Comm. Korean Math. Soc. **12**(1997), No. 2, pp. 251-256

SYSTEMS OF DERIVATIONS ON BANACH ALGEBRAS

EUN HWI LEE

ABSTRACT. We show that a strong system of derivations $\{D_0, D_1, , \cdots D_m\}$ on a commutative Banach algebra A is contained in the radical of A if it satisfies one of the following conditions for separating spaces ;

(1) $\mathfrak{S}(D_i) \subseteq rad(A)$ and $\mathfrak{S}(D_i) \subseteq K_{D_i}(rad(A))$ for all i, where

$$K_{D_i}(rad(A)) = \{x \in rad(A) : \text{for each } \ m \geq 1, \ D_i^m(x) \in rad(A)\}.$$

(2) $\mathfrak{S}(D_i^m) \subseteq rad(A)$ for all i and m.

(3) $\overline{x\mathfrak{S}(D_i)} = \mathfrak{S}(D_i)$ for all i and all nonzero x in $rad(A)$.

1. Introduction

A system of derivations of order m on a complex Banach algebra A is a set of $m+1$ linear operators $\{D_0, D_1, \cdots, D_m\}$ such that for $x, \ y \in A$ and $k = 0, \ 1, \ \cdots, \ m$

$$D_k(xy) = \sum_{j=0}^{k} \binom{k}{j} (D_j x)(D_{k-j} y).$$

A system of derivations $\{D_0, D_1, \cdots, D_m\}$ is strong if D_0 is an identity operator, and bounded if D_i is bounded for each $n = 1, 2, \cdots, m$ [2]. Note that in a strong system of derivations $\{D_0, D_1, \cdots, D_m\}$, D_1 is a derivation. We denote by $rad(A)$ the radical of a Banach algebra A. A system of derivations $\{D_0, D_1, \cdots, D_m\}$ on a Banach algebra A maps into its radical if $D_i(A) \subseteq rad(A)$ for $1 \leq i \leq m$.

Received August 16, 1996. Revised December 24, 1996.

1991 Mathematics Subject Classification: 46H99.

Key words and phrases: derivation, higher derivation, Banach algebra.

This paper was supported by research fund of Jeon-Ju University, 1996.

I. M. Singer and J. Wermer [6] proved that a bounded derivation on a commutative Banach algebra A maps into $rad(A)$ and M. P. Thomas [7] extended the Singer-Wermer theorem to arbitrary, not necessarily bounded derivations on commutative Banach algebras.

Results for non-commutative Banach algebras are contained in [4]. R. J. Loy [3] obtained that the result of the automatic continuity of derivations on semisimple Banach algebras can be extended to a strong system of derivations.

In this paper, we deal with a strong system of discontinuous derivations. It doesn't seem provable by the same method as for the proof of a derivation that every strong system of derivations on a commutative Banach algebra maps into its radical [2].

2. Systems of derivations

If S is a linear operator from a Banach space X into a Banach space Y, then the separating space $\mathfrak{S}(S)$ of S is defined by

$$\mathfrak{S}(S) = \{y \in Y : \text{there are } \; x_n \to 0 \text{ with } Sx_n \to y\}.$$

$\mathfrak{S}(S)$ gives us a measure of continuity of S since the closed graph theorem shows that S is continuous if and only if $\mathfrak{S}(S) = \{0\}$.

LEMMA 1. *Let $\{v_0, v_1, \cdots, v_m\}$ be a strong system of derivations on a commutative Banach algebra A. Then for each i and n, $\mathfrak{S}(D_i^n) \subseteq rad(A)$ if and only if $\phi \circ D_i^n$ is continuous for each $\phi \in \Phi_A$, where Φ_A is the set of all multiplicative linear functionals on A.*

PROOF. Note that for each i and n, $\mathfrak{S}(\phi \circ D_i^n) = \overline{\phi(\mathfrak{S}(D_i^n))}$ for $\phi \in \Phi_A$ [5]. Suppose that for each i and n, $\mathfrak{S}(D_i^n) \subseteq rad(A)$. Since $rad(A)$ is the intersection of all multiplicative linear functionals on A, for all $\phi \in \Phi_A$ $\mathfrak{S}(\phi \circ D_i^n) = \{0\}$. □

LEMMA 2. *Let A be a commutative Banach algebra and $\{D_0, D_1,, \cdots, D_m\}$ a strong system of derivations on A. Suppose that I is a prime ideal of A and $D_l(A) \subseteq I$ for each $l = 1, 2, \cdots, i-1$. Let*

$$K_{D_i}(I) = \{x \in I : \text{ for each } m \geq 1, D_i^m(x) \in I\}.$$

Then $K_{D_i}(I)$ is a prime ideal.

