REPRESENTATIONS OF THE BRAID GROUP $B_4$

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Abstract. In this work, the irreducible complex representations of degree 4 of $B_4$, the braid group on 4 strings, are classified. There are 4 families of representations: A two-parameter family of representations for which the image of $P_4$, the pure braid group on 4 strings, is abelian; two families of representations which are the composition of an irreducible representation of $B_3$, the braid group on 3 strings, with a certain special homomorphism $\pi : B_4 \longrightarrow B_3$; a family of representations which are the tensor product of 2 irreducible two-dimensional representations of $B_4$.

1. Introduction

The braid groups were first studied systematically by E. Artin in 1925 [1], and he continued his work in 1947 [2]. Among other results, he gave generators $\sigma_1, \ldots, \sigma_{n-1}$ and defining relations $\sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $1 \leq i \leq n - 2$ for $B_n$, the braid group on $n$ strings, and showed that it has a faithful representation as a group of automorphisms of a free group of rank $n$.

As for matrix representations of the braid groups, the first ones were given by W. Burau in 1936 [3]. For each $n$, the Burau representation is a homomorphism from $B_n$ to $\text{GL}_n(\mathbb{Z}[t^{\pm 1}])$, the group of invertible $n \times n$ matrices over a Laurent polynomial ring. The Burau representation is not irreducible, but it has a composition factor of degree $n - 1$, which is called the reduced Burau representation.

The problem of completely classifying the matrix representations of $B_n$ seems out of reach at the present time. A first step toward classifying the irreducible complex representations of $B_n$ was taken by E. Formanek [7]. Note that if the variable $t$ in the reduced Burau representation is
specialized to a nonzero complex number, then a representation of $B_n$ in $GL_{n-1}(\mathbb{C})$ is obtained. Moreover, except for a finite set of roots of unity, the representation so obtained is irreducible. The main result of Formanek [7] is that, with a few exceptions, any irreducible complex representation of $B_n$ of degree $n - 1$ or less is equivalent to the tensor product of a one-dimensional representation with a composition factor of a specialization of the reduced Burau representation.

Westbury [11] identified the set of equivalence classes of irreducible representations of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, which is the quotient of $B_3$ by the center of $B_3$, as a variety and classified its components by a one-to-one correspondence between them and ordered lists of 5 non-negative integers which satisfy certain conditions. Thus $B_4$ is the first of the braid groups whose representation theory is an open problem. It is not known if the Burau representation of $B_4$ is faithful, or even if $B_4$ has a faithful linear representation of any degree. It is known that the Burau representation of $B_n$ is faithful for $n \leq 3$ [9], and is not faithful for $n \geq 6$ [8], [10]. The group $B_4$ has particular interest because, letting $Z_4$ be its center, $B_4/Z_4$ is isomorphic to a subgroup of index 2 in $\text{Aut}(F_2)$, the automorphism group of a free group of rank 2. It was shown in [6] that $\text{Aut}(F_2)$ has a faithful linear representation if and only if $B_4$ has one.

The main result of this work is a classification, up to equivalence, of the irreducible complex representations of $B_4$ of degree 4. One consequence is that no irreducible four-dimensional complex representation of $B_4$ is faithful. Throughout this article, let $\eta$ denote a representation of $B_4$ of degree 4, and let $\mu$ denote a root of $t^2 - t + 1 = 0$ and $\mathcal{V} = (\eta(\sigma_1)\eta(\sigma_2))^3$.

2. Preliminaries

In this section, the fundamental results are listed.

**THEOREM 2.1.** [1, p. 51]. The braid group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with the relations:

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| \geq 2, \quad 1 \leq i, j \leq n - 1,
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 2.
\]

**EXAMPLE 2.2.**

1. $B_1 = \{1\}$ is the trivial group.
2. $B_2 = \langle \sigma_1 \rangle$ is infinite cyclic.
(3) \( B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 > \)
\[ = \langle \alpha, \beta \mid \alpha^3 = \beta^2 > \text{ where } \alpha = \sigma_1 \sigma_2, \beta = \sigma_1 \sigma_2 \sigma_1. \]
(4) \( B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1 \sigma_3 = \sigma_3 \sigma_1, \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3 > \).

Among the many important results, the following are essential to our analysis.

**Lemma 2.3.** Let \( B_n \) be the braid group on \( n \) strings.

(1) [4, p. 655]. Let \( \theta_n = \sigma_1 \cdots \sigma_{n-1} \). Then \( \theta_n \sigma_i \theta_n^{-1} = \sigma_{i+1} \) for \( i = 1, \ldots, n-2 \). Hence \( \theta_n \) and any \( \sigma_i \) generate \( B_n \).

(2) [4, p. 656]. For \( n \geq 3 \), the center of \( B_n \) is infinite cyclic, with generator \( \theta_n^n \).

There is an exceptional homomorphism of \( B_4 \) onto \( B_3 \) denoted by \( \pi \) that sends both \( \sigma_1 \) and \( \sigma_3 \) to \( \sigma_1 \). The next lemma collects the facts about this homomorphism.

**Lemma 2.4.** [6, p. 406]. Let \( \pi : B_4 \to B_3 \) be the homomorphism defined by \( \pi(\sigma_1) = \sigma_1, \pi(\sigma_2) = \sigma_1, \pi(\sigma_3) = \sigma_1. \)

(1) The kernel of \( \pi \) is \( F_2 = \langle \sigma_1 \sigma_3^{-1}, \sigma_2 \sigma_1 \sigma_3^{-1} \sigma_2^{-1} > \), a free group of rank 2.

(2) Let \( p = \sigma_1 \sigma_3^{-1}, q = \sigma_2 \sigma_1 \sigma_3^{-1} \sigma_2^{-1} \). Then the action of \( B_4 \) on \( \langle p, q > \) by conjugation is given by
\[
\sigma_1 p \sigma_1^{-1} = p, \quad \sigma_2 p \sigma_2^{-1} = q, \quad \sigma_3 p \sigma_3^{-1} = p,
\]
\[
\sigma_1 q \sigma_1^{-1} = q p^{-1}, \quad \sigma_2 q \sigma_2^{-1} = q p^{-1} q, \quad \sigma_3 q \sigma_3^{-1} = p^{-1} q.
\]

All the irreducible representations of \( B_3 \) and \( B_4 \) of degree 1.2.3 are listed now (cf. [7]). For \( y \in \mathbb{C}^* \), we define a one-dimensional representation,
\[
\chi(y) : B_n \to \mathbb{C}^*,
\]
where \( \chi(y)(\sigma_i) = y \) for \( 1 \leq i \leq n - 1 \).

**Theorem 2.5.** [7, Theorem 3]. For \( n \geq 2 \), the representations \( \chi(y) : B_n \to \mathbb{C}^* \) \( y \in \mathbb{C}^* \) are a complete set of one-dimensional representations of \( B_n \).

For the irreducible representations of \( B_4 \) of degree 2, the following theorem is in order.
Theorem 2.6. [7, Theorem 11]. Let $\rho : B_3 \to \text{GL}_2(\mathbb{C})$ be an irreducible representation. Then $\rho$ is equivalent to $\chi(y) \otimes \beta_3(z)$ for some $y, z \in \mathbb{C}^*$ where $z$ is not a root of $f_3(t) = t^2 + t + 1$.

$$\beta_3(z)(\sigma_1) = \begin{pmatrix} -z & 0 \\ -1 & 1 \end{pmatrix}, \quad \beta_3(z)(\sigma_2) = \begin{pmatrix} 1 & -z \\ 0 & -z \end{pmatrix}.$$  
Moreover, $\chi(y) \otimes \beta_3(z)$ is equivalent to the following representation $\rho' = \rho'(a, b)$,

$$\rho'(\sigma_1) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}, \quad \rho'(\sigma_2) = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$$

for $b \neq a\mu$, where $\mu$ is a root of $t^2 - t + 1 = 0$, if we conjugate with $\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix}$, where $-yg^2 = 1$ and $a = y, b = -yz$.

