ON STABILITY OF A TRANSMISSION PROBLEM

Hyeonbae Kang and Jin Keun Seo

Abstract. We investigate the behavior of the gradient of solutions to the refraction equation \( \text{div}((1 + (k - 1)\chi_D)\nabla u) = 0 \) under perturbation of domain \( D \). If \( u \) and \( u_h \) are solutions to the refraction equation corresponding to subdomains \( D \) and \( D_h \) of a domain \( \Omega \) in 2 dimensional plane with the same Neumann data on \( \partial \Omega \), respectively, we prove that \( \|\nabla (u - u_h)\|_{L^2(\Omega)} \leq C \sqrt{\text{dist}(D, D_h)} \) where \( \text{dist}(D, D_h) \) is the Hausdorff distance between \( D \) and \( D_h \). We also show that this is the best possible result.

1. Introduction and statement of results

Let \( \Omega \) be a simply connected bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\) with the \( C^2 \) smooth boundary and let \( D \) be a simply connected \( C^2 \) subdomain of \( \Omega \) with closure in \( \Omega \). Let \( k \neq 1 \) be a positive number. Consider the following Neumann problem

\[
P[D, g] = \begin{cases} 
\text{div}((1 + (k - 1)\chi_D)\nabla u) = 0 & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = g & \text{on } \partial \Omega, \\
\int_{\partial \Omega} g = 0, & g \in L^2(\partial \Omega) \\
\int_{\Omega} u = 0.
\end{cases}
\]

where \( \nu(x) \) is the unit normal to the boundary, \( \frac{\partial u}{\partial \nu} = \nu \cdot \nabla u \), and \( \chi_D \) is the characteristic function of \( D \).

In this paper we study stability of the solution to the transmission problem \( P[D, g] \) under the perturbation of \( D \). This study is motivated in relation to the inverse problem to \( P[D, g] \), namely the inverse conductivity problem.

Received May 17, 1997.
1991 Mathematics Subject Classification: 35B35.
Key words and phrases: conductivity problem, stability, layer potential.
The both authors are partially supported by GARC-KOSEF, BSRI'97, and KOSEF 95-0701-01-01-3.
Let $D_h$ be a bounded simply connected subdomain with $C^2$ smooth boundary defined by

$$\partial D_h : f(s) + h\omega_h(s)\nu(s) \quad (\partial D : x = f(s))$$

where $s$ is 1 dimensional local parameter, $\omega_h(s)$ is a $C^1$ function on $\partial D$ whose $C^1$ norm is bounded uniformly in $h$, and $\nu(s)$ is the outward unit normal to $\partial D$. Let $u$ and $u_h$ be solutions to $P[D, g]$ and $P[D_h, g]$, respectively. In [2] and [1], it is proved that

$$(1.1) \quad \|u - u_h\|_{L^1(\Omega)} \leq C h$$

if $p < 4$ and $\partial D$ and $\partial D_h$ are only $C^{1,1}$. (This result is for $n$-dimension $(n = 2, 3)$.) In [7], (1.1) is proved for $p = \infty$. Both results were used strongly in the study of the inverse problem (see [2, 1, 7]).

In this paper we investigate the behaviour of $\nabla u$ under perturbation of $D$. We prove that

$$\|\nabla (u - u_h)\|_{L^2(\Omega)} \leq C\sqrt{h}.$$ 

We also prove that $\sqrt{h}$ is the best possible result one can expect. To be precise, we have the following theorem:

**Theorem 1.1.** Let $\Omega$ be a simply connected bounded domain in $\mathbb{R}^2$ and $D$ and $D_h$ be as above. Let $D\Delta D_h$ be the symmetric difference of $D$ and $D_h$. Then, there exists a constant $C$ such that

$$(1.2) \quad \lim_{h \to 0} \frac{1}{h} \int_{\Omega \setminus D\Delta D_h} |\nabla (u - u_h)|^2 dx = 0,$$

$$(1.3) \quad \limsup_{h \to 0} \frac{1}{h} \int_{D\Delta D_h} |\nabla (u - u_h)|^2 dx \leq C \int_{\partial D} \left| \frac{\partial u}{\partial \nu^\pm} \right|^2 d\sigma.$$

