Comm. Korean Math. Soc. **12**(1997), No. 2, pp. 221-236

ON THE MULTIPLE POSITIVE SOLUTIONS TO A QUASILINEAR EQUATION

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ABSTRACT. In this paper we investigate the multiplicity of positive solutions to a quasilinear Neumann problem;

$$\begin{cases} \varepsilon^m div(|\nabla u|^{m-2}\nabla u) - u|u|^{m-2} + u|u|^{p-2} = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

making use of Ljusternik Schnirelmann category theory

1. Introduction

In this paper we investigate the multiplicity of positive solutions to a quasilinear Neumann problem;

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \varepsilon^m div(|\nabla u|^{m-2}\nabla u) - u|u|^{m-2} + u|u|^{p-2} = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $1 < m < N$, $N \geq 2$, $\varepsilon > 0$, $m < p < \frac{mN}{N-m}$ Ω is a smooth bounded domain in $\mathbb{R}^N$ and ν is the unit outer normal vector to $\partial\Omega$. It stems from a chemoactic aggregation model, which was initially studied by C.H.Lin, W.M.Nl, I.Takagi, [6, 7] in case of $m = 2$. In [12], Z-Q. Wang investigated the influence of the topology of Ω on the solutions of (P_ε) in case of $m = 2$.

Received June 2, 1996. Revised March 18, 1997.

1991 Mathematics Subject Classification: 35J70.

Key words and phrases: Lagrange multiplier rule, concentration compactness, Ljusternik Schnirelmann category.

*Supported partly by GARC, KOSEF 95-0701-04-01-3 and BSRI 96-1421, Ministry of Educations.

On the view of variational method, the critical points of the functional $J_\varepsilon : W^{1,m}(\Omega) \to \mathbb{R}$ defined by

$$J_\varepsilon(v) = \frac{1}{m}\int_\Omega \varepsilon^m |\nabla v|^m + |v|^m \, dx - \frac{1}{p}\int_\Omega |v|^p \, dx \tag{1.1}$$

are the solutions of (P_ε).

Let $E_\varepsilon(u) = \int_\Omega \varepsilon^m |\nabla u|^m + |u|^m \, dx$ and $V(u) = \int_\Omega |u|^p dx$ and set $M_1 = \{ u \in W^{1,m}(\Omega) \mid V(u) = 1 \}$. Suppose u is a critical point of E_ε on M_1, then by Lagrange multiplier rule $\tilde{u} = E_\varepsilon(u)^{1/(p-m)} u$ is a solution of $(\mathrm{P}\ _\varepsilon)$. This idea is not void since E_ε is coercieve and weakly lower semicontinuous (see [3]) and hence E_ε is bounded below so that $c_\varepsilon = \min_{u \in M_1} E_\varepsilon(u)$ is achieved , which is a critical value.

Now let $E_\varepsilon^{c_\varepsilon+\delta} = \{u \in M_1 \,|\, E_\varepsilon \le \varepsilon + \delta\}$. We are to show that $\mathrm{cat}(E_\varepsilon^{c_\varepsilon+\delta}) \ge 2\mathrm{cat}(\partial\Omega)$ for all sufficiently small ε and small $\delta(\varepsilon)$ (Lemma 4.2). Then Ljusternik Schnirelmann category theory (see e.g., Theorem 27.2,[8]) shows that there are at least $2\mathrm{cat}(\partial\Omega)$ critical points of E_ε. On the other hand these critical points do not change their sign (Theorem 4.3). Therefore there exist at least $2\mathrm{cat}(\partial\Omega)$ critical points of J_ε, which do not change sign. Consequently there are at least $\mathrm{cat}(\partial\Omega)$ positive solutions since (P_ε) is odd equation in u (Theorem 4.4).

2. Preliminary

There is 1-1 correspondence between the solutions of (P_ε) and the solutions of

$$(\mathrm{P}'_\varepsilon) \qquad \begin{cases} div(|\nabla u|^{m-2}\nabla u) - u|u|^{m-2} + u|u|^{p-2} = 0 & \text{in } \Omega_{1/\varepsilon} \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega_{1/\varepsilon} \end{cases}$$

where $\Omega_{1/\varepsilon} = \{ x \mid \varepsilon x \in \Omega \}$. In fact, the change of variable $u(x) = v(\varepsilon x)$ for each v solving (P_ε) gives the 1-1 correspondence. We associate $(\mathrm{P}'_\varepsilon)$ with the following functional

$$\tilde{E}_{1/\varepsilon}(v) = \int_{\Omega_{1/\varepsilon}} |\nabla v|^m + |v|^m dx, \qquad v \in M_1(\Omega_{1/\varepsilon}),$$

where $M_1(\Omega_{1/\varepsilon}) = \{u \in W^{1,m}(\Omega_{1/\varepsilon}) | \int_{\Omega_{1/\varepsilon}} |u|^p dx = 1\}$. Now define for each $v \in M_1(\Omega)$,

$$\sigma(v)(x) \overset{def}{=} \varepsilon^{N/p} v(\varepsilon x) .$$

It is easy to see that $\sigma(v) \in M_1(\Omega_{1/\varepsilon})$.