The irreducible representations of $B_4$ of degree 2 have been classified in the following theorem.

Theorem 2.7. [7, Theorem 12]. Let $\rho : B_4 \to \text{GL}_2(\mathbb{C})$ be an irreducible representation. Then $\rho$ is equivalent to one of the following, for some $y, z \in \mathbb{C}^*$.

1. $\chi(y) \otimes \hat{\beta}_4(z)$ where $z$ is either $\pm i$ or $-1$,

   $$\hat{\beta}_4(z)(\sigma_1) = \begin{pmatrix} -z & 0 \\ -1 & 1 \end{pmatrix}, \quad \hat{\beta}_4(z)(\sigma_2) = \begin{pmatrix} 1 & -z \\ 0 & -z \end{pmatrix}, \quad \hat{\beta}_4(z)(\sigma_3) = \begin{pmatrix} 1 & 0 \\ -z^2 & -z \end{pmatrix}.$$  

2. $(\chi(y) \otimes \beta_3(z))\pi$ where $z$ is not a root of $f_3(t) = t^2 + t + 1$ and $\pi : B_4 \to B_3$ is the special homomorphism, i.e.

   $$\beta_3(z)\pi(\sigma_1) = \begin{pmatrix} -z & 0 \\ -1 & 1 \end{pmatrix} = \beta_3(z)\pi(\sigma_3), \quad \beta_3(z)(\sigma_2) = \begin{pmatrix} 1 & -z \\ 0 & -z \end{pmatrix}.$$

Representations in (1) are not equivalent to representations in (2) except $\chi(y) \otimes \hat{\beta}_4(-1) = (\chi(y) \otimes \beta_3(-1))\pi$.

For the representations of $B_3, B_4$ of degree 3, we have the following theorems. Let $\mathbb{C}^3, \hat{\mathbb{C}}^3$ denote the sets of three-dimensional column and row vectors respectively.
Theorem 2.8. [7, Theorems 24, 25]. Let $B_3 = \langle \alpha, \beta \rangle$ where 
$\alpha = \sigma_1 \sigma_2$, $\beta = \sigma_1 \sigma_2 \sigma_1$. Let $y \in \mathbb{C^*}$, let $A = (a_1, a_2, a_3)^t \in \mathbb{C}^3$, and let
$B = (1, 1, 1) \in \mathbb{C}^3$ where $a_1 + a_2 + a_3 = 2$.

(1) $\tau(\alpha) = y^2 \text{diag}(1, \omega, \omega^2)$, $\tau(\beta) = y^3 (I - AB)$ defines a representation $\tau(y, a_1, a_2, a_3)$ of $B_3$.

(2) The representation $\tau(y, a_1, a_2, a_3)$ is irreducible if and only if $a_1 a_2 a_3 \neq 0$.

(3) Every irreducible representation $\tau : B_3 \to \text{GL}_3(\mathbb{C})$ is equivalent to some
$\tau(y, a_1, a_2, a_3)$.

Then $\tau(y, a_1, a_2, a_3)$, $\tau(y\omega, a_2, a_3, a_1)$ and $\tau(y\omega^2, a_3, a_1, a_2)$ are equivalent and these are the only equivalences among $\tau(y, a_1, a_2, a_3)$.

Theorem 2.9. [7, Theorem 13]. Let $\rho : B_4 \to \text{GL}_3(\mathbb{C})$ be an irreducible representation. Then $\rho$ is equivalent to one of the following, for some $y, z \in \mathbb{C^*}$.

(1) $\chi(y) \otimes \beta_4(z)$ where $z$ is not a root of $f_4(t) = t^3 + t^2 + t + 1$,

$\beta_4(z)(\sigma_1) = \begin{pmatrix} -z & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\beta_4(z)(\sigma_2) = \begin{pmatrix} 1 & -z & 0 \\ 0 & -z & 0 \\ 0 & -1 & 1 \end{pmatrix}$,

$\beta_4(z)(\sigma_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -z \\ 0 & 0 & -z \end{pmatrix}$.

(2) $\tau \pi$ where $\tau : B_3 \to \text{GL}_3(\mathbb{C})$ is an irreducible representation and $\pi : B_4 \to B_3$ is the special homomorphism.

(3) $\chi(y) \otimes \mathcal{E}(z)$ where $\mathcal{E}(z) : B_4 \to \text{GL}_3(\mathbb{C})$ is defined by

$\mathcal{E}(z)(\sigma_1) = \begin{pmatrix} 0 & 1 & 0 \\ z & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathcal{E}(z)(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & z & 0 \end{pmatrix}$,

$\mathcal{E}(z)(\sigma_3) = \begin{pmatrix} 0 & -1 & 0 \\ -z & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The eigenvalues of $\chi(y) \otimes \mathcal{E}(z)$ are $y, \pm y \sqrt{z}$, so distinct parameters $y, z$ give inequivalent representations. Representations in (1), (2) and
(3) are mutually inequivalent, except that \( \chi(y) \otimes \beta_1(1) \) is equivalent to \( \chi(y) \otimes \mathcal{E}(1) \).

3. Reduction of the classification of representations.

To classify all the irreducible representation \( \eta \) of \( B_4 \) in \( \text{GL}_4(\mathbb{C}) \), we consider the restriction of \( \eta \) to \( B_3 \) since it must be a representation of \( B_3 \) also. Since \( (\sigma_1\sigma_2)^3 \) is in the center of \( B_3 \), the image \( \mathcal{V} \) of \( (\sigma_1\sigma_2)^3 \) must centralize the image of \( B_3 \). Hence we consider the various possible Jordan canonical forms for \( \mathcal{V} \).

Let \( \lambda \) be a partition of a natural number, denoted \( \{1^{\lambda_1}2^{\lambda_2}\ldots s^{\lambda_s}\} \). Corresponding to \( \lambda \) is the nilpotent Jordan matrix \( J(\lambda) \) which has \( \lambda_i \) elementary \( i \times i \) nilpotent Jordan blocks for \( i = 1, \ldots, s \). For \( a \in \mathbb{C} \), set \( J(a, \lambda) = aI + J(\lambda) \). Now assume the Jordan canonical form of \( \mathcal{V} \) is the direct sum of Jordan blocks \( J(a_1, \lambda(a_1)), \ldots, J(a_k, \lambda(a_k)) \), where \( a_1, \ldots, a_k \) are the distinct eigenvalues. The centralizer of \( \mathcal{V} \) is the direct sum of the centralizers of the distinct \( J(a_i, \lambda(a_i)) \). Thus one needs to find the centralizer of a block \( J(a, \lambda) \), and we notice that this is the same as the centralizer of the nilpotent Jordan block \( J(0, \lambda) \). In other words, the structure of the centralizer does not depend on \( a \).

**Lemma 3.1.** Let \( \lambda = \{1^{\lambda_1}2^{\lambda_2}\ldots s^{\lambda_s}\} \). The centralizer of \( J(0, \lambda) \) has an invariant subspace of dimension \( \lambda_s \) where \( s \) is the largest part of \( \lambda \). In particular, if \( \lambda_s = 1 \), then the centralizer of \( J(0, \lambda) \) has a one-dimensional invariant subspace. Thus the centralizer of a matrix \( \mathcal{V} \) has a one-dimensional invariant subspace unless, for every Jordan block \( J(a, \lambda) \) occurring in the Jordan canonical form of \( \mathcal{V} \), the largest part of the partition \( \lambda \) has multiplicity two or more.