Here, $d\sigma$ is the line element on $\partial D$ and

$$\frac{\partial u}{\partial \nu^\pm}(P) = \lim_{t \to 0^+} \langle \nabla u(P \pm t\nu(P)), \nu(P) \rangle.$$

Moreover, if $\omega_h$ converges to $\omega$ uniformly as $h \to 0$, then

$$(1.4) \quad \lim_{h \to 0} \frac{1}{h} \int_{D\Delta D_h} |\nabla (u - u_h)|^2 dx = \frac{k - 1}{k} \int_{\partial D} \left| \frac{\partial u}{\partial \nu^+} \right|^2 |\omega| d\sigma.$$

If the Neumann data $g$ is not zero, then $\frac{\partial u}{\partial \nu^\pm}$ is not zero and hence we have
Corollary 1.2. There exists a constant $C$ independent of $h$ such that
\[ \| \nabla (u - u_h) \|_{L^2(\Omega)} \leq C \sqrt{h} \]
If $\omega_h$ converges to $\omega$ uniformly as $h \to 0$, then for small enough $h$
\[ \frac{1}{C} \sqrt{h} \leq \| \nabla (u - u_h) \|_{L^2(\Omega)}. \]

Corollary 1.2 says that $\sqrt{h}$ is the best possible.

The proof of Theorem 1.1 is based on our earlier result on the representation of the solution to $P[D,g]$ ([9]). So, we first review the representation formula in Section 2 and then prove Theorem 1.1 in Section 3.

One comment on a notation: the constants $C$ in estimates may differ from one step to another. However, those constants do not depend on the quantities to be estimated.

2. Representation of solutions

For this work, we assume that $\Omega$ is a simply connected bounded $C^2$ domain in $\mathbb{R}^2$ and $D$ be a simply connected subdomain with $C^2$ boundary which is compactly contained in $\Omega$. The single and double layer potentials on $D$ is defined by

\[ S_D f(X) = \frac{1}{2\pi} \int_{\partial D} \log |X - Q| f(Q) d\sigma_Q, \quad X \in \mathbb{R}^2, \]
\[ D_D f(X) = \frac{1}{2\pi} \int_{\partial D} \frac{\langle \nu_Q, X - Q \rangle}{|X - Q|^2} f(Q) d\sigma_Q, \quad X \in \mathbb{R}^2. \]

The following trace formula is well known (see [F] or [8]):

\[ (2.1) \quad \frac{\partial}{\partial \nu^\pm} S_D f(P) = (\pm \frac{1}{2} I + K_D^*) f(P) \quad (P \in \partial D) \]

where
\[ K_D f(P) = \frac{1}{2\pi} \int_{\partial D} \frac{\langle \nu_P, P - Q \rangle}{|P - Q|^n} f(Q) d\sigma_Q. \]

We denote by $K_D$ the dual of $K_D^*$. 
Let \( L^2_0(\partial \Omega) = \{ f \in L^2(\partial \Omega) : \int_{\partial \Omega} f d\sigma = 0 \} \). Then, the representation formula for the solution to the problem \( P[D, g] \) is as follows.

**Representation Formula [9].**

If \( u \) is the weak solution to the Neumann problem \( P[D, g] \), then there are unique harmonic functions \( H \in W^{1,2}(\Omega) \) and \( \varphi_D \in L^2_0(\partial D) \) so that \( u \) can be expressed as

\[
(2.2) \quad u(x) = H(x) + S_D \varphi_D(x) \quad \text{for} \quad x \in \Omega.
\]

Moreover, if \( f = u|_{\partial \Omega} \),

\[
(2.3) \quad H(x) = -S_\Omega g(x) + D_\Omega f(x)
\]
and

\[
(2.4) \quad \frac{k+1}{2(k-1)} I - K^*_D \varphi_D = \frac{\partial H}{\partial \nu}|_{\partial D} \quad \text{on} \quad \partial D.
\]

See [9] for proof. We remark that the representation formula holds for Lipschitz domains in \( \mathbb{R}^n, n \geq 2 \).

