtained in Section 4 of this paper. This is precisely the statement of Case (3) in our main theorem.

This completes the proof of Proposition 5.3 and gives the result for the intransitive case. \qed
6. Conclusion

We now collect the results of Sections 3, 4, and 5 and give a summary of the proof of our main theorem.

Proof of Theorem 1. Proposition 3.3 shows that no primitive subgroup is multiplicity-free. Propositions 4.7 and 4.10 give the results for the transitive but imprimitive case, i.e. cases (1) and (2) of the main theorem. Finally, the intransitive case is given by cases (3), (4), (5), (6), and (7) which we obtain in Proposition 5.3. □

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