We have 14 Jordan canonical forms of the \( 4 \times 4 \) matrix \( \mathcal{V} \) according to its partition type. Among these, there are only three for which the centralizer of \( \mathcal{V} \) does not have a one-dimensional invariant subspace, namely \( (A) \) \( \mathcal{V} \) has two distinct eigenvalues \( u, w \), each of multiplicity two with corresponding partitions \( (1^2), (1^2) \). The Jordan canonical form of \( \mathcal{V} \) is the diagonal matrix \( \text{diag}(u, u, w, w) \), \( u \neq w \). \( (B) \) \( \mathcal{V} \) has one eigenvalue \( w \) with corresponding partition \( (2^2) \). The Jordan canonical form of \( \mathcal{V} \) has two identical \( 2 \times 2 \) blocks \( (w, 1, 0, w) \). \( (C) \) \( \mathcal{V} \) has one eigenvalue \( w \) with corresponding partition \( (1^4) \), i.e., \( \mathcal{V} = wI \).
Theorem 3.2. Let \( \eta : B_4 \to \text{GL}_4(\mathbb{C}) \) be a representation, not necessarily irreducible. Then either \( \eta|_{B_3} \), the restriction of \( \eta \) to \( B_3 \), has a one-dimensional invariant subspace or \( \mathcal{V} \) has one of the following Jordan canonical forms
\[
\begin{pmatrix}
  u & 0 & 0 & 0 \\
  0 & u & 0 & 0 \\
  0 & 0 & w & 0 \\
  0 & 0 & 0 & w
\end{pmatrix}, \quad
\begin{pmatrix}
  w & 1 & 0 & 0 \\
  0 & w & 0 & 0 \\
  0 & 0 & w & 1 \\
  0 & 0 & 0 & w
\end{pmatrix}, \quad
\text{or}
\begin{pmatrix}
  w & 0 & 0 & 0 \\
  0 & w & 0 & 0 \\
  0 & 0 & w & 0 \\
  0 & 0 & 0 & w
\end{pmatrix},
\]
where \( u, w \) are distinct.

4. The restriction of \( \eta \) to \( B_3 \) has an invariant one-dimensional subspace.

Let \( \eta : B_4 \to \text{GL}_4(\mathbb{C}) \) be an irreducible representation and assume \( \eta|_{B_3} \) has an invariant one-dimensional subspace, \( \text{span} \{v\} \), i.e., \( \eta(\sigma)v = xv = \eta(\sigma_2)v \) for some \( x \in \mathbb{C}^* \). Define \( \theta = \sigma_1 \sigma_2 \sigma_3 \) \( \sigma_0 = \theta \sigma_3 \theta^{-1} \). \( V_i = \ker (\eta(\sigma_i) - x \cdot I) \) for \( 0 \leq i \leq 3 \).

Conjugation by \( \theta \) permutes \( \sigma_0, \sigma_1, \sigma_2, \sigma_3 \) cyclically. Thus left multiplication by \( \eta(\theta) \) permutes \( V_0, V_1, V_2, V_3 \) cyclically. In particular, all the \( V_i, 0 \leq i \leq 3 \), have the same dimension. Moreover, if \( v \) is an \( x \)-eigenvector for \( \eta(\sigma_1) \) and \( \eta(\sigma_2) \), then \( \eta(\theta)v \) is an \( x \)-eigenvector for \( \eta(\sigma_2) \) and \( \eta(\sigma_3) \).

Proposition 4.1. \( \dim V_1 = 2 \).

Proof. By one of the hypotheses, \( \dim V_1 \geq 1 \), since \( v \in V_1 \).

If \( \dim V_1 = 1 \), then \( \dim V_2 = 1 \) since as noted above, \( V_2 = \eta(\theta)(V_1) \). Since both \( v \) and \( \eta(\theta)v \) are in \( V_2 \), and \( \dim V_2 = 1 \), one must be a nonzero scalar multiple of the other, i.e., \( \eta(\theta)v = yv \) for some \( y \in \mathbb{C}^* \). But then \( \text{span} \{v\} \) is invariant under \( \eta(\sigma_1) \) and \( \eta(\theta) \), and hence invariant under \( B_4 \), since \( \sigma_1 \) and \( \theta \) generate \( B_4 \). This contradicts the irreducibility of \( \eta : B_4 \to \text{GL}_4(\mathbb{C}) \).

If \( \dim V_1 \geq 3 \), then \( \dim (V_1 \cap V_2 \cap V_3) \geq 1 \). This follows from two applications of the formula, \( \dim P + \dim Q = \dim (P \cap Q) + \dim (P + Q) \) for subspaces of a vector space, since \( V_1, V_2, V_3 \) are subspaces of \( \mathbb{C}^4 \). But any element of \( V_1 \cap V_2 \cap V_3 \) is a common \( x \)-eigenvector for \( \sigma_1, \sigma_2, \sigma_3 \) and again the irreducibility of \( \eta : B_4 \to \text{GL}_4(\mathbb{C}) \) is contradicted.

Finally, \( \dim V_1 = 2 \) since this is the only remaining possibility. \( \square \)
Now let \( W = \text{span} \{ \eta(\theta^3)v, v, \eta(\theta)v, \eta(\theta^2)v \} \). If \( \dim W \) were less than or equal to 3, then \( W \) would be a proper invariant subspace since \( W \) is invariant under \( \eta(\sigma_1) \) and \( \eta(\theta) \). Hence \( \dim W = 4 \) since \( \eta \) is irreducible. In other words, \( \{ \eta(\theta^3)v, v, \eta(\theta)v, \eta(\theta^2)v \} \) forms a basis for \( W \). Since \( \theta^4 \) is in the center of \( B_4 \) and \( \eta \) is irreducible, \( \eta(\theta^4) = z \cdot I \) for some \( z \in \mathbb{C}^* \). Since the action of \( \eta(\theta) \) on \( W \) sends \( v \) to \( \eta(\theta)v \), \( \eta(\theta^2)v \) to \( \eta(\theta^3)v \), and \( \eta(\theta^3)v \) to \( z \cdot v \), we have

\[
\eta(\theta) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
z & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix},
\]

with respect to the ordered basis \( \{ \eta(\theta^3)v, v, \eta(\theta)v, \eta(\theta^2)v \} \). The action of \( \eta(\sigma_1) \) on \( W \) sends \( \eta(\theta^3)v \) to \( x \cdot \eta(\theta^3)v \), \( v \) to \( x \cdot v \) and \( \eta(\sigma_1)(V_3) \subseteq V_3 \) since \( \sigma_1, \sigma_3 \) commute. Thus there are \( a_1, a_2, a_3, a_4 \in \mathbb{C} \) such that the matrices of \( \eta(\sigma_1), \eta(\sigma_2), \eta(\sigma_3) \) with respect to the ordered basis \( \{ \eta(\theta^3)v, v, \eta(\theta)v, \eta(\theta^2)v \} \), are

\[
\eta(\sigma_1) = \begin{pmatrix}
x & 0 & 0 & 0 \\
0 & x & 0 & 0 \\
0 & 0 & a_1 & a_2 \\
0 & 0 & a_3 & a_4 \\
\end{pmatrix}, \quad \eta(\sigma_2) = \eta(\theta \sigma_1 \theta^{-1}) = \begin{pmatrix}
a_4 & 0 & 0 & a_3 \\
0 & x & 0 & 0 \\
0 & 0 & x & 0 \\
a_2 & 0 & 0 & a_1 \\
\end{pmatrix},
\]

and

\[
\eta(\sigma_3) = \eta(\theta \sigma_2 \theta^{-1}) = \begin{pmatrix}
a_1 & a_2 x^{-1} & 0 & 0 \\
a_3 x & a_4 & 0 & 0 \\
0 & 0 & x & 0 \\
0 & 0 & 0 & x \\
\end{pmatrix}.
\]

Now comparing \( \eta(\theta) \) and \( \eta(\sigma_1 \sigma_2 \sigma_3) \), we have \( a_1 = 0, a_4 = 0, \) and \( a_3 = x^{-2}, z = a_3^2 \). By conjugating with the diagonal matrix \( \text{diag} \left( 1, a_2^{-2}, a_2^{-2}, a_2^{-1} \right) \), and setting \( u = a_2 x^{-2} \), we get the following theorem.