**Lemma 2.1 [9].** If \( u \) is the weak solution to the Neumann problem \( P[D, g] \), then

\[
(2.6) \quad \varphi_D = (k-1) \frac{\partial n}{\partial \nu} = \frac{k-1}{k} \frac{\partial u}{\partial \nu^+}
\]

### 3. Proofs

Let \( D \) and \( D_h \) be as in Section 1. Write \( \partial D_h \) as

\[
\partial D_h : \zeta + h \omega_h(\zeta) \nu(\zeta), \quad \zeta \in \partial D
\]
if slight abuse of notations is allowed, where \( \omega_h(\zeta) \) is a \( C^1 \) function on \( \partial D \) whose \( C^1 \) norm is uniformly bounded and \( \nu(\zeta) \) is the outward unit normal to \( \partial D \) at \( \zeta \). Let \( u \) and \( u_h \) be the weak solutions of \( P[D, g] \) and \( P[D_h, g] \), respectively. By the representation formula (2.2), the solutions \( u \) and \( u_h \) can be expressed uniquely as:

\[
(3.1) \quad u = H + S_D \varphi_D \quad \text{and} \quad u_h = H_h + S_{D_h} \varphi_{D_h} \quad \text{in} \ \Omega
\]
where \( H, \varphi_D, H_h, \) and \( \varphi_{D_h} \) satisfy the relations (2.3) and (2.4). To make the notations short, we put

\[
\varphi = \varphi_D, \ S = S_D, \ K^* = K^*_D, \ \varphi_h = \varphi_{D_h}, \ S_h = S_{D_h}, \ K^*_h = K^*_{D_h}.
\]
Lemma 3.1. There is a positive constant $C$ such that

\begin{equation}
\| \nabla (H - H_h) \|_{L^\infty(\Omega)} < Ch.
\end{equation}

Lemma 3.1 is proved in [7].

By identifying the real 2-D vector $(v_1, v_2)$ with the complex number $v_1 + iv_2$, we can see that

\begin{equation}
\nabla S \varphi(z) = \frac{1}{2\pi} \int_{\partial D} \frac{\varphi(\zeta)}{\bar{z} - \zeta} d|\zeta|
\end{equation}

and

\begin{equation}
\mathcal{K} \varphi(z) = \frac{1}{2\pi} \Im \int_{\partial D} \frac{\varphi(\zeta)}{z - \zeta} d\zeta.
\end{equation}

Here $\Im$ is the imaginary part. Let $\Phi_h$ be the diffeomorphism from $\partial D$ onto $\partial D_h$ defined by $\Phi_h(\zeta) = \zeta + h\omega_h(\zeta)\nu(\zeta)$.

Lemma 3.2. There is a positive constant $C$ such that

\[ \| \varphi_h \circ \Phi_h - \varphi \|_{L^2(\partial D)} \leq C. \]

if $h$ is small enough.

Proof. Let $\lambda = \frac{k+1}{2(k-1)}$. Since $(\lambda I - \mathcal{K}^*)$ is invertible on $L^2(\partial D)$ [5], we have from (2.4) that

\[ \| \varphi_h \circ \Phi_h - \varphi \|_{L^2(\partial D)} \]

\begin{align*}
&\leq C \| (\lambda I - \mathcal{K}^*)(\varphi_h \circ \Phi_h - \varphi) \|_{L^2(\partial D)} \\
&\leq C \| (\lambda I - \mathcal{K}_h^* \varphi_h) \circ \Phi_h - (\lambda I - \mathcal{K}^*)\varphi \|_{L^2(\partial D)} \\
&\quad + C \| (\mathcal{K}_h^* \varphi_h) \circ \Phi_h - \mathcal{K}^*(\varphi_h \circ \Phi_h) \|_{L^2(\partial D)} \\
&\leq C \left\| \frac{\partial H_h}{\partial \nu} \circ \Phi_h - \frac{\partial H}{\partial \nu} \right\|_{L^2(\partial D)} \\
&\quad + C \left\| (\mathcal{K}_h^* \varphi_h) \circ \Phi_h - \mathcal{K}^*(\varphi_h \circ \Phi_h) \right\|_{L^2(\partial D)}.
\end{align*}

Since the first term in the most right hand side of the above inequalities is $O(h)$ by Lemma 3.1, Lemma 3.2 follows from the following lemma. \hfill $\Box$
**Sublemma.** For any function \( g \in L^2(\partial D_h) \)
\[
\| (K_h^* g) \circ \Phi_h - K^*(g \circ \Phi_h) \|_{L^2(\partial D)} \leq C h \| g \|_{L^2(\partial D_h)},
\]
if \( h \) is small enough.