LEMMA 2.1. *For any* $v \in M_1(\Omega)$, $\tilde{E}_{1/\varepsilon}(\sigma(v)) = \varepsilon^{-N(p-m)/p} E_\varepsilon(v)$ *and*

$$\min_{v \in M_1(\Omega_{1/\varepsilon})} \tilde{E}_{1/\varepsilon}(v) = \varepsilon^{-N(p-m)/p} \min_{v \in M_1(\Omega)} E_\varepsilon(v) .$$

PROOF. It suffices to show the first equality.

$$\begin{aligned} \tilde{E}_{1/\varepsilon}(\sigma(v)) &= \int_{\Omega_{1/\varepsilon}} |\nabla(\varepsilon^{N/p} v(\varepsilon x))|^m + |\varepsilon^{N/p} v(\varepsilon x)|^m dx \\ &= \varepsilon^{Nm/p} \int_\Omega (\varepsilon^m |\nabla v|^m + v^m) \varepsilon^{-N} dy \\ &= \varepsilon^{-N(p-m)p} E_\varepsilon(v) . \end{aligned}$$

□

For each $S \subset \mathbb{R}^N$, Define $M_\alpha(S) = \{ u \in W^{1,m}(S) \,|\, \int_S |v|^p dx = \alpha \}$, and

(i) $$m(r, \alpha) = \min\{ \int_{\Omega_r} |\nabla u|^m + |u|^m dx \mid u \in M_\alpha(\Omega) \},$$

(ii) $$m(+, \alpha) = \min\{ \int_{\mathbb{R}^N_+} |\nabla u|^m + |u|^m dx \mid u \in M_\alpha(\mathbb{R}^N_+) \},$$

(iii) $$m(\infty, \alpha) = \min\{ \int_{\mathbb{R}^N} |\nabla u|^m + |u|^m dx \mid u \in M_\alpha(\mathbb{R}^N) \} ,$$

where $\mathbb{R}^N_+ = \{ x = (x_1, \cdots, x_N) \mid x_N \geq 0 \}$. We remark that there exists positive, radially symmetric, nonincreasing solution ω of the quasilinear elliptic equation

(I) $$-div(|\nabla u|^{m-2} |\nabla u) + u|u|^{m-2} - u|u|^{p-2}$$

so that ω minimizes the energy $J(v)$, i.e, the functional on $W^{1,m}(\mathbb{R}^N)$ corresponding to the equation (I), which is defined by

$$J(v) = \frac{1}{m}\int_{\mathbb{R}^N} |\nabla v|^m + |v|^m \, dx - \frac{1}{p}\int_{\mathbb{R}^N} |v|^p \, dx .$$

Here ω minimizes the enrgy J in the sense that $J(v) \geq J(\omega)$ for any nontrival v solving (I). Confer [1, 2] for details in case m=2 and [11] in geneal case. We call ω the ground state solution. From these definitions, it is easy to observe the following lemma.

LEMMA 2.2. *For $r \geq 1$, $\alpha > 0$, we have*

(1) $m(\infty, 1) = \int_{\mathbb{R}^N} |\nabla\tilde{\omega}|^m + |\tilde{\omega}|^m dx$, *where ω is a ground state solution of (I) and $\tilde{\omega} = \omega/\|\omega\|_{L_p(\mathbb{R}^N)}$*

(2) $m(r,\alpha) = \alpha^{m/p} m(r,1)$ *(r may be $+$ or $+\infty$),*

(3) $m(+\infty, 2) = 2m\,(+,1)$.

PROOF. Let u be a minimizer of $E(u) = \int_{\mathbb{R}^N} |\nabla u|^m + |u|^m dx$ on $M := \{u \in W^{1,m}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^p = 1\}$. Then $\tilde{u} = E(u)^{1/(p-m)} u$ is again a solution of (I). It is easy to see that $J(\tilde{u}) = (p-m)/pmE(u)^{p/(p-m)}$. On the other hand, if ω is a ground state solution of (I), $\tilde{\omega} := \omega/\|\omega\|_{L_p} \in M$ and $E(\tilde{\omega}) = (\int_{\mathbb{R}^N} |\omega|^p)^{-m/p} E(\omega)$. Since $\int_{\mathbb{R}^N} \omega^p dx = E(u)$ and $E(u) = pm/(p-m)J(\omega)$, and since ω is a energy minimizing solution, we have

$$E(\tilde{\omega}) = \left(\frac{pm}{p-m} J(\omega)\right)^{(p-m)/p} \leq \left(\frac{pm}{p-m} J(\tilde{\omega})\right)^{(p-m)/p} = E(u) .$$

Hence $\tilde{\omega}$ is a minimizer of $E(u)$ on M and this shows (1). (2) and (3) are easy to see. □

3. Some asymptotic estimates

In this section we give the asymptotic estimates of m_ε as $\varepsilon \to 0$. We denote a minimizer corresponding to $m_\varepsilon = \min_{u \in M_1} E_\varepsilon(u)$ by u_ε. Let $v_\varepsilon = \sigma(u_\varepsilon)$. In the following, we also write v_ε to denote $\chi_{\Omega_{1/\varepsilon}} v_\varepsilon$, where χ_A is a characteristic function.