**Theorem 4.2.** Let \( \xi : B_4 \to GL_4(\mathbb{C}) \) be an irreducible representation. If the restriction of \( \xi \) to \( B_3 \) has an invariant one-dimensional subspace, then \( \xi \) is equivalent to the following representation \( \eta = \eta(x, u) \), for some \( x, u \in \mathbb{C}^* \), \( u \neq x^2 \), defined by

\[
\eta(\sigma_1) = \begin{pmatrix}
x & 0 & 0 & 0 \\
0 & x & 0 & 0 \\
0 & 0 & x & 0 \\
0 & 0 & u & 0 \\
\end{pmatrix}, \quad \eta(\sigma_2) = \begin{pmatrix}
0 & 0 & 0 & u \\
0 & x & 0 & 0 \\
0 & 0 & x & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \eta(\sigma_3) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
u & 0 & 0 & 0 \\
0 & 0 & x & 0 \\
0 & 0 & 0 & x \\
\end{pmatrix}.
\]
Distinct choices of $x$, $u$ give inequivalent representations.

All nonzero $x$, $u$ define representations, and $\eta$ is irreducible unless $u = x^2$. If $u = x^2$, then $(x^2, x^3, 1, x)^t$ is a common eigenvector for $\eta$.

**Theorem 4.3.** The image of the pure braid group $P_4$ of the representation $\eta$ in Theorem 4.2 consists of the diagonal matrices, and is abelian.

**Proof.** The pure braid group $P_4$ is the normal subgroup of $B_4$ generated by $\sigma_i^2$ for $1 \leq i \leq 3$ [2, Theorem 17]. Since $\eta(\sigma_i^2)$ and $\eta(\sigma_j D\sigma_j^{-1})$ are diagonal matrices for any diagonal matrix $D$, $1 \leq i, j \leq 3$, the image of the pure braid group $P_4$ under the representation $\eta$ consists of diagonal matrices, and hence is abelian. $\square$

**Remark 4.1.** For the representation $\eta$ in Theorem 4.2, we have $V = \text{diag} \left( x^2 u^2, x^6, x^2 u^2, x^2 u^2 \right)$.

1. If $u \neq \pm x^2$, then $V$ has eigenvalues, $x^2 u^2$ of multiplicity 3, and $x^6$ of multiplicity 1.
2. If $u = x^2$, then the representation is reducible.
3. If $u = -x^2$, then $V$ is a scalar matrix.

Thus any irreducible representation of $B_4$ of degree 4 whose restriction to $B_3$ has an invariant one-dimensional subspace, has $V$ either a scalar matrix or a diagonal matrix with one eigenvalue of multiplicity 3 and another eigenvalue of multiplicity 1.

5. The Jordan canonical form of $V$ is the diagonal matrix $\text{diag}(u, u, w, w)$, $u \neq w$.

Let’s assume $V$ is the diagonal matrix $\text{diag} \left( u, u, w, w \right)$, $u \neq w$. Since $\eta(\sigma_1)$ and $\eta(\sigma_2)$ commute with $V$, they have the form, $\eta(\sigma_i) = \begin{pmatrix} \rho(\sigma_i) & 0 \\ 0 & \tau(\sigma_i) \end{pmatrix}$, $i = 1, 2$, where $\rho, \tau : B_3 \rightarrow \text{GL}_2(\mathbb{C})$ are representations. If either $\rho$ or $\tau$ is reducible, then $\eta|_{B_3}$, the restriction of $\eta$ to $B_3$, has an invariant one-dimensional subspace, but then $V$ can not have Jordan canonical form $\text{diag}(u, u, w, w)$, $u \neq w$, by Remark 4.1. Therefore the representations $\rho, \tau$ are irreducible. Then, by Theorem 2.6, there exist $a, b, c, d$ such that $b \neq a \mu, d \neq c \mu$ (where $\mu$ is a primitive cube root of
-1), and

\[
\eta(\sigma_1) = \begin{pmatrix}
a & 1 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 1 \\
0 & 0 & 0 & d \\
\end{pmatrix}, \quad \eta(\sigma_2) = \begin{pmatrix}
b & 0 & 0 & 0 \\
-ab & c & 0 & 0 \\
0 & 0 & d & 0 \\
0 & 0 & -cd & c \\
\end{pmatrix}.
\]

**Definition 5.1.** A matrix is called non-derogatory if it generates its own centralizer over \( \mathbb{C} \).

If \( X \) is non-derogatory, then any matrix which centralizes \( X \) is a polynomial in \( X \) over \( \mathbb{C} \). In particular, if \( \eta(\sigma_1) \) is non-derogatory, then \( \eta(\sigma_3) \) has the form,

\[
\begin{pmatrix}
* & * & 0 & 0 \\
* & * & 0 & 0 \\
0 & 0 & * & * \\
0 & 0 & * & * \\
\end{pmatrix},
\]

and thus \( \eta \) is reducible.

**Remark 5.1.** An \( n \times n \) matrix is non-derogatory if it has only 1 Jordan block for each eigenvalue.

For \( n = 4 \), the following are the "types" of non-derogatory matrices.

\[
\begin{pmatrix}
s & 0 & 0 & 0 \\
0 & t & 0 & 0 \\
0 & 0 & u & 0 \\
0 & 0 & 0 & w \\
\end{pmatrix}, \quad \begin{pmatrix}
t & 0 & 0 & 0 \\
0 & u & 0 & 0 \\
0 & 0 & v & 0 \\
0 & 0 & 0 & w \\
\end{pmatrix}, \quad \begin{pmatrix}
u & 1 & 0 & 0 \\
0 & u & 0 & 0 \\
0 & 0 & w & 0 \\
0 & 0 & 0 & w \\
\end{pmatrix}, \quad \begin{pmatrix}
u & 1 & 0 & 0 \\
0 & w & 1 & 0 \\
0 & 0 & w & 1 \\
0 & 0 & 0 & w \\
\end{pmatrix}, \quad \begin{pmatrix}
u & 1 & 0 & 0 \\
0 & w & 1 & 0 \\
0 & 0 & w & 1 \\
0 & 0 & 0 & w \\
\end{pmatrix}, \quad \begin{pmatrix}
u & 1 & 0 & 0 \\
0 & w & 1 & 0 \\
0 & 0 & w & 1 \\
0 & 0 & 0 & w \\
\end{pmatrix},
\]

where \( s, t, u, w \) are distinct complex numbers. Let

\[
\eta(\sigma_3) = \begin{pmatrix}
n_{11} & n_{12} & n_{13} & n_{14} \\
n_{21} & n_{22} & n_{23} & n_{24} \\
n_{31} & n_{32} & n_{33} & n_{34} \\
n_{41} & n_{42} & n_{43} & n_{44} \\
\end{pmatrix}.
\]
The commutativity of \( \sigma_1 \) and \( \sigma_3 \) implies

\[
\eta(\sigma_1)\eta(\sigma_3) = \begin{pmatrix}
    an_{11} + n_{21} & an_{12} + n_{22} & an_{13} + n_{23} & an_{14} + n_{24} \\
    bn_{21} & bn_{22} & bn_{23} & bn_{24} \\
    cn_{31} + n_{41} & cn_{32} + n_{42} & cn_{33} + n_{43} & cn_{34} + n_{44} \\
    dn_{41} & dn_{42} & dn_{43} & dn_{44}
\end{pmatrix}
\]

= \begin{pmatrix}
    an_{11} & n_{11} + bn_{12} & cn_{13} & n_{13} + dn_{14} \\
    an_{21} & n_{21} + bn_{22} & cn_{23} & n_{23} + dn_{24} \\
    an_{31} & n_{31} + bn_{32} & cn_{33} & n_{33} + dn_{34} \\
    an_{41} & n_{41} + bn_{42} & cn_{43} & n_{43} + dn_{44}
\end{pmatrix} = \eta(\sigma_3)\eta(\sigma_1).