**Proof.** By duality and boundedness of \( \Phi'_h \), it suffices to show that
\[
\| (K_h g) \circ \Phi_h - K(g \circ \Phi_h) \|_{L^2(\partial D)} \leq C h \| g \|_{L^2(\partial D_h)}
\]
for any \( g \in L^2(\partial D_h) \). By (3.4),
\[
(K_h g) \circ \Phi_h(z) - K(g \circ \Phi_h)(z)
\]
\[
= \frac{1}{2\pi} \Im \left[ \int_{\partial D_h} \frac{g(\zeta)}{\Phi_h(z) - \zeta} d\zeta - \int_{\partial D} \frac{(g \circ \Phi_h)(\zeta)}{z - \zeta} d\zeta \right]
\]
\[
= \frac{1}{2\pi} \Im \left[ \int_{\partial D} \left( \frac{\Phi'_h(\zeta)}{\Phi_h(z) - \Phi_h(\zeta)} - \frac{1}{z - \zeta} \right) (g \circ \Phi_h)(\zeta) d\zeta \right]
\]
\[
= \frac{1}{2\pi} \Im \left[ \int_{\partial D} \left( \frac{1}{\Phi_h(z) - \Phi_h(\zeta)} - \frac{1}{z - \zeta} \right) (g \circ \Phi_h)(\zeta) d\zeta \right]
\]
\[
+ \frac{1}{2\pi} \Im \left[ \int_{\partial D} \frac{h(\omega_h \nu)'(\zeta)}{\Phi_h(z) - \Phi_h(\zeta)} (g \circ \Phi_h)(\zeta) d\zeta \right]
\]
\[
:= I(z) + II(z).
\]
Since the Cauchy transform on \( C^2 \)-curves (in fact, on Lipschitz curves) is bounded on \( L^2 \), we have
\[
\int_{\partial D} |II(\zeta)|^2 d|\zeta| \leq C h^2 \| g \|_{L^2(\partial D_h)}^2.
\]
Note that
\[
I(z) = \frac{1}{2\pi} \Im \int_{\partial D} \frac{1}{z - \zeta} \sum_{j=1}^{\infty} h^{j} \left( \frac{(\omega_h \nu)(z) - (\omega_h \nu)(\zeta)}{z - \zeta} \right)^j (g \circ \Phi_h)(\zeta) d\zeta.
\]
It is proven in [3] that
\[
\int_{\partial D} |I(\zeta)|^2 d|\zeta| \leq \sum_{j=1}^{\infty} (h C \| (\omega_h \nu)' \|_{L^\infty(\partial D)})^{2j} \| g \|_{L^2(\partial D_h)}^2
\]
\[
\leq C' h^2 \| g \|_{L^2(\partial D_h)}^2
\]
if \( h \) is small enough. This completes the proof. \( \square \)

Finally the following lemma leads us to Theorem 1.1.
Lemma 3.3. There exists a constant $C$ such that
\begin{equation}
\lim_{h \to 0} \frac{1}{h} \int_{\Omega \setminus D \Delta D_h} |\nabla (S\varphi - S_h\varphi_h)(z)|^2 dV(z) = 0
\end{equation}
\begin{equation}
\limsup_{h \to 0} \frac{1}{h} \int_{D \Delta D_h} |\nabla (S\varphi - S_h\varphi_h)(z)|^2 dV(z) \leq C \int_{\partial D} |\varphi(\zeta)|^2 d\sigma(\zeta).
\end{equation}
Moreover, if $\omega_h \to \omega$ uniformly as $h \to 0$, then
\begin{equation}
\lim_{h \to 0} \frac{1}{h} \int_{D \Delta D_h} |\nabla (S\varphi - S_h\varphi_h)(z)|^2 dV(z) = \int_{\partial D} |\varphi(\zeta)|^2 |\omega(\zeta)| d\sigma(\zeta).
\end{equation}