PROPOSITION 3.1. *As* $\varepsilon \to 0$,

$$m_\varepsilon = \varepsilon^{N(p-m)/p}\left(m(+,1)+o(1)\right).$$

To show this proposition, we need to define some functions. Let η be smooth, nonincreasing function defined on $[0,+\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$, $\eta(t) = 0$ for $t \ge 2$ and $|\eta'| \le 2$. We also define $\eta_r(t) = \eta(t/r)$. Let ρ be a (to be chosen properly) positive constant. Define for each $y \in \partial\Omega$,

$$\psi_\varepsilon(y)(x) = \eta_\rho(|x-y|)\,\omega\left(\frac{x-y}{\varepsilon}\right) \qquad x \in \Omega, \tag{3.1}$$

and define $\varphi_\varepsilon(y) = \frac{\psi_\varepsilon(y)}{\|\psi_\varepsilon\|_{L^p}}$. Note then that $\varphi_\varepsilon(y) \in M_1(\Omega)$.

PROOF. It is obvious that $m_\varepsilon \le \varepsilon^{N(p-m)/p}(m(+,1)+o(1))$, if we take the function φ_ε into consideration. So it remains to show $m_\varepsilon \ge \varepsilon^{N(p-m)/p}(m(+,1)+o(1))$. Suppose to the contrary that there exists α, $0 \le \alpha \le m(+,1)$ satisfying

$$\liminf_{\varepsilon\to 0} \varepsilon^{-N(p-m)/p} m_\varepsilon = \alpha < m(+,1) \tag{3.2}$$

i.e., there exists $\varepsilon_n \to 0$ and $u_{\varepsilon_n} \in M_1(\Omega)$ (in the following, we write u_n for the sake of simplicity) such that

$$m_{\varepsilon_n} = E_{\varepsilon_n}(u_n), \qquad \lim_{n\to\infty} \varepsilon_n^{-N(p-m)/p} E_{\varepsilon_n}(u_n) = \alpha.$$

To lead a contradiction, we investigate the concentration compactness of v_n $(= \chi_{\Omega_{1/\varepsilon_n}}\sigma(u_n))$. For the concentration compactness theory, confer [9]. We show that $(v_n)_{n\ge1}$ does not provide vanishing-case or dichotomy-case i.e., there happens compactness with $(v_n)_{n\ge1}$. First, we assume the following two lemmas whose proofs are deferred for the time being. □

LEMMA 3.2. *Suppose that for any* $R > 0$, $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)} v_n^p\, dx = 0$, *where* v_n *is as above, then we obtain* $\lim_{n\to\infty}\int_{\mathbb{R}^N} v_n^p\, dx = 0$.

LEMMA 3.3. *$(v_n)_{n\geq 1}$ does not make dichotomy.*

Lemma 3.2 shows that $(v_n)_{n\geq 1}$ is not vanishing. Due to above two lemmas we can assume the compactness of the sequence $(v_n)_{n\geq 1}$ that is, there exists $y_n \in \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R > 0$ satisfying

$$\int_{y_n + B_R} \chi_n \,|\, v_n \,|^p \, dx \geq 1 - \varepsilon.$$

We assume the following lemma whose proof is deferred for the time being.

LEMMA 3.4. *There exists $\bar{c} > 0$ such that* $\operatorname{dist}(y_n, \partial\Omega_n) \leq \bar{c}$.

By virtue of Lemma 3.4, there exists $t_n \in \partial\Omega$ such that

$$\operatorname{dist}(y_n, \tilde{q}_n) \leq \bar{c}, \text{ where } \tilde{q}_n = t_n/\varepsilon_n \in \partial\Omega_n.$$

Now let's choose a unitary matrix U_n such that $\tilde{\Omega}_n = U_n(\Omega_n - \tilde{q}_n)$ has y^N as inner normal direction of $\partial\Omega_n$ at the origin. Then it is easy to see the following lemma.

LEMMA 3.5. *For any fixed $R_1 > 0$, $U_n(\Omega_n - \tilde{q}_n) \cap B_{r_1}(0)$ converges to $B^+_{R_1}(0) = \{ x \in B_{R_1}(0) \,|\, x^N \geq 0 \}$ in the following sense; for any $\delta > 0$, there exist K_1 and there exists n_δ such that*

$$\{ x \in B^+_{R_1}(0) \,|\, x^N \geq \delta \} \subset U_n(\Omega_n - \tilde{q}_n) \qquad \text{for } n \geq n_\delta \tag{3.3}$$

$$|\{ x \in U_n(\Omega_n - \tilde{q}_n) \cap B_{R_1}(0) \,|\, x^N \leq \delta \}| \leq K_1 \delta. \tag{3.4}$$

Finally we assume the following lemma.

