ON THE SQUARE OF BROWNIAN DENSITY PROCESS

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Abstract. The square of Brownian density process, \( Q^\lambda \) is defined where \( \lambda \) is a parameter. Applying limit theorems of stochastic integrals w.r.t. martingale measure, we prove a weak limit theorem for \( Q^\lambda \) in \( D_{S'(\mathbb{R}^d)}[0, 1] \).

1. Introduction

Let \( \{X^\alpha, \alpha \in N\} \) be a family of i.i.d. standard Brownian motions in \( \mathbb{R}^d \) with initial distribution given by a Poisson point process \( \Pi^\lambda \) of parameter \( \lambda \). Set for any test function \( \phi \),

\[
(1.1) \quad \eta_t(\phi) = \sum_{\alpha} \phi(X_t^\alpha), \quad \eta_t^2(\phi) = \sum_{\alpha, \beta} \phi(X_t^\alpha)\phi(X_t^\beta)
\]

We will first symmetrize this, then throw away the terms with \( \alpha = \beta \) to get a new process, \( Q^\lambda_t \) of which limit we want to consider.

Let \( \{\xi^\alpha, \alpha \in A\} \) be i.i.d. random variables independent of the \( X^\alpha \) such that \( P\{\xi^\alpha = 1\} = P\{\xi^\alpha = -1\} = \frac{1}{2} \).

Define for any test function \( \psi \) on \( \mathbb{R}^d \times \mathbb{R}^d \)

\[
(1.2) \quad Q^\lambda_t(\psi) = \frac{1}{\lambda} \sum_{\alpha \neq \beta} \xi^\alpha \xi^\beta \psi(X^\alpha, X^\beta).
\]

The study of this process is related to the intersection local times of super processes and is said (by Dynkin and Mandelbaum[2]), and Walsh[5]) to be connected with \( U \)-statistics.

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Define

\begin{align}
\tilde{\eta}_t(\phi) = & \sum_{\alpha \in A} \xi^\alpha \phi(X^\alpha_t) \\
W_t(A) = & \frac{1}{\lambda} \sum_{\alpha} \int_0^t I_A(X_s^\alpha) dX_s^\alpha \\
\tilde{W}_t(A) = & \frac{1}{\lambda} \sum_{\alpha} \xi^\alpha \int_0^t I_A(X_s^\alpha) dX_s^\alpha
\end{align}

Obviously, \(E[\tilde{\eta}(\phi)] = 0\). It is known (see [G], [5]) that if \(\phi \in L^1(R^d)\), then

\begin{align}
E[\eta_t(\phi)] = & \lambda \int \phi(x) dx, \quad \text{and Var } \eta_t(\phi) \sim O(\lambda)
\end{align}

Now, let \(\lambda_n\) be the sequence of parameter values and we consider the corresponding processes, \(\eta^n, \tilde{\eta}^n, W^n, \text{ and } \tilde{W}^n\).

Let \(\tilde{\Pi}^n(dx) = \frac{1}{\sqrt{\lambda_n}}(\Pi^x(dx) - \lambda_n dx)\) be the normalized initial measure. It is known that \(\tilde{\Pi}^n \Rightarrow V^0\), where \(V^0\) is a white noise based on \(R^d\).

Walsh [5] studied the limiting behavior of \(Q^\lambda\) and proved the following theorem in his famous note.

**Theorem 1.1.** [5] The process \(Q^\lambda\) converges weakly in \(D(S'(R^2))$, $[0, 1]$ to a solution of the SPDE

\[
\frac{\partial Q}{\partial t}(x, y) = \frac{1}{2} \Delta Q(x, y) + \eta(x) \nabla_2 \cdot W^0_{ys} + \eta(y) \nabla_1 \cdot W^0_{xs}
\]

\[
Q_0 = V^0 \times V^0
\]

where \(V^0\) and \(W^0\) are independent white noises on \(R^d\) and \(R^d \times R_+\) respectively.