Proof. It is easy to see that for each $\delta > 0$
\[ \lim_{h \to 0} \frac{1}{h} \int_{\text{dist}(z, \partial D) > \delta} |\nabla (S\varphi - S_h\varphi_h)(z)|^2 dV(z) = 0. \]
So, we assume, from the beginning, that $\Omega = \{z = \zeta + t\nu(\zeta) : |t| < \delta, \zeta \in \partial D\}$ for some $\delta$. ($\delta$ is chosen so that the normal projection from $\Omega$ onto $\partial D$ is well-defined.) Let $\epsilon > 0$ be a fixed number to be determined later and let $U$ be the tubular neighborhood of $\partial D$ defined by $U = \{z = \zeta + t\nu(\zeta) : |t| < \epsilon, \zeta \in \partial D\}$. If $z \in \Omega \setminus U$, then by (3.3)
\[
\nabla (S_h\varphi_h - S\varphi)(z)
= \frac{1}{2\pi} \left[ \int_{\partial D} \frac{\varphi(\zeta)}{\bar{z} - \zeta} d|\zeta| - \int_{\partial D_h} \frac{\varphi_h(\zeta)}{\bar{z} - \zeta} d|\zeta| \right]
= \frac{1}{2\pi} \left[ \int_{\partial D} \frac{\varphi(\zeta) - \varphi_h \circ \Phi_h(\zeta)}{\bar{z} - \zeta} d|\zeta| + h \int_{\partial D} \frac{(\omega_h \nu(\zeta))}{(\bar{z} - \zeta)(\bar{z} - \Phi_h(\zeta))} \varphi_h \circ \Phi_h(\zeta) d|\zeta| \right.
+ \left. \int_{\partial D} \frac{\varphi_h \circ \Phi_h(\zeta)}{\bar{z} - \Phi_h(\zeta)} \left[ |\Phi'_h(\zeta)| - 1 \right] d|\zeta| \right]
= I_1(z) + I_2(z) + I_3(z).
\]
Suppose $z = \xi + t\nu(\xi) \in \Omega \setminus D$, $\xi \in \partial D$. If $N$ is the smallest integer such that $2^N t > \max\{|z - \zeta| : \zeta \in \partial D\}$, then $N \leq C \log \frac{1}{t}$ and
\[
|I_1(z)| \leq \sum_{j=1}^{N} \int_{\xi - 2^{j-1} t < \zeta < 2^j t} \left| \frac{\varphi(\zeta) - \varphi_h \circ \Phi_h(\zeta)}{|z - \zeta|} \right| d|\zeta|
\leq C |\log t| M(\varphi - \varphi_h \circ \Phi_h)(\xi).
\]
where $M$ is the Hardy-Littlewood maximal operator on $\partial D$. Since $M$ is bounded on $L^2(\partial D)$ ([10]), it follows from Lemma 3.2 that

$$
\int_{\Omega \setminus U} |I_1(z)|^2 dV = \int_{t < |t| < \delta} \int_{\partial D} |I_1(\xi + t\nu(\xi))|^2 d|\xi| dt
\leq C \int_{t < |t| < \delta} |\log |t|^2 dt \int_{\partial D} |M(\varphi - \varphi_h \circ \Phi_h)(\xi)|^2 d|\xi|
\leq C \|\varphi - \varphi_h \circ \Phi_h\|_{L^2(\partial D)}^2 \leq Ch^2.
$$

(3.8)

For $I_2(z)$, we have

$$
|I_2(z)| \leq Ch \int_{\partial D} \frac{1}{|z - \xi|^2} d|\xi| \leq Ch t^{-1},
$$

and hence

(3.9)

$$
\int_{\Omega \setminus U} |I_2(z)|^2 dV \leq Ch^2 \int_{r < |t| < \delta} t^{-2} dt \leq C\epsilon^{-1}.
$$

Since $|\Phi_h'(\xi)| - 1 = O(h)$, we have

(3.10)

$$
|I_3(z)| \leq Ch \int_{\partial D} \frac{1}{|z - \xi|} d|\xi| \leq Ch \log \frac{1}{t}.
$$

Thus, we have

(3.11)

$$
\int_{\Omega \setminus U} |I_3(z)|^2 dV \leq Ch^2.
$$

Combining (3.8), (3.9), and (3.11), we have

(3.12)

$$
\frac{1}{h} \int_{\Omega \setminus U} |\nabla (S_h \varphi - S_h \varphi_h)(z)|^2 dV \leq Ch \epsilon^{-1}.
$$