LEMMA 3.6. *$\| v_n \|_{L^\infty(\Omega_n)}$ is uniformly bounded.*

Now by Lemma 3.4, if we put $R_1 = R + \bar{c}$, we have $\int_{B_{R_1}(\tilde{q}_n)} v_n^p \, dx \geq 1 - \varepsilon$. From (3.3), (3.4) and Lemma 3.6, for given $\varepsilon > 0$, there exist $\delta_1 > 0$ and n_{δ_1} such that

$$\int_{\{ x \in B^+_{R_1}(0) \,|\, x^N \geq \delta_1 \}} v_n^p (U_n^{-1} x + \tilde{q}_n) \, dx \geq 1 - 2\varepsilon \qquad \text{for } n \geq n_{\delta_1}. \tag{3.5}$$

Let $x_{\delta_1} = (0, \cdots, 0, \delta_1)$ and

$$\tilde{v}_n(x) = \eta_{R_1}(|\, x\,|) v_n(U_n(x + x_{\delta_1}) + \tilde{q}_n), \quad x \in R^N_+.$$

Then we obtain

$$\int_{\Omega_n} (|\, \nabla v_n\,|^m + v_n^m)\, dx - \int_{R^N_+} (|\, \nabla \tilde{v}_n\,|^m + |\, \tilde{v}_n\,|^m)\, dx \geq -2\varepsilon. \tag{3.6}$$

Now for n large enough, by (3.5) and (3.6)

$$\begin{aligned}
\alpha &= \lim_{n\to\infty} \int_{\Omega_n} |\, \nabla v_n\,|^m + v_n^m\, dx \\
&\geq \lim_{n\to\infty} \int_{R^N_+} |\, \nabla \tilde{v}_n\,|^m + |\, \tilde{v}_n\,|^m\, dx - 2\varepsilon \\
&\geq (1-\varepsilon)^{m/p} m(+,1) - 2\varepsilon.
\end{aligned}$$

Hence $A \geq m(+,1)$, which contradicts our assumption (3.2).

Now we give the proofs of the lemmas used in proposition 3.1.

Proof of Lemma 3.2. Let $\bar{p} = 1 + \frac{m-1}{m} p$, $\quad 1 < \bar{p} < p$ and follow the proof of Lemma 2.2 of [12]. Then we get a positive constant C independent of n and ε_n, satisfying for all $\alpha, 1 < \alpha < \frac{N}{N-1}$,

$$\int_{\Omega_n} |v_n|^{\alpha \bar{p}} dx \leq C \varepsilon_n^{\alpha - 1}. \tag{3.7}$$

When $\frac{1}{m} - \frac{1}{p} < \frac{1}{N}$, take $\alpha = \frac{p}{\bar{p}}$. Then $\alpha < \frac{N}{N-1}$, which shows our assertion. If $\frac{1}{m} - \frac{1}{p} \geq \frac{1}{N}$, take any $\alpha \in (0, \frac{N}{N-1})$ and let $A_n = \{x \in \Omega_n \,|\, v_n(x) \geq 1\}$ and $B_n = \Omega_n - A_n$. Then $|A_n| \to 0$ and therefore,

$$\begin{aligned}
\int_{\Omega_n} |v_n|^p dx &= \int_{A_n} |v_n|^p dx + \int_{B_n} |v_n|^p dx \\
&\leq C^p |A_n| + \int_{B_n} |v_n|^{\alpha \bar{p}} dx \to 0
\end{aligned}$$

□

PROOF OF LEMMA 3.3. Suppose not. Then

$$Q_n(t) = \sup_{y \in \mathbb{R}^N} \int_{y+B_t} v_n^p \, dx, \quad Q(t) = \lim_{n \to \infty} Q_n(t)$$

satisfy

$$\lim_{t \to \infty} Q(t) = \lambda, \quad \text{with } \ 0 < \lambda < 1.$$

Hence for any $\varepsilon > 0$, there exists $R_0 > 0$ such that $Q(R_0) \geq \lambda - \varepsilon/4$ and there exist $y_n \in \mathbb{R}^N$, $n_0 \in \mathbb{N}$ such that

$$Q_n(R_0) = \int_{y_n + B_{R_0}} v_n^p \, dx \ \geq \ \lambda - \varepsilon/2, \qquad \text{for all } n \geq n_0.$$

Also by definition, there exists a sequence $R_n \to \infty$ such that

$$Q_n(2R_n) \ \leq \ \lambda + \varepsilon/2.$$

Let $\xi = 1 - \eta$ and set

$$v_n^1(x) = \chi_n(x)\eta\left(\frac{|\, x - y_n \,|}{R_0}\right) v_n(x),$$
$$v_n^2(x) = \chi_n(x)\xi\left(\frac{|\, x - y_n \,|}{R_n}\right) v_n(x).$$