In particular, $D_i(P) \subseteq P$ for every minimal prime ideal P.

PROOF. Take $D_i^0(x) = x \quad (x \in A)$. Since for each $x \in K_{D_i}(I)$ and $y \in A$

$$D_i^m(xy) = \sum_{j_1=0}^{i} \sum_{j_2=0}^{i} \cdots \sum_{j_m=0}^{i} \binom{i}{j_1}\binom{i}{j_2} \cdots \binom{i}{j_m}(D_{j_m} D_{j_m-1} \cdots D_{j_0}(x))(D_{i-j_m} D_{i-j_{m-1}} \cdots D_{i-j_0}(y)),$$

$D_i^m(xy) \in I$. Thus $K_{D_i}(I)$ is an ideal.

Let $a_1,\ a_2 \in K_{D_i}(I)$ for some $a_1,\ a_2 \in A$ and let $a_1 \notin K_{D_i}(I)$. We must show that $a_2 \in K_{D_i}(I)$. Since $a_1 \notin K_{D_i}(I)$ there is an integer $t \geq 0$ such that for each $s < t$, $D_i^s(a_1) \in I$ but $D_i^t(a_1) \notin I$. Now by induction on r we prove that for each $r \geq 0$, $D_i^r(a_2) \in I$. Then $a_2 \in K_{D_i}(I)$.

For $r = 0$, note that

$$\begin{aligned} D_i^t(a_1a_2) &= \sum_{j_1=0}^{i} \cdots \sum_{j_t=0}^{i} \binom{i}{j_1} \cdots \binom{i}{j_t}(D_{j_t} \cdots D_{j_0}(a_1))(D_{i-j_t} \cdots D_{i-j_0}(a_2)) \\ &= D_o^t(a_1)a_2 + \sum_{j_1=0}^{i-1} \cdots \sum_{j_t=0}^{i-1} \binom{i}{j_1} \cdots \binom{i}{j_t}(D_{j_t} \cdots D_{j_0}(a_1))(D_{i-j_t} \cdots D_{i-j_0}(a_2)) \end{aligned}$$

and $a_1,\ a_2 \in K_{P_i}(I)$. Then $D_i^t(a_1)a_2 \in I$. Since I is a prime ideal and $D_i^t(a_1) \notin I,\ a_2 \in I$.

Let $D_i^0 a_2, \cdots, D_i^{r-1} a_2 \in I$. We proceed the proof for r. Note that

$$\begin{aligned} D_i^{t+r}(a_1a_2) &= \sum_{j_1=0}^{i} \cdots \sum_{j_{t+r}=0}^{i} \binom{i}{j_1} \cdots \binom{i}{j_{t+r}}(D_{j_{t+r}} \cdots D_{j_0}(a_1)) (D_{i-j_{t+r}} \cdots D_{i-j_0}(a_2)) \\ &= N_{t+r} D_i^{t+r}(a_1)a_2 + N_{t+r-1} D_i^{t+r-1}(a_1) D_i(a_2) + \cdots \\ &\quad + N_t D_i^t(a_1) D_i^r(a_2) + \cdots + N_0 a_1 D_i^{t+r}(a_2) + \sum_{j_1=1}^{i-1} \cdots \end{aligned}$$

$$\sum_{j_{t+r}=1}^{i-1} \binom{i-1}{j_1} \cdots \binom{i-1}{j_{t+r}} (D_{j_{t+r}} \cdots D_{j_0}(a_1))(D_{i-j_{t+1}} \cdots D_{i-j_0}(a_2))$$

for some $N_i (i = 0, 1, 2, \cdots, t + r)$.