Now suppose that $z = \xi + t\nu(\xi) \in U$. Put $\xi_h = \xi + h\omega_h(\xi)\nu(\xi)$. Put $S^\epsilon = \{\xi \in \partial D : |\xi - \xi| < \epsilon\}$ and $S_h^\epsilon = \{\xi + h\omega(\xi)\nu(\xi) \in S^\epsilon\}$. Then,

$$
\nabla (S_h \varphi - S \varphi)(z) = \frac{1}{2\pi} \left[ \int_{\partial D \setminus S^\epsilon} \frac{\varphi(\zeta)}{|z - \zeta|} d|\zeta| - \int_{\partial D \setminus S_h^\epsilon} \frac{\varphi_h(\zeta)}{|z - \zeta|} d|\zeta| \right]
+ \frac{1}{2\pi} \left[ \int_{S^\epsilon} \frac{\varphi(\zeta) - \varphi(\xi)}{|z - \zeta|} d|\zeta| - \int_{S_h^\epsilon} \frac{\varphi_h(\zeta) - \varphi_h(\xi_h)}{|z - \zeta|} d|\zeta| \right]
+ \frac{\varphi_h(\xi_h)}{2\pi} \left[ \int_{S^\epsilon} \frac{1}{|z - \zeta|} d|\zeta| - \int_{S_h^\epsilon} \frac{1}{|z - \zeta|} d|\zeta| \right].
$$

$$
+ \frac{[\varphi(\xi) - \varphi_h(\xi_h)]}{2\pi} \int_{S^\epsilon} \frac{1}{|z - \zeta|} d|\zeta|.
$$

:= II_1(z) + II_2(z) + II_3(z) + II_4(z).
In the same way to derive (3.8), one can see that

\[(3.13) \quad \frac{1}{h} \int_U |I_{I_1}(z)|^2 dV \leq C \varepsilon^{-1}.\]

Since \(\varphi\) is \(C^\alpha\) for every \(\alpha < 1\) (see [4]), \(|I_{I_2}(z)| \leq C \varepsilon^\alpha\) independently of \(h\). Hence, we have

\[(3.14) \quad \frac{1}{h} \int_U |I_{I_2}(z)|^2 dV \leq C \varepsilon^{2\alpha+1} h^{-1} \quad \text{for every } \alpha < 1.\]

By Lemma 3.2 and the estimate used in (3.10), we have

\[(3.15) \quad \frac{1}{h} \int_U |I_{I_3}(z)|^2 dV = \frac{1}{h} \int_{-\varepsilon}^{\varepsilon} \int_{\partial D} |I_4(\xi + t \nu(\xi))|^2 d|\xi| dt \leq \frac{C}{h} \int_{-\varepsilon}^{\varepsilon} \int_{\partial D} |\varphi(\xi) - \varphi_h(z_h)|^2 \log t^2 d|\xi| dt \leq C h.\]

We now deal with \(I_{I_3}(z)\). Put \(\overline{W_h(z)} = \frac{1}{2\pi} \left[ \int_{S^r} \frac{1}{z - \zeta} d|\zeta| - \int_{S_h^r} \frac{1}{z - \zeta} d|\zeta| \right].\)

For each \(z \in U\), we may assume that \(S^r\) is a graph by taking \(\varepsilon\) small enough if necessary, namely,

\(S^r : \zeta = x + i g(x), \quad -\varepsilon < x < \varepsilon, \quad g \in C^2,\)

g(0) = g'(0) = 0, and \(z = it(|t| < h)\). Put

\[J(\zeta) = \frac{|1 + ig'(x)|}{1 + ig'(x)} \quad \text{and} \quad J_h(\zeta) = \frac{|1 + ig'(x) + h(\omega_h \nu)'(x)|}{1 + ig'(x) + h(\omega_h \nu)'(x)}.\]

Let \(\Gamma_h\) (\(\Gamma'_h\), resp.) be the straight line connecting the left (right, resp.) endpoints of \(S^r\) and \(S_h^r\) and let \(C_h\) be the positively oriented closed
curve composed of \( S' \), \( S'_h \), \( \Gamma_h \), and \( \Gamma_h' \). Then

\[
2\pi \overline{W_h(z)} = \int_{S'} \frac{J(\zeta)}{z - \zeta} d\zeta - \int_{S'_h} \frac{J_h(\zeta)}{z - \zeta} d\zeta
= \int_{C_h} \frac{1}{z - \zeta} d\zeta - \int_{\Gamma_h + \Gamma'_h} \frac{1}{z - \zeta} d\zeta
+ \int_{S'} \frac{J(\zeta) - 1}{z - \zeta} d\zeta - \int_{S'_h} \frac{J_h(\zeta) - 1}{z - \zeta} d\zeta.
\]