Then we obtain

$$\lambda - \varepsilon/2 \leq \int_{y_0 + B_{R_0}} v_n^p \, dx \leq \int_{\mathbb{R}^N} (v_n^1)^p \, dx \leq \int_{y_n + B_{2R_n}} (v_n)^p \, dx \leq \lambda + \varepsilon/2.$$

Hence

$$\left| \int_{\mathbb{R}^N} |\, v_n^1 \,|^p \, dx - \lambda \right| \leq \frac{1}{2}\varepsilon, \qquad \left| \int_{\mathbb{R}^N} |\, v_n^2 \,|^p \, dx - (1 - \lambda) \right| < \frac{1}{2}\varepsilon.$$

Thus

$$
\left\{
\begin{array}{lll}
\int_{\mathbb{R}^N} |\, v_n^1 \,|^p \, dx & = \lambda + \varepsilon_n^1 & |\, \varepsilon_n^1 \,| \le \varepsilon \, , \\
\int_{\mathbb{R}^N} |\, v_n^2 \,| \, dx & = (1-\lambda) + \varepsilon_n^2 & |\, \varepsilon_n^2 \,| \le \varepsilon \, .
\end{array}
\right.
\tag{3.8}
$$

Moreover, we can choose R_0 so large for fixed $\varepsilon > 0$ that

$$
\begin{aligned}
\int_{\mathbb{R}^N} (|\, \nabla v_n \,|^m + |\, v_n \,|^m) \, dx - \int_{\Omega_n} (|\, \nabla v_n^1 \,|^m + |\, v_n^1 \,|^m) \, dx \\
- \int_{\Omega_n} (|\, v_n^2 \,|^m + |\, v_n^2 \,|^m) \, dx \ge -2\varepsilon.
\end{aligned}
\tag{3.9}
$$

Now by (3.7), (3.8), (3.9) and Lemma 3.2,

$$
\begin{aligned}
\alpha &= \lim_{n\to\infty} m\left(\frac{1}{\varepsilon_n}, 1\right) \\
&= \lim_{n\to\infty} \int_{\Omega_n} |\, \nabla v_n \,|^m + v_n^m \\
&\ge \lim_{n\to\infty} \left(\int_{\Omega_n} |\, \nabla v_n^1 \,|^m + |\, v_n^1 \,|^m \, dx + \int_{\Omega_n} |\, \nabla v_n^2 \,|^m + |\, v_n^2 \,|^m \, dx \right) - 2\varepsilon \\
&\ge \lim_{n\to\infty} \left(m\left(\frac{1}{\varepsilon_n}, \lambda - 2\varepsilon\right) + m\left(\frac{1}{\varepsilon_n}, (1-\lambda) - 2\varepsilon\right)\right) - 2\varepsilon \\
&= \lim_{n\to\infty} \left((\lambda - 2\varepsilon)^{m/p} m\left(\frac{1}{\varepsilon_n}, 1\right) + (1 - \lambda - 2\varepsilon)^{m/p} m\left(\frac{1}{\varepsilon_n}, 1\right)\right) - 2\varepsilon \\
&= \left((\lambda - 2\varepsilon)^{m/p} + (1 - \lambda - 2\varepsilon)^{m/p} \right) \alpha - 2\varepsilon.
\end{aligned}
$$

Letting $\varepsilon \to 0$, we have

$$
\alpha \ge \left(\lambda^{m/p} + (1-\lambda)^{m/p} \right) \alpha > \alpha,
$$

which leads a contradiction. □

PROOF OF LEMMA 3.4. Suppose it is not true i.e., dist $(y_n, \partial\Omega_n) \to +\infty$. Since

$$\int_{y_n+B_R} \chi_n \,|\, v_n \,|^p \, dx \geq 1 - \varepsilon. \tag{3.10}$$

we may assume $y_n \in \Omega_n$. Now for a fixed $\varepsilon > 0$, there exists $R > 0$ such that (3.10) holds. We may assume $y_n + B_{2R} \subset \Omega_n$ for sufficiently large n. Put $w_n = \eta\left(\frac{x-y_n}{R}\right) v_n(x)$. Then $w_n(x) \in W^{1,m}(\mathbb{R}^N)$. Also as noted above, we can choose so large R satisfying

$$\int_{\Omega_n} |\, \nabla v_n \,|^m + |\, v_n \,|^m \, dx - \int_{\mathbb{R}^N} |\, \nabla w_n \,|^m + w_n^m \, dx \geq -2\varepsilon.$$