Since $a_1, a_2 \in K_{D_i}(I)$, $D_i^{t+r}(a_1 a_2) \in I$. From the assumption and induction hypothesis, $N_t D_i^t(a_1) D_i^r(a_2) \in I$ for some integer N_t and so $D_i^r(a_2) \in I$. Hence we have the result.

Now note that $K_{D_i}(P) \subseteq P$ and $D(K_{D_i}(P)) \subseteq K_{D_i}(P)$. By minimality of P, $D_i(P) \subseteq P$. □

LEMMA 3. *Let $\{D_0, D_1, \cdots, D_m\}$ be a strong system of derivations on a commutative Banach algebra A and $D_l(A) \subseteq rad(A)$ for $l = 1, 2, \cdots, i-1$. Then $D_i(A) \subseteq rad(A)$ if*

$$\mathfrak{S}(D_i) = \mathfrak{S}(D_i) \cap rad(A) \subseteq K_{D_i}(rad(A)).$$

PROOF. Denote $K = K_{D_i}(rad(A))$. If $\mathfrak{S}(D_i) = \mathfrak{S}(D_i) \cap rad(A) \subseteq K$ then the operator $\overline{D_i}$ from A into $A/\overline{K}$ defined by $\overline{D_i}(a) = D_i(a) + \overline{K}$ $(a \in A)$, where $\overline{K}$ denotes the closure of K, is continuous, by [5, Lemma 1.4]. By Lemma 2, $D_i(K) \subseteq K$. Thus it follows that $\overline{D_i}(K) = 0$ and so $\overline{D_i}\ \overline{K} = 0$.

Now define a linear map $\overline{\overline{D_i}}$ from $A/\overline{K}$ into $A/\overline{K}$ by $\overline{\overline{D_i}}(x + \overline{K}) = D_i(x) + \overline{K}$ $(x \in A)$. Then $\overline{\overline{D_i}}$ is continuous and so $\overline{\overline{D_i}}(A/\overline{K}) \subseteq rad(A)/\overline{K}$[4].

Since $\overline{K} \subseteq rad(A)$, $D_i(A) \subseteq rad(A)$. □

LEMMA 4. *Let $\{D_0, D_1, \cdots, D_m\}$ be a strong system of derivations on a commutative Banach algebra A and $D_l(A) \subseteq rad(A)$ for $l = 1, 2, \cdots, i-1$. Then for every m, $\mathfrak{S}(D_i^m) \subseteq rad(A)$ if and only if $D_i(A) \subseteq rad(A)$.*

PROOF. If $D_i(A) \subseteq rad(A)$, then $\mathfrak{S}(D_i^m) \subseteq rad(A)$ because $rad(A)$ is a closed ideal. Conversely, if $\mathfrak{S}(D_i^m) \subseteq rad(A)$ for every m, by Lemma 1, $\phi \circ D_i^m$ is continuous for every $\phi \in \Phi_A$. Let $x \in \mathfrak{S}(D_i) \cap rad(A)$. Then $\phi \circ D_i^m(x) = 0$ for every m and so $D_i^m(x) \in rad(A)$. Therefore $x \in K_{D_i}(rad(A))$. By Lemma 3, $D_i(A) \subseteq rad(A)$. □

By Lemma 3, 4 and induction we have the following result.

THEOREM 5. *Let A be a commutative Banach algebra and $\{D_0, D_1, \cdots, D_m\}$ be a system of derivations on A.*

(1) *If $\mathfrak{S}(D_i) \subseteq rad(A)$ and $\mathfrak{S}(D_i) \leq K_{D_i}(rad(A))$ for all i, then $D_i(A) \subseteq rad(A)$ for all i.*
(2) *If $\mathfrak{S}(D_i^m) \subseteq rad(A)$ for all i and m, then $D_i(A) \subseteq rad(A)$ for all i.*

THEOREM 6. *Let A be a commutative Banach algebra. If $\{D_0, D_1, \cdots, D_m\}$ is a strong system of derivations such that $\overline{x\mathfrak{S}(D_i)} = \mathfrak{S}(D_i)$ for each $i = 1, 2, \cdots, n$ and all nonzero $x \in rad(A)$, then $D_i(A) \subseteq rad(A)$ for all $i = 1, 2, \cdots, n$.*