Let’s consider the following cases according to the number of distinct eigenvalues of \( \eta(\sigma_1) \).

1. If \( \eta(\sigma_1) \) has a single eigenvalue i.e. \( a = b = c = d \), then \( \mathcal{V} \) is a scalar matrix. Contradiction to the hypotheses.

2. \( \eta(\sigma_1) \) has 2 eigenvalues.
   a. \( a = b = c \neq d \).
      
      The commutativity of \( \sigma_1 \) and \( \sigma_3 \) implies \( n_{21} = n_{23} = n_{24} = n_{31} = n_{41} = n_{42} = n_{43} = 0, n_{11} = n_{22}, n_{13} = (a - d)n_{14}, \) and \( n_{34} = \frac{n_{33} - n_{34}}{a - d} \). The equality of the \((4, 1)\) entries of \( \eta(\sigma_2 \sigma_3 \sigma_2) = \eta(\sigma_3 \sigma_2 \sigma_3) \) forces \( a^3dn_{32} = 0 \). But if \( n_{32} \) vanishes, then \( \eta \) is reducible.
   
   b. The same reasoning as in (2a) applies when any three of \( a, b, c, d \) are equal, since the role of \( a, b, c, d \) are symmetric.
   
   c. \( a = b \neq c = d \).
      
      \( \eta(\sigma_1) \) is non-derogatory.
   
   d. \( a = c \neq b = d \).
      
      Again we have a scalar matrix for \( \mathcal{V} \). Contradiction to the hypotheses.

3. \( \eta(\sigma_1) \) has 3 eigenvalues.
   a. \( a = b \) and \( b, c, d \) are distinct.
      
      \( \eta(\sigma_1) \) is non-derogatory.
   
   b. \( a = c \) and \( b, c, d \) are distinct.
      
      The commutativity of \( \sigma_1 \) and \( \sigma_3 \) implies \( n_{21} = n_{23} = n_{24} = n_{31} = n_{41} = n_{42} = n_{43} = 0, n_{12} = \frac{n_{11} - n_{22}}{a - b}, n_{14} = \frac{n_{13}}{a - d}, n_{32} = \frac{n_{31}}{a - b}, \) and \( n_{34} = \frac{n_{33} - n_{34}}{a - d} \). Comparing \((2, 4)\) entries of \( \eta(\sigma_2 \sigma_3 \sigma_2) = \eta(\sigma_3 \sigma_2 \sigma_3) \), we have \( abn_{13}(n_{22} - a) = 0 \). If \( n_{13} \) is equal to \( 0 \), then the representation is reducible. Thus \( n_{22} = a \). From the equality of the \((4, 2)\) entries of \( \eta(\sigma_2 \sigma_3 \sigma_2) = \eta(\sigma_3 \sigma_2 \sigma_3) \), we have
\[ abn_{31}(n_{41} - a) = 0. \text{ Thus } n_{41} = a, \text{ for if } n_{31} \text{ vanishes, then the corresponding representation becomes reducible. Then the equality of the } (2, 3)\text{-entries of } \eta(\sigma_2) \sigma_3 \sigma_2 = \eta(\sigma_3) \sigma_2 \sigma_3 \text{ forces } abn_{13}(a^2 - ad + d^2) = 0. \text{ So } a^2 - ad + d^2 = 0. \text{ Thus } d = a\mu = c\mu, \text{ where } \mu \text{ is a root of } \mu^2 - \mu + 1 = 0, \text{ and } \gamma : B_3 \rightarrow \text{GL}_2(\mathbb{C}) \text{ is reducible (cf. Theorem 2.6). Contradiction to the hypotheses.} \\
(4) \text{ All eigenvalues of } \eta(\sigma_1) \text{ are distinct to each other. Then } \eta(\sigma_1) \text{ is non-derogatory again.} \\

Thus we have shown

\textbf{Theorem 5.2.} Let } \eta : B_4 \rightarrow \text{GL}_4(\mathbb{C}) \text{ be an irreducible representation. Then the Jordan canonical form of } V = (\eta(\sigma_1) \eta(\sigma_2))^3 \text{ is not a diagonal matrix diag}(u, u, w, w) \text{ where } u \neq w. \\

6. \text{ The Jordan canonical form of } V \text{ has two identical } 2 \times 2 \text{ blocks } (w, 1, 0, w). \\

When } V \text{ has two identical } 2 \times 2 \text{ blocks } (w, 1, 0, w), \text{ we first conjugate the representations so that } V \text{ is equal to } \begin{pmatrix} w & I \\ 0 & w \cdot I \end{pmatrix} \text{ where } I \text{ is the } 2 \times 2 \text{ identity matrix. The centralizer } C \text{ of } V \text{ in } M_4(\mathbb{C}) \text{ is the set of block matrices } \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \text{ where } A, B \in M_2(\mathbb{C}). \text{ Since } \eta(\sigma_1) \text{ and } \eta(\sigma_2) \text{ commute with } V, \text{ the image } \eta(B_3) \subseteq C. \text{ Moreover, the map } \gamma : C \rightarrow M_2(\mathbb{C}) \text{ that sends } \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \text{ to } A \text{ is a } \mathbb{C}-\text{algebra homomorphism. For the representation } \rho = \eta|_{B_3} \text{ of } B_3 \text{ onto } C \cap \text{GL}_4(\mathbb{C}), \text{ we consider the composition } \delta = \gamma \rho : B_3 \rightarrow \text{GL}_2(\mathbb{C}). \text{ If } \delta \text{ is reducible, } \eta(\sigma_1) \text{ and } \eta(\sigma_2) \text{ have a one-dimensional invariant subspace, and then } V \text{ can not have Jordan canonical form with two identical } 2 \times 2 \text{ blocks } (w, 1, 0, w) \text{ by the Remark 4.1. Thus } \delta \text{ is irreducible. Then by Theorem 2.6,}

\[
\eta(\sigma_1) = \begin{pmatrix} a & 1 & * & p \\ 0 & b & * & q \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad \eta(\sigma_2) = \begin{pmatrix} b & (0 & * & *) \\ -ab & c & * & * \\ 0 & (0 & b & 0) \\ 0 & (0 & -ab & a) \end{pmatrix}.
\]
where \( b \neq a\mu \) (where \( \mu \) is a primitive cube root of \(-1\)), \( a, b \in \mathbb{C}^* \), \( p, q \in \mathbb{C} \), and each "*" denotes an element of \( \mathbb{C} \). By conjugating with

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -q & p \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

we may assume

\[
\eta(\sigma_1) = \begin{pmatrix}
a & 1 & c & 0 \\
0 & b & d & 0 \\
0 & 0 & a & 1 \\
0 & 0 & 0 & b
\end{pmatrix}, \quad \eta(\sigma_2) = \begin{pmatrix}
b & 0 & g & h \\
-\alpha b & a & k & l \\
0 & 0 & b & 0 \\
0 & 0 & -\alpha b & a
\end{pmatrix},
\]

where \( b \neq a\mu, a, b \in \mathbb{C}^* \). The braid relation \( \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \) and \( V \) as in the hypotheses implies

\[
d = b(c - \frac{a}{3}),
\quad h = \frac{(b - a)g - d}{ab},
\quad k = abh - bc = (b - a)g - d - bc,
\quad l = c - g.
\]