We dare to say that the proof in [5] is somehow wrong and try to give an alternative proof using our previous theorem w.r.t. martingale measure.
DEFINITION 1.1. Let $(R^d, B(R^d), \nu)$ be a $\sigma-$finite measure space. A white noise based on $\nu$ is a random set function $W$ on the sets $A \in B(R^d)$ of finite $\nu-$measure such that

1. $W(A)$ is a $N(0, \nu(A))$ random variable,
2. if $A \cap B = \emptyset$, then $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$.

Let $S'(R^d)$ be the dual of Schwartz space, $S(R^d)$ which is the space of infinitely differentiable functions vanishing at infinity.

The following definition is for the martingale measure established by Walsh\[5].

DEFINITION 1.2. Let $(\Omega, \mathcal{F}, \mu)$ be a filtered space, and $B(\mathbb{R}^d)$ be the Borel $\sigma-$field. Let $M(\cdot, \cdot)$ be a random real-valued function on $\mathbb{R}^d \times \mathbb{R}_+$. $M$ is called an $(\mathcal{F}, \mu)$-martingale measure if it satisfies the following properties.

1. For each $A \in B(\mathbb{R}^d)$, $M(A, \cdot)$ is a $(\mathcal{F}, \mu)$-square integrable martingale and $M(A, 0) = 0$.
2. For any $A, B \in B(\mathbb{R}^d)$ such that $A \cap B = \emptyset$, $M(A \cup B, t) = M(A, t) + M(B, t)$, $\mu$-a.s. for every $t > 0$.
3. For every $t > 0$, $M(\cdot, t)$ is a $\sigma$-finite $L^2$-valued measure in a certain sense. (See in detail \[5\]).

For $A, B \in B(\mathbb{R}^d)$, there exists a unique predictable process, $(M(A), M(B))_t$, such that $M(A, t)M(B, t) - (M(A), M(B))_t$ is a martingale.

DEFINITION 1.3. Let the covariance functional of martingale measure, $M$ be $\text{Cov}_t(A, B) = (M(A), M(B))_t$ where $A, B \in B(\mathbb{R}^d)$. Define a set function $U$ by

$$U(A \times B \times (s, t]) = \text{Cov}_t(A, B) - \text{Cov}_s(A, B)$$

DEFINITION 1.4. A martingale measure is worthy if there is a $\sigma$-finite $L^2$-valued measure $K(\Gamma, \omega)$, $\Gamma \in B(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+)$. $\omega \in \Omega$ such that for fixed $A, B \{K(A \times B \times (0, t], t \geq 0) \text{ is predictable, and } |U(\Gamma)| \leq K(\Gamma)\}$. We call $K$ a dominating measure.

The processes $W^n$ in (1.3) are good examples of martingale measure.
\textbf{Proposition 1.1.} [1] If $\lambda_n \to \infty$, then for each $\phi(x, y) \in S(R^{2d})$, along the appropriate subsequence
\[
\frac{1}{\sqrt{\lambda_n}} \int \phi(x, y) \tilde{\eta}_s^n(dx) \tilde{W}^n(ds, dy) \longrightarrow \int \phi(x, y) \tilde{\eta}_s(dx) \tilde{W}(ds, dy),
\]
where $\tilde{W}$ is a white noise based on Lebesgue measure, and $\tilde{\eta}$ is the $S'(R^d)$-valued Gaussian process.

\section{Main theorem}

The following is our version of Theorem 1.1.

\textbf{Theorem 2.1.} The process \{\(Q_t^\lambda\)} is relatively compact in \(D_{S'(R^d)} [0, \infty)\), and converges to the solution of the following equation:
\begin{equation}
Q_t(\psi) = Q_0(\psi) + \frac{1}{2} \int_0^t Q_s(\Delta \psi) ds + \int_{R^d \times [0, t]} \int_{R^d} \psi(x, y) \tilde{\eta}_s(dy) \tilde{W}(ds, dx) \\
+ \int_{R^d \times [0, t]} \int_{R^d} \psi(x, y) \tilde{\eta}_s(dx) \tilde{W}(ds, dy),
\end{equation}
for every $\psi \in S(R^{2d})$.