Hence with $\lambda_n = \int_{\mathbb{R}^N} |\, w_n \,|^p \, dx$ we have

$$\begin{aligned}\int_{\Omega_n} |\, \nabla v_n \,|^m + v_n^m \, dx &\geq \int_{\mathbb{R}^N} |\, \nabla w_n \,|^m + w_n^m \, dx - 2\varepsilon \\ &\geq m(\infty, \lambda_n) - 2\varepsilon \\ &\geq \lambda^{m/p} m(\infty, 1) - 2\varepsilon \\ &\geq (1-\varepsilon)^{m/p} 2^{1-m/p} m(+, 1) - 2\varepsilon\end{aligned}$$

Then $A \geq m(+, 1)$ which is against the assumption (3.2). □

PROOF OF LEMMA 3.6. Note that $\left(E_\varepsilon(u_n)^{1/(p-m)} u_n\right)_{n\geq 1}$ satisfy $(\mathrm{P}_{\varepsilon_n})$ and hence they are uniformly bounded and that from Lemma 2.1,

$$E_{\varepsilon_n}(u_n) = \varepsilon_n^{N(p-m)/p} \tilde{E}_{1/\varepsilon_n}(v_n).$$

Since $v_n(x) = \varepsilon_n^{N/p} u_n(\varepsilon_n x)$,

$$\begin{aligned}E_{\varepsilon_n}(u_n)^{1/(p-m)} u_n(x) &= \left(\varepsilon_n^{N(p-m)/p}\right)^{1/(p-m)} \left(\tilde{E}_{1/\varepsilon_n}(v_n)\right)^{1/(p-m)} \varepsilon_m^{-N/p} v_n \\ &= \tilde{E}_{1/\varepsilon_n}(v_n)^{1/(p-m)} v_n.\end{aligned}$$

Since $\tilde{E}_{1/\varepsilon_n}(v_n)$ tends to $\alpha + o(1)$, $(v_n)_{n\geq 1}$ is uniformly bounded. □

Looking closely into the proof of proposition 3.1, we easily obtain the following result.

LEMMA 3.7. *Suppose* (ε_n) *be a sequence of positive numbers which decreases to* 0. *If* $v_n \in M_1(\Omega_{1/\varepsilon_n})$ *satisfies*

$$\int_{M_1(\Omega_{1/\varepsilon_n})} |\nabla v_n|^m + |v_n|^m \, dx \to m(+,1)$$

as $n \to \infty$, *then there exists a subsequence of* v_n *and a positive constant* $\bar{c}$ *independent of* n *such that for any* δ, *there exists* R *satisfying*

$$\int_{B_R(y_n)\cup\Omega_{1/\varepsilon_n}} |v_n|^p dx \geq 1-\delta \quad \text{and} \quad dist(y_n, \partial\Omega_{1/\varepsilon_n}) \geq \bar{c}\,.$$

PROPOSITION 3.8. $\varphi_\varepsilon : \partial\Omega \to M_1(\Omega)$ *is a continuous function and*

$$E_\varepsilon\left(\varphi_\varepsilon(y)\right) = \varepsilon^{N(p-m)/p}(m(+,1) + o(1))$$

uniformly on $\partial\Omega$

PROOF. It is similar as the proof for Proposition 2.2,[12]. □

4. Multiplicity of positive solution of $(\mathbf{P}_\varepsilon)$

Now we introduce a mass-centre, $c(u)$ of $u \in M_1(\Omega)$ by means of $L_p(\Omega)$, that is $c(u) = \int_\Omega |u|^p x \, dx$. It is obvious that $c(u)$ is continuous.

PROPOSITION 4.1. *Let* $\rho > 0$ *be given in the proof of Proposition 3.1. Then there exist* $\delta_1 > 0$ *and* $\varepsilon_1 > 0$ *such that for all* ε, $0 < \varepsilon < \varepsilon_1$

$$c(u) \in N_\rho(\partial\Omega), \qquad \text{for all } u \in E_\varepsilon^{m_\varepsilon + \delta_1 \varepsilon^{N(p-m)/p}}$$

PROOF. Suppose the assertion is not true. Then there exist $\varepsilon_n \to 0$ and $\delta_n \to 0$ satisfying

$$\text{(4.1)} \qquad u_n \in E_{\varepsilon_n}^{m_{\varepsilon_n} + \delta_n \varepsilon_n^{N(p-m)/p}} \quad \text{but} \quad c(u_n) \notin N_\rho(\partial\Omega).$$

Then from (4.1), we have

$$m_{\varepsilon_n} \varepsilon_n^{-N(p-m)/p} \leq \varepsilon_n^{-N(p-m)/p} E_\varepsilon(u_n) \leq m_{\varepsilon_n} \varepsilon_n^{-N(p-m)/p} + \delta_n\,.$$

By Proposition 3.1, we obtain

$$\varepsilon_n^{-N(p-m)/p} E_\varepsilon(u_n) \to m(+,1) \quad \text{as} \quad n \to \infty.$$