To determine \( \eta(\sigma_3) \), let

\[
\eta(\sigma_3) = \begin{pmatrix}
n_{11} & n_{12} & n_{13} & n_{14} \\
n_{21} & n_{22} & n_{23} & n_{24} \\
n_{31} & n_{32} & n_{33} & n_{34} \\
n_{41} & n_{42} & n_{43} & n_{44}
\end{pmatrix}.
\]

Then we have

\[
\eta(\sigma_1)\eta(\sigma_3) = \begin{pmatrix}
D & E \\
F & G
\end{pmatrix},
\]
where

\[
D = \begin{pmatrix} an_{11} + n_{21} + cn_{31} & an_{12} + n_{22} + cn_{32} \\ bn_{21} + bcn_{31} - \frac{ab}{3}n_{31} & bn_{22} + bcn_{32} - \frac{ab}{3}n_{32} \end{pmatrix},
\]

\[
E = \begin{pmatrix} an_{13} + n_{23} + cn_{33} & an_{14} + n_{24} + cn_{34} \\ bn_{23} + bcn_{33} - \frac{ab}{3}n_{33} & bn_{24} + bcn_{34} - \frac{ab}{3}n_{34} \end{pmatrix},
\]

\[
F = \begin{pmatrix} an_{31} + n_{32} + n_{41} \\ bn_{41} \end{pmatrix},
\]

\[
G = \begin{pmatrix} an_{33} + n_{34} + n_{43} \\ bn_{43} \end{pmatrix}.
\]

And

\[
\eta(\sigma_3)\eta(\sigma_1) = \begin{pmatrix} an_{11} + n_{12} & cn_{11} + bcn_{12} - \frac{ab}{3}n_{12} + an_{13} & n_{13} + bn_{14} \\ an_{21} & n_{22} + bn_{22} - \frac{ab}{3}n_{22} + an_{23} & n_{23} + bn_{24} \\ an_{31} + bn_{32} & cn_{31} + bcn_{32} - \frac{ab}{3}n_{32} + an_{33} & n_{33} + bn_{34} \\ an_{41} + bn_{42} & cn_{41} + bcn_{42} - \frac{ab}{3}n_{42} + an_{43} & n_{43} + bn_{44} \end{pmatrix}.
\]

Comparing (3, 1)- and (4, 4)-entries of \(\eta(\sigma_1)\eta(\sigma_3)\) and \(\eta(\sigma_3)\eta(\sigma_1)\), we have

(1) \[n_{41} = n_{43} = 0.\]

With the equality of (1, 1)- and (2, 2)-entries of \(\eta(\sigma_1)\eta(\sigma_3)\) and \(\eta(\sigma_3)\eta(\sigma_1)\), we have

(2) \[n_{21} = -cn_{31}.\]

and

(3) \[n_{21} = bn_{32}(c - \frac{a}{3}).\]

Then with the equality of (2, 1)-entries of \(\eta(\sigma_1)\eta(\sigma_3)\) and \(\eta(\sigma_3)\eta(\sigma_1)\), we have

\[
0 = (b - a)n_{21} + bn_{31}(c - \frac{a}{3})
\]

\[
= (b - a)(-cn_{31}) + bn_{31}(c - \frac{a}{3})
\]

\[
= n_{31}(-bc + ac + bc - \frac{ab}{3})
\]

\[
= an_{31}(c - \frac{b}{3}).
\]

(1) \(c - \frac{b}{3} \neq 0\).

Then \(n_{31} = 0\) implies \(n_{21} = 0\) and \(bn_{32}(c - \frac{a}{3}) = 0\) by Eq (2) and Eq (3).
(a) $n_{32} = 0$.

If $n_{31} = n_{32} = 0$, then $(a - b)n_{32} - n_{31} + n_{42} = 0$ from the equality of (3, 2)-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$ implies $n_{42} = 0$. Therefore $\eta(\sigma_3)$ has the form,

$$
\begin{pmatrix}
* & * & * \\
* & * & * \\
0 & 0 & * \\
0 & 0 & *
\end{pmatrix},
$$

and the corresponding representation is reducible.

(b) $c - \frac{a}{3} = 0$.

In the equality of (3, 2)-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, we have $n_{42} = (b - a)n_{32}$. Now the equality of (4, 1)-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$ implies $abn_{32}(a^2 - (a - b)n_{11}) = 0$. If $n_{32}$ vanishes, then the representation is reducible by the previous case. So assume $n_{32} \neq 0$, then $n_{11} = \frac{a^2}{a - b}$. In the equality of (3, 1)-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have $abn_{32}(-b + n_{11}) = 0$. Thus $b = n_{11} = \frac{a^2}{a - b}$. Hence $a^2 - ab + b^2 = 0$. Contradiction to our initial assumption $b \neq a\mu$.

(2) $c - \frac{b}{3} = 0$.

The equality of (4, 3)-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$ forces $b(b - a)n_{42} = 0$.

(a) $n_{42} = 0$.

If we compare the (3, 2)-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, we get $n_{31} = (a - b)n_{32}$. In the equality of (4, 1)-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have $abn_{32}(b^2 - (a - b)n_{44}) = 0$. Since $n_{32} = 0$ implies reducibility, assume $n_{44} = \frac{b^2}{b - a}$. Then in the equality of (4, 2)-entries of $\eta(\sigma_2\sigma_3\sigma_2) = \eta(\sigma_3\sigma_2\sigma_3)$, we have $abn_{32}(-a + n_{44}) = 0$. Therefore $c = n_{44} = \frac{b^2}{b - a}$ implies $b^2 - ab + a^2 = 0$. Contradiction to our initial assumption $b \neq a\mu$ again.

(b) $b = a$.

In Eq (3), we have $n_{21} = bn_{32}(c - \frac{a}{3}) = in_{32}(c - \frac{b}{3}) = 0$. From Eq (2), we have $n_{31} = 0$ since $c = \frac{b}{3} \neq 0$. In the equality of (3, 2)-entries of $\eta(\sigma_1)\eta(\sigma_3)$ and $\eta(\sigma_3)\eta(\sigma_1)$, $n_{32}(a - b) + n_{42} - n_{31} = 0$ forces $n_{42} = 0$ and the vanishing of $n_{42}$ makes this case belong to the previous case.

Hence we have shown
THEOREM 6.1. Let $\eta : B_4 \rightarrow GL_4(\mathbb{C})$ be an irreducible representation. Then the Jordan canonical form of $\mathcal{V} = (\eta(\sigma_1)\eta(\sigma_2))^3$ is not equal to

$$
\begin{pmatrix}
w & 1 & 0 & 0 \\
0 & w & 0 & 0 \\
0 & 0 & w & 1 \\
0 & 0 & 0 & w
\end{pmatrix}.
$$

7. The Jordan canonical form of $\mathcal{V}$ is a scalar matrix.

Since scalar matrices are central in $GL_4(\mathbb{C})$, all commutators $[(\sigma_1\sigma_2)^3, w] = (\sigma_1\sigma_2)^3w(\sigma_1\sigma_2)^{-3}w^{-1}$ will lie in the kernel of the representation for any word $w$ in $B_4$, so it will really be a representation $\eta$ of $B_4/N$, where $N$ is the normal subgroup of $B_4$ generated by all such commutators.