\textbf{Proof.} Using Ito’s formula for $Q_t^\lambda(\psi)$ and letting
\[
\chi(x) = (\nabla_1 \psi)(x, x) + (\nabla_2 \psi)(x, x),
\]
we rewrite $Q_t^\lambda(\psi)$ as the following:
\begin{equation}
Q_t^\lambda(\psi) = Q_0^\lambda(\psi) + \frac{1}{2} \int_0^t Q_s^\lambda(\Delta \psi) ds + \frac{1}{\sqrt{\lambda}} \int_{R^d \times [0, t]} \tilde{\eta}_s[\nabla_1 \psi(x, \cdot)] \tilde{W}(dx, ds) \\
+ \frac{1}{\sqrt{\lambda}} \int_{R^d \times [0, t]} \tilde{\eta}_s[\nabla_2 \psi(\cdot, y)] \tilde{W}(dy, ds) - \frac{1}{\sqrt{\lambda}} \int_{R^d \times [0, t]} \chi(x) W(dx, ds)
\end{equation}

Let
\[
p_t(x, x') = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x - x'|^2}{2t}}, \quad G_t(x, x'; y, y') := p_t(x, x')p_t(y, y')
\]
\[
G_t(\psi, x, y) = \int_{R^{2d}} p_t(x, x')p_t(y, y') \psi(x', y') dx'dy'
\]
Then $G$ is the green function on $R^{2d}$ for this problem. Write $Q^\lambda = \tilde{Q}^\lambda + R^\lambda$, where

$$
(2.2) \quad \tilde{Q}_t^\lambda(\psi) = Q_0^\lambda(\psi) + \frac{1}{2} \int_0^t \tilde{Q}_s^\lambda(\Delta \psi) \, ds \\
+ \frac{1}{\sqrt{\lambda}} \int_{R^d \times [0, t]} \tilde{\eta}_s(\nabla_1 \psi(x, \cdot)) \tilde{W}(ds, dx) \\
+ \frac{1}{\sqrt{\lambda}} \int_{R^d \times [0, t]} \tilde{\eta}_s(\nabla_2 \psi(\cdot, y)) \tilde{W}(ds, dy)
$$

$$
R_t^\lambda(\psi) = \frac{1}{2} \int_0^t R_s(\Delta \psi) \, ds - \frac{1}{\sqrt{\lambda}} \int_{R^d \times [0, t]} \chi(x) W(ds, dx).
$$

$$
= -\frac{1}{\lambda} \int_{R^d \times [0, t]} (\nabla_1 G_{t-s}(\psi)(y, y) + \nabla_2 G_{t-s}(\psi)(y, y)) W(ds, dy).
$$

Define a pair of martingale measures on $R^{2d}$ by

$$
M_{1,t}^n(\psi) = \int_{R^d \times [0, t]} (\int_{R^d} \psi(x, y) \tilde{\eta}_s^n(\psi, d(x)) \tilde{W}_\lambda(ds, dy),
$$

$$
M_{2,t}^n(\psi) = \int_{R^d \times [0, t]} (\int_{R^d} \psi(x, y) \tilde{\eta}_s^n(dy)) \tilde{W}_\lambda(ds, dx).
$$

Define

$$
M_{2,t}(\psi) = \int_{R^d \times [0, t]} \int_{1^n} \psi(x, y) \tilde{\eta}_s(dy) \tilde{W}_\lambda(ds, dx)
$$

We write (2.2);

$$
(2.3) \quad \tilde{Q}_t^\lambda(\psi) = Q_0^\lambda(\psi) + \frac{1}{2} \int_0^t \tilde{Q}_s^\lambda(\Delta \psi) \, ds + \frac{1}{\sqrt{\lambda}} M_{1,t}(\nabla_2 \psi) + \frac{1}{\sqrt{\lambda}} M_{2,t}(\nabla_1 \psi).
$$