Let $v_n = \sigma(u_{\varepsilon_n})$ i.e., $\varepsilon_n^{N/p} u_{\varepsilon_n}(\varepsilon_n x)$. Then $v_n(x) \in M_1(\Omega_{1/\varepsilon_n})$ and

$$\int_{\Omega_{1/\varepsilon_n}} |\nabla v_n|^m + |v_n|^m dx \to m(+,1).$$

For simplicity we write $u_{\varepsilon_n} = u_n$ and $\Omega_{1/\varepsilon_n} = \Omega_n$. By Lemma 3.7, we can choose $y_n \in R^N$ and a constant $\bar{c} > 0$ independent of n such that for each $\delta > 0$ there exists $R_\delta > 0$ satisfying

$$\int_{B_R(y_n)\cap\Omega_n} |v_n|^p dx \geq 1 - \delta$$

with $\text{dist}(y_n, \partial\Omega_n) < \bar{c}$. Thus, there is $q_n \in \partial\Omega$ such that $\text{dist}(y_n, q_n/\varepsilon_n) \leq \bar{c}$. Then there exists $R_1 > 0$ such that for sufficiently large n,

$$\int_{B_{R_1}(q_n/\varepsilon_n)\cap\Omega_n} |v_n|^p dx \geq 1 - \delta.$$

We may assume $q_n \to q \in \partial\Omega$ and $z_n = c(u_n) \to 0$. Note that

$$\begin{aligned}
\int_{\Omega_n} x|v_n|^p dx &= \int_{\Omega_n} \varepsilon_n^N u_n^p(\varepsilon_n x) x dx \\
&= \frac{1}{\varepsilon_n} \int_\Omega u_n^p(x) x dx \\
&= \frac{z_n}{\varepsilon_n}.
\end{aligned}$$

Since $q \neq 0$, we may assume $q_1 > 0$, where $q = (q^1, \cdots, q^N)$. Let $\gamma = \min\{ y^1 \mid y \in \partial\Omega \}$. Then for n large enough, we have

$$\begin{aligned}
\frac{z_n^1}{\varepsilon_n} &= \int_{\Omega_n} x^1 |v_n| dx \\
&= \int_{B_{R_1}(q_n/\varepsilon_n)} |v_n|^p x^1 dx + \int_{\Omega_n \setminus B_{R_1}(q_n/\varepsilon_n)} |v_n|^p x^1 dx \\
&\geq (\frac{z_n^1}{\varepsilon_n} - R_1)(1-\delta) - \frac{|\gamma|}{\varepsilon_n}\delta.
\end{aligned}$$

Therefore,

$$z_n^1 \geq (q_n^1 - R_1\varepsilon_n)(1-\delta) - |\gamma|\delta \to q^1 > 0,$$

which leads a contradiction. □

LEMMA 4.2. *Let $\delta_1 > 0$ be given as in the above Proposition 4.1. For any δ, $0 < \delta < \delta_1$, there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon < \varepsilon_\delta$ such that*

$$cat\,(E_\varepsilon^{m_\varepsilon + \delta\varepsilon^{N(p-m)/p}}) \;\geq\; 2cat\,(\partial\Omega).$$

PROOF. Let δ_1 be chosen as above. Then there exists $\varepsilon_\delta > 0$ such that for ε, $0 < \varepsilon < \varepsilon_\delta$, it follows that

$$\varphi_\varepsilon \,:\, \partial\Omega \to E_\varepsilon^{m_\varepsilon+\delta_\varepsilon} \cap M_1^+$$
$$c \,:\, E_\varepsilon^{m_\varepsilon+\delta_\varepsilon} \to N_\rho(\partial\Omega), \qquad \delta_\varepsilon = \varepsilon^{N(p-m)/p}\delta$$

are both continuous maps, where $M_1^+ = \{u \in M_1(\Omega) \mid u \geq 0 \quad a.e.\}$. Note further that

$$c \circ \varphi_\varepsilon(y) \in N_{2\rho}(y), \qquad \text{for all } y \in \partial\Omega. \tag{4.2}$$

Let $P : N_{2\rho}(\partial\Omega) \to \partial\Omega$ be a homotopy with $P|_{\partial\Omega} = id_{\partial\Omega}$. Set $A_+ = E_\varepsilon^{m_\varepsilon+\delta_\varepsilon} \cap M_1^+$ and assume cat $(A_+) = k$. Then there exist k closed and contractible subsets of A_+, say $A_1, A_2, \cdots, A_k$ such that

$$A_+ = \bigcup_{i=1}^{k} A_i.$$

Let $B_i = \varphi_\varepsilon^{-1}(A_i) \subset \partial\Omega, \quad i = 1, \cdots, k$. Then $\bigcup_{i=1}^k B_i = \partial\Omega$. Therefore

$$\text{cat}\,(\partial\Omega) \leq \sum_{i=1}^{k} \text{cat}_{\partial\Omega}(B_i)\,.$$

We assert that every non-empty B_i is contractible. Since A_i is contractible in A_+, there exists a contraction $H_i \in C([0,\, 1] \times A_i,\, A_+)$ such that

$$\begin{cases} H_i(0,\zeta) \;=\; \zeta & \text{for all } \zeta \in A_i \\ H_i(1,\zeta) \;=\; \zeta_i \in A_+ & \text{for all } \zeta \in A_i\,. \end{cases}$$