Now assume $\eta : B_4 \rightarrow GL_4(\mathbb{C})$ is an irreducible representation and $\mathcal{V}$ is a scalar matrix. There is a short exact sequence $1 \rightarrow F_2 \rightarrow B_4 \rightarrow B_3 \rightarrow 1$, where the map between $B_4$ and $B_3$ is the special homomorphism $\pi$ which sends $\sigma_1, \sigma_3$ to $\sigma_1$ and fixes $\sigma_2$, and $F_2 = \langle \sigma_1\sigma_3^{-1}, \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} \rangle$ is a free group of rank 2 [6, p. 406]. For any word $w \in B_4$, $\pi(w) \in B_3$ and $(\sigma_1\sigma_2)^3$ is in the center of $B_3$. Hence $\pi(N) = 1$ implies $N \subseteq F_2$. Factoring out $N$ gives rise to an exact sequence $1 \rightarrow K = F_2/N \rightarrow B_4/N \rightarrow B_3 \rightarrow 1$. To figure out $F_2/N$, set $p = \sigma_1\sigma_3^{-1}$, $q = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$. Then $[(\sigma_1\sigma_2)^3, \sigma_3] = qpq^{-1}p \in N$ and $\sigma_2(qpq^{-1}p)\sigma_2^{-1} = qp^{-1}qp \in N$. Hence $F_2/N$ is isomorphic to the quaternion group of order 8. There are 5 irreducible representations of the quaternion group, four of them, $\phi_i$, $1 \leq i \leq 4$, are of degree 1 and one of them, $\psi$, is of degree 2. They are as follows, $\phi_1(p) = 1$, $\phi_1(q) = 1$, $\phi_2(p) = 1$, $\phi_2(q) = -1$, $\phi_3(p) = -1$, $\phi_3(q) = 1$, $\phi_4(p) = -1$, $\phi_4(q) = -1$, and $\psi(p) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\psi(q) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For any irreducible representation $\eta : B_1/N \rightarrow GL_4(\mathbb{C})$, the restriction of $\eta$ to $K$, $\eta|_K$ is the direct sum of irreducible representations of the same degree, which are conjugates, with the same multiplicity, by Clifford’s theorem [5, (49.2) Theorem, (49.7) Theorem], since $K$ is a normal subgroup of $B_4/N$. And the conjugacy classes of irreducible representations of $K$ are $\{\phi_1\}, \{\phi_2, \phi_3, \phi_4\}, \{\psi\}$. 
**Lemma 7.1.** Let \( \eta : B_4 \to \text{GL}_4(\mathbb{C}) \) be an irreducible representation and \( \eta|_K \) is the direct sum of irreducible representations. Then one of the following occurs.

(A) \( \eta|_K = \phi_1 \oplus \phi_1 \oplus \phi_1 \oplus \phi_1 \).

(B) \( \eta|_K = \psi \oplus \psi \).

First of all, we consider the case, in which \( \eta|_K \) is the direct sum of 4 one-dimensional representations.

**Proposition 7.2.** Let \( \eta : B_4 \to \text{GL}_4(\mathbb{C}) \) be an irreducible representation and \( \eta|_K \) is the direct sum of 4 one-dimensional representations. Then \( \eta = \tau \pi \) where \( \pi : B_4 \to B_3 \) is the special homomorphism and \( \tau : B_3 \to \text{GL}_4(\mathbb{C}) \) is an irreducible representation of \( B_3 \) of degree 4.

**Proof.** By Lemma 7.1, \( \eta|_K = \phi_1 \oplus \phi_1 \oplus \phi_1 \oplus \phi_1 \). Since \( \eta(p) = \eta(\sigma_1 \sigma_3^{-1}) = I \), we have \( \eta(\sigma_1) = \eta(\sigma_3) \). Hence \( \eta = \tau \pi \) where \( \pi : B_4 \to B_3 \) is the special homomorphism and \( \tau : B_3 \to \text{GL}_4(\mathbb{C}) \) is an irreducible representation. Since \( B_3 \) has a presentation \( \langle \alpha, \beta \mid \alpha^3 = \beta^2 \rangle \), \( \tau(\alpha^3) = \tau(\beta^2) = z \cdot I \) for some \( z \in \mathbb{C}^* \). Thus \( \tau(\alpha) \) and \( \tau(\beta) \) are diagonalizable. For some \( a, b \in \mathbb{C}^* \), the eigenvalues of \( \tau(\alpha) \) belong to the set \( \{a, aw, a\omega^2\} \) where \( \omega = \frac{-1 + \sqrt{-3}}{2} \), a cube root of unity and the eigenvalues of \( \tau(\beta) \) belong to the set \( \{b, -b\} \). But \( \alpha^3 = \beta^2 \) implies \( a^3 = b^2 \). Set \( y = a^{-1}b \). If \( \tau(\beta) \) has only one eigenvalue, then it is a scalar matrix and this would imply that \( \tau \) is reducible. So we may suppose that \( \tau(\beta) \) is conjugate to either \( \text{diag} (b, b, b, -b) \) or \( \text{diag} (b, b, -b, -b) \). But if \( \tau(\beta) \) is conjugate to \( \text{diag} (b, b, b, -b) \), then \( \tau(\alpha) \) and \( \tau(\beta) \) have a common eigenvector since \( \tau(\alpha) \) has an eigenvalue of multiplicity 2 by comparing the dimensions of the null spaces of each eigenvalues. If \( \tau(\alpha) \) has an eigenvalue of multiplicity 3, then \( \tau(\alpha) \) and \( \tau(\beta) \) have a common eigenvector again contradicting to the irreducibility of \( \tau \). Hence \( \tau(\alpha) \) is conjugate to a diagonal matrix with either one eigenvalue of multiplicity 2 and two of multiplicity 1 or two eigenvalues of multiplicity 2, i.e. we may assume \( \tau(\beta) = y^2 \text{diag} (1, 1, -1, -1) \) and \( \tau(\alpha) \) is conjugate to either \( y^2 \omega^i \text{diag} (1, 1, \omega, \omega^2) \) or \( y^2 \omega^i \text{diag} (1, 1, \omega, \omega) \) where \( 1 \leq i \leq 3 \). But since \( y^2 \omega^i = (y \omega^i)^2 \), they correspond to a different choice of \( y \).

According to [11], the set of equivalence classes of irreducible representations of \( \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z} \) of dimension \( n \) is a nonempty variety and the components of this variety are indexed by ordered lists of non-negative integers, \((n_1, n_2; m_1, m_2, m_3)\), which satisfy the following conditions, \( n_1 + n_2 = n, m_1 + m_2 + m_3 = n \) and \( n_1, n_2 \) are greater than or
equal to all of \( m_1, m_2, m_3 \). Furthermore, if an irreducible representation of \( \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z} = \langle s \rangle \ast \langle t \rangle \) is given by \( s \mapsto S \) and \( t \mapsto T \), then for each non-zero scalar \( c \), there is an irreducible representation of \( B_3 \) given by \( s \mapsto c^3 S \) and \( t \mapsto c^2 T \). Thus we have two families of representations corresponding to indices \((2, 2; 2, 1, 1)\) and \((2, 2; 2, 2, 0)\), which are the composition of an irreducible representation of \( B_3 \) with the special homomorphism \( \pi : B_4 \to B_3 \).

For the second case, we need to consider the restriction of \( \eta \) to \( K \) is the direct sum of 2 two-dimensional representations, i.e. \( \eta|_K = \psi \oplus \psi \).