Now we apply Theorem 5.1 of Walsh[5], replacing $\lambda$ with $\lambda_n$, for every $\psi \in S(R^{2d})$.

$$
(2.4) \quad \tilde{Q}_t^{\lambda_n}(\psi)
$$

$$
= Q_0^{\lambda_n} (G_t \psi) + \frac{1}{\sqrt{\lambda_n}} \int_{R^{2d} \times [0, t]} \nabla_1 G_{t-s}(\psi, x, y) M_2^n(dx, dy, ds)
$$

$$
+ \frac{1}{\sqrt{\lambda_n}} \int_{R^{2d} \times [0, t]} \nabla_2 G_{t-s}(\psi, x, y) M_1^n(dx, dy, ds).
$$
The following argument shows the relative compactness of \( \{Q_t^{\lambda_u}\} \).
Define
\[
V_{i,t}^n(\phi) \equiv \frac{1}{\sqrt{\lambda_n}} \int_{\mathbb{R}^d \times [0,t]} \nabla_j G_{t-s}(\psi, x, y) M_i^n(dx, dy, ds)
\]
for \( i, j = 1, 2 \). Then
\[
(2.5) \\
V_{2,t}^n(\psi) \\
= \frac{1}{\sqrt{\lambda_n}} \int_{\mathbb{R}^d \times [0,t]} \nabla_1 G_{t-s}(\psi, x, y) M_2^n(dx, dy, ds) \\
= \frac{1}{\sqrt{\lambda_n}} \int_{\mathbb{R}^d \times [0,t]} \nabla_1 \left( \int_s^t G_{u-s}(\Delta \psi, x, y) + (\psi, x, y) du \right) M_2^n(dx, dy, ds) \\
= \frac{1}{\sqrt{\lambda_n}} (M_{2,t}^n(\nabla_1 \psi) \\
+ \int_0^t \left[ \int_0^u \int_{\mathbb{R}^d} \nabla_1 G_{u-s}(\Delta \psi, x, y) M_2^n(dx, dy, ds) \right] du)
\]
Recalling that \( M_{2,u}^n(\psi) = \int_0^u \int_{\mathbb{R}^d} \tilde{\eta}_s^n(\psi(x, \cdot)) \tilde{W}^\lambda(ds, dx) \), let
\[
V_{2,t}^n(\psi) = \frac{1}{\sqrt{\lambda_n}} M_{2,t}^n(\nabla_1 \psi(x, y)) \\
+ \frac{1}{\sqrt{\lambda_n}} \int_0^t \left[ \int_{\mathbb{R}^d \times [0,u]} \tilde{\eta}_s^n(\nabla_1 G_{t-s}(\Delta \psi, x, \cdot)) \tilde{W}^\lambda(ds, dx) \right] du.
\]
\[\square\]

**Lemma 2.2.** For any \( \phi, \phi \in L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d), \frac{\partial \phi}{\partial x_i} \in L^2(\mathbb{R}^d) \) for \( i = 1, \ldots, d \),
\[
E[\tilde{\eta}_t^n(\phi)^2] \leq \lambda_n ||\phi||_2^2 + \lambda_n t ||\nabla \phi||_2^2
\]