Define a map $\overline{H} : [0, 2]\times \to \partial\Omega$ by

$$\overline{H}(t,y) = \begin{cases} P(y - t[y - c \circ H_i(0, \cdot) \circ \varphi_\varepsilon(y)]) & 0 \le t \le 1, \ y \in B_i \\ P \circ c \circ H_i(t-1, \cdot) \circ \varphi_\varepsilon(y) & 1 \le t \le 2, \ y \in B_i \end{cases}$$

By (4.2), $\overline{H}$ is well-defined and it is easy to see that $\overline{H}(0, y) = y$ for all $y \in B_i$ and $\overline{H}(2, y) = P \circ c(\zeta_i)$. This shows $\mathrm{cat}_{\partial\Omega}(B_i) = 1$ and therefore we obtain

$$\mathrm{cat}\,(\partial\Omega) \le k = \mathrm{cat}\,(A_+).$$

In a similar manner with $-\varphi_\varepsilon$, we have for $A_- = E_\varepsilon^{m_\varepsilon+\delta_\varepsilon} \cap M_1^-(\Omega)$,

$$\mathrm{cat}\,(A_-) \ge \mathrm{cat}\,(\partial\Omega)\,.$$

Since A_- and A_+ are disjoint in $E_\varepsilon^{m_\varepsilon+\delta_\varepsilon}$, we have

$$\mathrm{cat}\,(E_\varepsilon^{m_\varepsilon+\delta_\varepsilon}) \ge 2\mathrm{cat}(\partial\Omega).$$

□

LEMMA 4.3. *If u is a critical point of E_ε on M_1 with $E_\varepsilon(u) < 2^{(p-m)/p}m_\varepsilon$, then u does not change sign.*

PROOF. Suppose $u = u_+ + u_-$ with $u_\pm \ne 0$. Then $m_\varepsilon \le E_\varepsilon(u/\|u_+\|_{L^p})$, that is

$$\|u_+\|_{L^p}^m m_\varepsilon \le \int_\Omega \varepsilon^m |\nabla u_+|^m + u_+^m \, dx. \tag{4.3}$$

Since $E_\varepsilon(u)^{1/p-m}u$ is a solution of (P $_\varepsilon$), we obtain by taking u_+ as a test function

$$\int_\Omega \varepsilon^m |\nabla u_+|^m + u_+^m dx = E_\varepsilon(u) \int_\Omega u_+^p \, dx. \tag{4.4}$$

By (4.3), (4.4) and the given condition, we have $\|u_+\|_{L^p}^p > \frac{1}{2}$ and similarly $\|u_-\|_{L^p}^p > \frac{1}{2}$. Then

$$1 = \|u\|_{L^p}^p = \|u_+\|_{L^p}^p + \|u_-\|_{L^p}^p > \frac{1}{2} + \frac{1}{2} = 1\,,$$

which is absurd. □

THEOREM 4.4. *Suppose $p \geq 2$. Then for all sufficiently small ε, (P $_\varepsilon$) has at least cat $(\partial\Omega)$ distinct non-constant positive solutions.*

PROOF. Note that $m_\varepsilon = \varepsilon^{N(p-m)/p}(m(+,1) + o(1))$. For $\delta_1 > 0$ in Proposition 4.1, choose $\delta_0 > 0$ satisfying

$$\delta_0 < \min(\delta_1, (2^{p-m/p} - 1)m(+,1)).$$

Since $m(+,1) = m_\varepsilon \varepsilon^{-N(p-m)/p} + o(1)$ as $\varepsilon \to 0$, we can find $\varepsilon_0 (< \varepsilon_1)$ satisfying for all $\varepsilon < \varepsilon_0$.

$$\delta_0 \; < \; (2^{p-m/p} - 1)m_\varepsilon \varepsilon^{-N(p-m)/p},$$

equivalently,

$$m_\varepsilon + \delta_0 \varepsilon^{N(p-m)/p} < 2^{(p-m)/p} m_\varepsilon . \tag{4.5}$$

For this δ_0, we can choose $\bar{\varepsilon}(< \varepsilon_0)$ satisfying for all $0 < \varepsilon < \bar{\varepsilon}$,

$$\text{cat}\,(E_\varepsilon^{m_\varepsilon + \delta_\varepsilon}) \geq 2\text{cat}\,(\partial\Omega), \qquad \delta_\varepsilon = \delta_0 \varepsilon^{N(p-m)/p}$$

By (4.6) and Ljusternik-Schnirelmann category Theorem, there exist at least 2cat $(\partial\Omega)$ critical points. By (4.5) and Lemma 4.3, these critical points do not change sign. Consequently there exist at least cat $(\partial\Omega)$ positive solutions since (P$_\varepsilon$) is odd. □

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