**Proposition 7.3.** Let \( \eta : B_4 \to \text{GL}_4(\mathbb{C}) \) be an irreducible representation and \( \eta|_K \) is the direct sum of 2 two-dimensional representations. Then \( \eta \) is equivalent to a representation \( \rho \pi \otimes \hat{\beta}_4(i) \). where \( \pi \) is the special homomorphism, \( \rho : B_3 \to \text{GL}_2(\mathbb{C}) \) is the irreducible representation defined by \( \rho(\sigma_1) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}, \ \rho(\sigma_2) = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix} \) for \( a, b \in \mathbb{C}^* \), \( b \neq a\mu \), and \( \hat{\beta}_4(i) : B_4 \to \text{GL}_2(\mathbb{C}) \) is the irreducible representation defined by

\[
\hat{\beta}_4(i) (\sigma_1) = \begin{pmatrix} -i & 0 \\ -1 & 1 \end{pmatrix}, \ \hat{\beta}_4(i) (\sigma_2) = \begin{pmatrix} 1 & -i \\ 0 & -i \end{pmatrix}, \ \hat{\beta}_4(i) (\sigma_3) = \begin{pmatrix} 1 & 0 \\ 1 & -i \end{pmatrix}.
\]

**Proof.** By conjugation, we may assume

\[
\eta(p) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \ \eta(q) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\]

Since \( \eta(\sigma_1) \) and \( \eta(\sigma_3) \) commute with \( \eta(p) \), they have the form

\[
\begin{pmatrix}
* & * & 0 & 0 \\
* & * & 0 & 0 \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{pmatrix}.
\]

Let \( \eta(\sigma_1) = \begin{pmatrix} L & 0 \\ 0 & L_1 \end{pmatrix} \) where \( L, L_1 \) are the upper left and the lower right \( 2 \times 2 \) matrices respectively. Since \( \sigma_3 = p^{-1}\sigma_1 \),

\[
\eta(\sigma_3) = \eta(p^{-1})\eta(\sigma_1) = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & L_1 \end{pmatrix} = \begin{pmatrix} -iL & 0 \\ 0 & iL_1 \end{pmatrix}.
\]
In [6, p. 406], $\sigma_1 q \sigma_1^{-1} = qp^{-1} = q \sigma_3 \sigma_1^{-1}$. Then $\sigma_1 q = q \sigma_3$ implies

\[
\begin{pmatrix}
L & 0 \\
0 & L_1
\end{pmatrix}
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\begin{pmatrix}
-iL & 0 \\
0 & iL_1
\end{pmatrix}.
\]

Thus $L_1 = -iL$. Hence we have $\eta(\sigma_1) = \begin{pmatrix} L & 0 \\ 0 & -iL \end{pmatrix}$, $\eta(\sigma_3) = \begin{pmatrix} -iL & 0 \\ 0 & L \end{pmatrix}$.

Let $\eta(\sigma_2) = \begin{pmatrix} M & M_1 \\ M_2 & M_3 \end{pmatrix}$ where $M$ and $M_i$'s are $2 \times 2$ matrices. Since $q \sigma_2 = \sigma_2 q$, we have $\eta(\sigma_2) = \begin{pmatrix} M & M_1 \\ iM & -iM_1 \end{pmatrix}$. Furthermore $\sigma_2 q \sigma_2^{-1} = qp^{-1} q = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}$ implies

\[
\begin{pmatrix}
M & M_1 \\
iM & -iM_1
\end{pmatrix}
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
= 
\begin{pmatrix}
-iI & 0 \\
0 & iI
\end{pmatrix}
\begin{pmatrix}
M & M_1 \\
iM & -iM_1
\end{pmatrix}.
\]

Therefore $M_1 = iM$ and consequently $\eta(\sigma_2) = \begin{pmatrix} M & iM \\ iM & M \end{pmatrix}$. The braid relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ implies $LML = (1 + i)MLM$. If we let $M' = (1 + i)M$, then $LM' L = M'LM'$. The solutions of the last equation (up to equivalence) are $L = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$, $M' = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$ for $a, b \in \mathbb{C}^*$. Then $M = \frac{1}{1+i} M' = \frac{1}{1+i} \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}$. We now have

\[
\eta(\sigma_1) = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -ia & -i \\ 0 & 0 & 0 & -ib \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},
\]

\[
\eta(\sigma_2) = \begin{pmatrix} \frac{1}{1+i} & 0 & \frac{ib}{1+i} & 0 \\ -ab & \frac{1}{1+i} & \frac{1+i}{1+i} & \frac{1}{1+i} \\ \frac{1+i}{1+i} & 0 & \frac{1-i}{1+i} & 0 \\ -i & ib & \frac{1+i}{1+i} & \frac{1+i}{1+i} \end{pmatrix} = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix},
\]

\[
\eta(\sigma_3) = \begin{pmatrix} -ia & -i & 0 & 0 \\ 0 & -ib & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & b \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}.
\]
These representations are irreducible for nonzero \( a \) and \( b, b \neq a \mu \). This family of representations is the tensor product of 2 two-dimensional representations of \( B_4 \) and is equivalent to \( \rho \pi \otimes \widehat{\beta}_4(i) \).

\[ \square \]

8. Conclusion.

Combining Theorem 4.2, Proposition 7.2, and Proposition 7.3, we have the main theorem.

Theorem 8.1. Let \( \xi : B_4 \to \text{GL}_4(\mathbb{C}) \) be an irreducible representation. Then \( \xi \) is equivalent to one of the following irreducible representations.

1. A representation \( \eta = \eta(x, u) \), where \( x, u \in \mathbb{C}^* \), \( u \neq x^2 \), defined by

\[
\eta(\sigma_1) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & u & 0 \end{pmatrix}, \quad \eta(\sigma_2) = \begin{pmatrix} 0 & 0 & 0 & u \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}, \quad \eta(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}.
\]

2. A representation \( \tau \pi \) where \( \pi : B_4 \to B_3 \) is the special homomorphism and \( \tau : B_3 \to \text{GL}_4(\mathbb{C}) \) is an irreducible representation of \( B_3 \) of degree 4.

3. A representation \( \rho(a, b)\pi \otimes \widehat{\beta}_4(i) \), where \( \pi \) is the special homomorphism, \( \rho(a, b) \) is the irreducible representation of \( B_3 \) into \( \text{GL}_2(\mathbb{C}) \) defined by

\[
\rho(a, b)(\sigma_1) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}, \quad \rho(a, b)(\sigma_2) = \begin{pmatrix} b & 0 \\ -ab & a \end{pmatrix}
\]

for \( a, b \in \mathbb{C}^*, b \neq a \mu \), and \( \widehat{\beta}_4(i) : B_4 \to \text{GL}_2(\mathbb{C}) \) is the irreducible representation defined by

\[
\widehat{\beta}_4(i)(\sigma_1) = \begin{pmatrix} -i & 0 \\ -1 & 1 \end{pmatrix}, \quad \widehat{\beta}_4(i)(\sigma_2) = \begin{pmatrix} 1 & -i \\ 0 & -i \end{pmatrix}, \quad \widehat{\beta}_4(i)(\sigma_3) = \begin{pmatrix} 1 & 0 \\ 1 & -i \end{pmatrix}.
\]

There are no equivalences between representations under distinct headings (1), (2) and (3) except \( \eta(x, -x^2) \) in (1) is equivalent to \( \rho(x, xi)\pi \otimes \widehat{\beta}_4(i) \) in (3).

For representations in (1), the image of \( P_4 \), the pure braid group on 4 strings, consists of diagonal matrices, and hence is abelian. In fact, if \( x, u \in \mathbb{C}^* \) generate a free abelian group of rank 2 under multiplication, then the kernel of \( \eta(x, u) \) is exactly \([P_4, P_1]\), the commutator subgroup of \( P_4 \). For representations in (2), the kernel contains \( F_2 \), the kernel
of the special homomorphism $\pi : B_4 \to B_3$. For representations in (3), the image of $F_2$ is the quaternion group $Q_8$, and the kernel of the representation $\rho(a,b)\pi \otimes \hat{\beta}(i)$ is the kernel of a specific homomorphism $F_2 \to Q_8$.

References


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