**Proof.** By Proposition 8.4[5], we have
\[
(2.6) \\
\tilde{\eta}_t^n(\phi) = \tilde{\eta}_0^n(\phi) + \frac{1}{2} \int_0^t \tilde{\eta}_s^n(\Delta \phi) ds + \sqrt{\lambda_n} \int_0^t \int_{\mathbb{R}^d} \nabla \phi(x) \cdot W(dx, ds)
\]
Define $G_t(\phi, y) \equiv \int (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \phi(x) dx$. Then by Theorem 5.1[5], the solution of (2.6) is
\begin{equation}
\tilde{\eta}_t^n(\phi) = \int_{R^d} G_t(\phi, y) \tilde{\Pi}_n^\lambda(dy) + \sqrt{\lambda_n} \int_0^t \int_{R^d} G_{t-s}(\nabla \phi, y) W(dy, ds)
\end{equation}
Since the two terms on the right hand side of (2.7) are orthogonal, we have
\begin{align*}
E[\tilde{\eta}_t^n(\phi)^2] &= E[(\int_{R^d} G_t(\phi, y) \tilde{\Pi}_n^\lambda(dy))^2] + \lambda_n \int_0^t \int_{R^d} |G_{t-s}(\nabla \phi, y)|^2 dy ds \\
&= \lambda_n \int_{R^d} G_t^2(\phi, y) dy + \lambda_n \int_0^t \int_{R^d} |G_{t-s}(\nabla \phi, y)|^2 dy ds
\end{align*}
(2.6)
Note that since $G_t(\phi, y) = f * \phi(y)$ and $\|f_t\|_1 = 1$
\begin{align*}
|G_t(\phi, y)|_2 &\leq \|\phi\|_2 \\
|\nabla G_{t-s}(\phi, y)|_2 &= \|G_{t-s}(\nabla \phi, y)\|_2 \leq \|\nabla \phi\|_2,
\end{align*}
by Schwartz’s inequality. Hence
\[
E[|\tilde{\eta}_t^n(\phi)|^2] \leq \lambda_n \|\phi\|^2_2 + \lambda_n t \|\nabla \phi\|^2_2.
\]
Let $h(x) = (1 + x^2)^{-1}, x \in R^d$, and define an increasing process $k_n$ by
\[
k_n(t) = \int_0^t \int_{R^d} h(y) \tilde{\eta}_t^n(dy) dx ds.
\]
Recall that if $\psi \in S(R^{2d})$ then $G_t(\psi, x, y) \in S(R^{2d})$.

**Lemma 2.3.** For each $T > 0$, $\psi(x, y) \in S(R^{2d})$,
\[
E[\sup_{0 \leq t \leq T} (V_t^n(\psi))^2]
\leq \{8\|\nabla_1 \psi(x, y)(h^2(y)h(x))^{-1}\|_\infty \\
+ 2T^2 \cdot \sup_{0 \leq t \leq T} \|\nabla_1 G_t(\Delta \psi)(h^2(y)h(x))^{-1}\|_\infty \} \cdot \frac{E[k_n(T)]}{\lambda_n}.
\]
Proof. In (2.5)
\[
\sup_t V_t^n(\psi)^2 \leq \frac{1}{\lambda_n} 2 \sup_t (M_{2,t}^n(\nabla_1 \psi))^2 \\
+ 2 \sup_t T(\int_0^t (\int_0^u \int_{R^2a} \nabla_1 G_{u-s}(\Delta \phi) M_2^n(dx,dy,ds)^2 du)
\]
by the Schwartz inequality. By Doob's inequality
\[
E[\sup_{t \leq T} V_t^n(\phi)^2] \leq \frac{1}{\lambda_n} 8E[M_{2,T}^n(\nabla_1(\psi))^2]
\]
(2.8)
\[
+ 2T \int_0^T E[\int_0^u \int_{R^2a} \nabla_1 G_{t-s}(\Delta \psi) M^n(dx,dy,ds)^2 du]
\]
(2.9)
Walsh (p410[5]) shows that the covariance measure for \(M_2^n\) is
\[
\tilde{\eta}_s^n(dy)\tilde{\eta}_s^n(dy')\delta_{x'}(x') dx dx' ds I,
\]
where \(I\) is the identity matrix, hence its dominating measure (defined in Definition1.4) is
\[
K(dx dy dx' dy' ds) = \tilde{\eta}_s(dy)\tilde{\eta}_s(dy')\delta_{x'}(x') dx dx' ds
\]
Then by theorem 2.5[5]
\[
\leq \frac{8}{\lambda_n} E\left[ \int (\int \nabla_1 \psi(x,y)\tilde{\eta}_s^n(dy))^2 dx ds \right]
= \frac{8}{\lambda_n} E[||\nabla_1 \psi(x,y)(h^2(y)h^2(x))^{-1}||_\infty \int_{R^d \times [0,T]} (\int h(y)\tilde{\eta}_s^n(dy))^2 h^2(x) dx dx ds]
\]
\[
\leq \frac{8}{\lambda_n} \int \nabla_1 \psi(x,y)(h^2(y)h^2(x))^{-1}||_\infty E[k_n(T)]
\]
\[= \frac{2T}{\lambda_n} \int_0^T E\left[ \int_0^u \left( \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta \psi) M^n(dx, dy, ds) \right)^2 du \right]
\]
\[= \frac{2T}{\lambda_n} \int_0^T E\left[ \int_0^u \left( \int_{R^{2d}} (\tilde{\eta}_n^u(\nabla_1 G_{u-s}(\Delta \psi)) \tilde{W}^\lambda(dx, ds) \right)^2 \right] du\]