By the Cauchy integral formula,

\[
\left| \int_{C_h} \frac{1}{z - \zeta} d\zeta \right| = \begin{cases} 
2\pi & \text{if } z \in D \Delta D_h \\
0 & \text{if } z \in \Omega \setminus \overline{D \Delta D_h}.
\end{cases}
\]

It is easy to see that

\[
\left| \int_{\Gamma_h + \Gamma'_h} \frac{1}{z - \zeta} d\zeta \right| \leq C h e^{-1}.
\]

Note that

\[
|J(\zeta) - 1| \leq C |g'(x)| \leq C |x| \leq C |\zeta|, \quad \zeta \in \partial D
\]
\[
|J_h(\zeta) - 1| \leq C |g'(x)| + C h |\omega'_h(x)| \leq C (|\zeta| + h), \quad \zeta \in \partial D_h.
\]

It then follows that for \( z = \xi + t \nu(\xi) \),

\[
\left| \int_{S'} \frac{J(\zeta) - 1}{|z - \zeta|} d\zeta \right| \leq C \int_{S'} \frac{|\zeta|}{|z - \zeta|} d|\zeta| \leq C e,
\]
\[
\left| \int_{S'_h} \frac{J_h(\zeta) - 1}{z - \zeta} d\zeta \right| \leq C \int_{S'_h} \frac{|\zeta| + h}{|z - \zeta|} d|\zeta| \leq C (e + h \log \frac{1}{|h \omega_h(\xi) - t|}).
\]

So far, we proved that

\[
(3.16) \quad |W_h(z)| \leq C (e + h e^{-1} + h \log \frac{1}{|h \omega_h(\xi) - t|})
\]
if \( z = \xi + t\nu(\xi) \notin D\Delta D_h \), and

\[ ||W_h(z)|| - 1|| \leq C(\epsilon + he^{-1} + h\log \frac{1}{|(h - s)\omega_h(\xi)|}) \]

if \( z = \xi + s\omega_h(\xi)\nu(\xi) \in D\Delta D_h \). It then follows from (3.16) that

\[ \frac{1}{h} \int_{U \setminus D \Delta D_h} |II_3(z)|^2 dV \]
\[ = \frac{1}{h} \int_{h|\omega_h(\xi)| \leq t < \epsilon} \int_{\partial D} |\varphi_h(\xi_h)W_h(\xi + t\nu(\xi))|^2 d|\xi| dt \]
\[ \leq C(\epsilon^3 h^{-1} + he^{-1} + h). \]

On the other hand, from (3.17), we have

\[ \left| \frac{1}{h} \int_{D \Delta D_h} |II_3(z)|^2 dV - \int_{\partial D} |\varphi(\xi)|^2 |\omega_h(\xi)| d|\xi| \right| \]
\[ \leq \frac{1}{h} \int_{\partial D} \int_0^h \left| I_3(\xi + s\omega_h(\xi)\nu(\xi)) \right|^2 - |\varphi(\xi)|^2 ds |\omega_h(\xi)| d|\xi| \]
\[ \leq C \int_{\partial D} |\varphi_h(\xi_h) - \varphi(\xi)|^2 d|\xi| \]
\[ + \frac{C}{h} \int_{\partial D} \int_0^h \left| W_h(\xi + s\omega_h(\xi)\nu(\xi)) \right|^2 - 1 ds |\varphi(\xi)|^2 |\omega_h(\xi)| d|\xi| \]
\[ \leq C(h^2 + \epsilon + he^{-1} + \int_{\partial D} |h\omega_h(\xi)| \log \frac{1}{|h\omega_h(\xi)|} d|\xi|). \]

Take \( \epsilon = h^{2/3} \). Then (3.12)-(3.15), (3.18), and (3.19) prove Lemma 3.3.

\[ \square \]

References


Hyeonbae Kang
Department of Mathematics
Korea University
Seoul 136-701, Korea
E-mail: kang@semi.korea.ac.kr

Current address of Hyeonbae Kang
Department of Mathematics
Seoul National University
Seoul 151-742, Korea

Jin Keun Seo
Department of Mathematics
Yonsei University
Seoul 120-749, Korea
E-mail: seoj@bubble.yonsei.ac.kr