PROOF. By Thomas's theorem [7], $D_1(A) \subseteq rad(A)$. Suppose that $D_l(A) \subseteq rad(A)$ for all $l = 1, 2, \cdots, i-1$. By hypothesis, $\mathfrak{S}(D_i) \subseteq rad(A)$ and so $\phi \circ D_i$ is continuous for all $\phi \in \Phi_A$. Suppose that $\phi \in \Phi_A$ and $y \in \phi \circ D_i(\mathfrak{S}(D_i))$. Let $x \in \mathfrak{S}(D_i)$ with $y = \phi \circ D_i(x)$. Since $\overline{x\mathfrak{S}(D_i)} = \mathfrak{S}(D_i)$, there is a sequence $\{y_n\}$ in $\mathfrak{S}(D_i)$ such that $x = \lim_{n\to\infty} xy_n$. Then

$$\begin{aligned} y &= \phi \circ D_i(x) \\ &= \lim_{n\to\infty} \phi \circ D_i(xy_n) \\ &= \lim_{n\to\infty} \phi\Big(\sum_{j=0}^{i} \binom{i}{j} (D_j x)(D_{i-j} y_n)\Big) \\ &= \lim_{n\to\infty} \phi\Big(xD_i(y_n) + (D_i x)y_n + \sum_{j=1}^{i-1} \binom{i}{j} (D_i x)(D_{i-j} y_n)\Big) \\ &\subseteq \phi(rad(A)) = \{0\}. \end{aligned}$$

Thus $\phi \circ D_i(\mathfrak{S}(D_i)) = \{0\}$ for all $\phi \in \Phi_A$. Since $\overline{\phi \circ D_i(\mathfrak{S}(D_i))} = \mathfrak{S}(\phi \circ D_i^2) = \overline{\phi(\mathfrak{S}(D_i)^2)}$ for all $\phi \in \Phi_A$, $\mathfrak{S}(D_i^2) \subseteq rad(A)$. Suppose that $\mathfrak{S}(D_i^r) \subseteq rad(A)$ for each $r \leq m$ and $y \in \phi \circ D^m(\mathfrak{S}(D_i))$ for $\phi \in \Phi_A$. Then there are $x \in \mathfrak{S}(D_i)$ and a sequence $\{y_n\}$ in $\mathfrak{S}(D_i)$ such that

$y = \phi \circ D_i^m(x)$ and $x = \lim_{n\to\infty} xy_n$. Therefore

$$\begin{aligned} y &= \phi \circ D_i^m(x) \\ &= \lim_{n\to\infty} \phi \circ D_i^m(xy_n) \\ &= \lim_{n\to\infty} \sum_{j_1=0}^{i} \cdots \sum_{j_m=0}^{i} \binom{i}{j_1} \cdots \binom{i}{j_m} \phi(D_{j_m} \cdots D_{j_1}(x))\phi(D_{i-j_m} \\ &\quad \cdots D_{i-J_1}(y)) \\ &\subseteq \phi(rad(A)) = \{0\}. \end{aligned}$$

By induction, $\mathfrak{S}(D_i^m) \subseteq rad(A)$ for all $m \geq 1$. By Lemma 5, $D_i(A) \subseteq rad(A)$. □

References

[1] F. Gulick, *Systems of derivations*, Trans. Amer. Math. Soc. **149** (1970), 465-488.

[2] K. W. Jun and Y. W. Lee, *The image of a continuous strong higher derivation is contained in the radical*, Bull. Korean. Math. Soc. **33** (1996), 229-232.

[3] R. J. Loy, *Continuity of higher derivations*, Proc. Amer. Math. Soc. **37** (1973), 505-510.

[4] M. Mathieu, *Where to find the image of a derivation*, Banach Center Pub **30** (1994), 237-249.

[5] A. M. Sinclair, *Automatic continuity of linear operators*, London Math. Soc., Lecture Note Series **21** (1976).

[6] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260-264.

[7] M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of Math.(2) **128** (1988), 435-460.

Department of Mathematics
Jeon-Ju University
Chon-Ju 560-759, Korea