(2.9) \[\leq \frac{2T}{\lambda_n} \sup_{0 \leq t \leq T} \| \nabla_1 G_t(\Delta \psi)(h^2(y)h^2(x))^{-1} \|_{\infty} \cdot \int_0^T E[k_n(u)] du\]

\[\leq 2T^2 \cdot \sup_{0 \leq t \leq T} \| \nabla_1 G_t(\Delta \psi)(h^2(y)h^2(x))^{-1} \|_{\infty} \cdot \frac{E[k_n(T)]}{\lambda_n} \]

\[\square\]

**Lemma 2.4.** For each \( T > 0 \), \( \sup_n \frac{1}{\lambda_n} E[k_n(T)] \leq (\|h\|^2 T + \frac{1}{2} T^2 \|\nabla h\|_2^2) \cdot \|h\|_2 \)

*Proof.*

\[E[k_n(T)] = \int_0^T E\left[ (\int h(y) \tilde{\eta}_s^y dy) \right] \int_{R^{2d}} h(x) dx ds\]
\[\int_0^T E\left[ (\int h(y) \tilde{\eta}_s^y dy) \right] ds \leq \int_0^T (\lambda_n \|h\|_2^2 + \lambda_n s \|\nabla h\|_2^2) ds\]
\[= \lambda_n (\|h\|_2^2 \cdot T + \frac{1}{2} T^2 \|\nabla h\|_2^2),\]

by Lemma 2.2. Therefore,

\[\sup_n \frac{1}{\lambda_n} E[k_n(T)] \leq (\|h\|^2 T + \frac{1}{2} T^2 \|\nabla h\|_2^2) \cdot \|h\|_1. \]

\[\square\]

**Lemma 2.5.** For \( t \leq T \), and \( \psi \in S(R^{2d}) \), \( \{\bar{Q}^{\psi,n}_t(\psi)\} \) is relatively compact.

*Proof.* In (2.5), let

\[U^n_t \equiv \frac{1}{\sqrt{\lambda_n}} \int_0^t \left[ \int_0^\infty \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta \psi)(x, y, s) M^n_2(dx, dy, ds) \right] dv\]

and

\[S_n \equiv \frac{1}{\sqrt{\lambda_n}} \sup_{v \leq T} \left| \int_0^\infty \int_{R^{2d}} \nabla_1 G_{u-s}(\Delta \psi)(x, y, s) M^n_2(dx, dy, ds) \right|\]
Then for \( t \leq T \), for any \( \delta > 0 \), and \( 0 \leq u \leq \delta \),
\[
|U_{t+u}^n - U_t^n| \leq \delta \cdot S_n
\]

By an argument similar to the proof of Lemma 2.3
\[
E[S_n^2] \leq C \cdot \frac{1}{\lambda_n} E[k_n(T)]. \quad \text{for some constant } C
\]

Hence if we let \( \gamma_n(\delta) = (\delta \cdot S_n)^2 \)
\[
E[|U_{t+u}^n - U_t^n|^2 | \mathcal{F}_t^n] \leq E[\gamma_n(\delta) | \mathcal{F}_t^n]
\]
and by Lemma 2.4
\[
\limsup_{\delta \to 0} \frac{1}{n} E[\gamma_n(\delta)] = \limsup_{\delta \to 0} \delta \cdot C \left( \frac{1}{\lambda_n} E[k_n(T)] \right) = 0
\]

It is obvious that \( U_t^n \) satisfies the condition (a) of Th. 3.7.2 in [3], so by Th. 3.8.6 in the book \( U_t^n \), the last row in (2.5), is relatively compact. By the same way, we can show the relative compactness of the third term, \( V_{1,t}^n(\psi) \) in (2.4). Since \( V_{1,t}^n(\psi) \), \( V_{2,t}^n(\psi) \) are continuous and \( \{\tilde{Q}_0^n(\psi)\} \) is relative compact, \( \{\tilde{Q}_t^n(\psi)\} \) is relative compact.

**Proof of Theorem 2.1, continued.** It is known that (Prop. 8.16[5])
\[
(\tilde{\Pi}^n, \tilde{\Pi}^n \times \tilde{\Pi}^n, \tilde{W}^n, \frac{1}{\sqrt{\lambda_n}} \tilde{\eta}^n, R_n^\lambda) \Rightarrow (V^0, V^0 \times V^0, \tilde{W}, \tilde{\eta}, 0)
\]

Since
\[
\frac{1}{\sqrt{\lambda_n}} \int_{R^d} \psi(x, y) \tilde{\eta}_s^n(dy) \Rightarrow \int_{R^d} \psi(x, y) \tilde{\eta}_s(dy) \quad \text{on } D_{C^0(R^d)}[0, T]
\]
for any test function \( \psi(x, y) \in \mathcal{S}(R^{2d}) \), by Proposition 1.1,
\[
\frac{1}{\sqrt{\lambda_n}} M^n_{2,t}(\psi) = \frac{1}{\sqrt{\lambda_n}} \int_{R^d \times [0,t]} \int_{R^d} \psi(x, y) \tilde{\eta}_s^n(dy) \tilde{W}^n(ds, dx)
\Rightarrow \int_{R^d \times [0,t]} \int_{R^d} \psi(x, y) \tilde{\eta}_s(dy) \tilde{W}(ds, dx)
= M_{2,t}(\psi)
\]
Thus the two terms of $V_{2,n}^n(\psi)$ in (2.5) converge.

Furthermore, $Q_0^n$ is known to be $\bar{\Pi}_n \times \bar{\Pi}_n$, and hence $Q_0^n(\psi) \Rightarrow Q_0(\psi)$, where $Q_0 = V^0 \times V^0$. Since $R^n \Rightarrow 0$, $\tilde{Q}^n_t(\psi)$ is relatively compact, and

$$\frac{1}{\sqrt{\lambda_n}} M^n_{1,t}(\nabla_2 \psi) \Rightarrow M_{1,t}(\nabla_2 \psi), \quad \frac{1}{\sqrt{\lambda_n}} M^n_{2,t}(\nabla_1 \psi) \Rightarrow M_{2,t}(\nabla_1 \psi),$$

in (2.3), $\tilde{Q}^n_t(\psi)$ converges to $Q_t(\psi)$ satisfying

$$Q_t(\psi) = Q_0(\psi) + \frac{1}{2} \int_0^t Q_s(\Delta \psi) ds + M_1(\nabla_2(\psi)) + M_2(\nabla_1(\psi)).$$

Since $R^n \Rightarrow 0$, $Q^n_t(\psi) \Rightarrow Q_t(\psi)$, where $Q_t$ is a possible limit of $\tilde{Q}^n_t$ and in fact, the unique solution of (2.0).

References

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