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POSITIVELY EXPANSIVE ENDOMORPHISMS ON SUBSHIFTS OF FINITE TYPE

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ABSTRACT. It is shown that if S is a positively expansive endomorphism on a one–sided mixing SFT (X, T), then (X, S) is conjugate to a one–sided mixing SFT, and the Parry measures of (X, T) and (X, S) are identical.

1. Introduction

Recently Blanchard and Maass [3] obtained remarkable results on positively expansive one–sided cellular automata. They considered a positively expansive endomorphism S on a one–sided full shift (X, T), and proved that (X, S) is a mixing subshift of finite type which is shift equivalent to a full shift and that the Parry measures of (X, T) and (X, S) are identical.

In this article, it is attempted to generalize their results to endomorphisms on one–sided subshifts of finite type. In Section 3, we prove that the Parry measure of a transitive subshift of finite type is invariant under a surjective endomorphism. Also we reprove a theorem which was originally proved by Nasu [4]: any positively expansive endomorphism on a transitive subshift of finite type is a subshift of finite type. The main results appear in Section 4. There we show that any positively expansive endomorphism on a mixing subshift of finite type is mixing and that the Parry measures coincide. Finally in Section 5, we provide some examples.

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During the preparation of this article we were informed that M. Boyle and Fiebigs [2] independently proved our results. It seems that their proof is rather abstract.

2. Preliminaries

Let $\mathcal{A}$ be a finite set equipped with the discrete topology. Then the product space $\mathcal{A}^{\mathbb{N}}$ is compact Hausdorff, and hence metrizable. The map

$$\sigma : \mathcal{A}^{\mathbb{N}} \ni \langle a_i \rangle_{i=0}^{\infty} \mapsto \langle a_{i+1} \rangle_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{N}} \qquad (\langle a_i \rangle_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{N}})$$

is called the *shift map*, and the system $(\mathcal{A}^{\mathbb{N}}, \sigma)$ is called the *full $\mathcal{A}$–shift.* If X is a closed σ–invariant subset of $\mathcal{A}^{\mathbb{N}}$ for some finite set $\mathcal{A}$, then the system $(X, \sigma|_X)$ is said to be a *subshift.*

The concept of the subshift can be described in the following invariant way. For a partition $\mathcal{A}$ of a set X let $\pi = \pi_{\mathcal{A}}$ be the natural projection from X onto $\mathcal{A}$, defined by $x \in \pi(x)$ for $x \in X$. Let (X, d) be a compact metric space and let T be a continuous map from X onto itself. We say that (X, T) is a subshift if there is a partition $\mathcal{A}$ of X, called an *alphabet* for T, satisfying

(i) $\#\mathcal{A} < \infty$,
(ii) for all $a \in \mathcal{A}$, a is both open and closed in X, and
(iii) for all $x, y \in X$ if $\pi(T^i x) = \pi(T^i y)$ for all $i \in \mathbb{N}$, then $x = y$.

If $\mathcal{A}$ is an alphabet for T, then a compactness argument shows that

$$\bigcup_{n=0}^{\infty} \bigvee_{i=0}^{n} T^{-i}\mathcal{A}$$

is a basis for the topology of X. Let us denote the image of X under the map

$$X \ni x \mapsto \langle \pi(T^i x) \rangle_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{N}}$$

by $X_{\mathcal{A}}$. Then $X_{\mathcal{A}}$ is a closed σ–invariant subset of $\mathcal{A}^{\mathbb{N}}$, and (X, T) and $(X_{\mathcal{A}}, \sigma)$ are conjugate systems. A subshift (X, T) is said to be of *finite type*, or simply an SFT, if there is an alphabet $\mathcal{A}$, called a *Markov* alphabet for (X, T), such that

$$X_{\mathcal{A}} = \{\xi \in \mathcal{A}^{\mathbb{N}} : \xi_i \cap T^{-1}\xi_{i+1} \neq \emptyset \ \forall i \in \mathbb{N}\}.$$

It is clear that for any alphabet $\mathcal{A}$ if $\mathcal{A}$ refines a Markov alphabet, then $\mathcal{A}$ is also Markov.

A subshift (X, T) is called *transitive* if, for every ordered pair a, b of nonempty open sets in X, there is an $n > 0$ for which $a \cap T^{-n}(b) \neq \emptyset$; a subshift (X, T) is called *mixing* if, for every ordered pair a, b of nonempty open sets in X, there is an $N > 0$ such that $a \cap T^{-n}(b) \neq \emptyset$ for all $n \geq N$.

The following lemma was first proved by W. Parry [5].

LEMMA 2.1. *Let (X, T) be a subshift. Then (X, T) is an SFT if and only if T is an open map.*

PROOF. If (X, T) is an SFT and $\mathcal{A}$ is a Markov alphabet for T, then it is clear that $\sigma : X_{\mathcal{A}} \to X_{\mathcal{A}}$ is open.

Conversely, let $\mathcal{A}_0$ be an alphabet for (X, T), and assume that T is open. Let $a \in \mathcal{A}_0$. Then $T(a)$ is an open subset of X which is also compact. Since $\bigcup_{n=0}^{\infty} \bigvee_{i=0}^{n} T^{-i}\mathcal{A}_0$ is a basis for the topology of X, there is a positive integer $N(a)$ such that the open compact set $T(a)$ is a union of members of $\bigvee_{i=0}^{N(a)} T^{-i}\mathcal{A}_0$. Since $\#\mathcal{A} < \infty$, there is a positive integer N such that for all $a \in \mathcal{A}_0$, $T(a)$ is a union of members of $\bigvee_{i=0}^{N} T^{-i}\mathcal{A}_0$. Let $\mathcal{A} = \bigvee_{i=0}^{N} T^{-i}\mathcal{A}_0$, then it is easy to see that for all $a \in \mathcal{A}$

$$T(a) = \bigcup \{b \in \mathcal{A} : a \cap T^{-1}b \neq \emptyset\},$$

from which our assertion follows. □

If (X, T) is an SFT with Markov alphabet $\mathcal{A}$, we define $M : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$ as follows. For $a, b \in \mathcal{A}$

$$M(a, b) = \begin{cases} 1, & \text{if} \quad a \cap T^{-1}b \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix M is called the *transition matrix* for the system $(X, T, \mathcal{A})$. In this case, it is clear that (X, T) is conjugate to the topological Markov subshift (X_M, σ_M).

3. Endomorphisms on transitive SFTs

Suppose that (X, T) is a transitive SFT. Then there is a unique T-invariant probability measure μ, called the *Parry measure* for (X, T), such that the measure theoretic entropy of T with respect to μ is equal to the topological entropy of T [1]. If $\mathcal{A}$ is a Markov alphabet and if M is the transition matrix, then for $w = a_0 \cap T^{-1}(a_1) \cap \cdots \cap T^{-n}(a_n) \in \bigvee_{i=0}^{n} T^{-i}\mathcal{A}$, $w \neq \emptyset$, $\mu(w)$ is given by

$$(*) \qquad \mu(w) = L(a_0)R(a_n)\lambda^{-n},$$

where λ is the Perron eigenvalue of M, and $L, R : \mathcal{A} \to (0,\infty)$ are left and right eigenvectors of M corresponding to the eigenvalue λ which are normalized so that

$$\sum_{a\in\mathcal{A}} L(a)R(a) = 1.$$

A continuous map $S : X \to X$ which commutes with T is called an *endomorphism* on (X,T).

THEOREM 3.1. *Let S be an endomorphism on a transitive SFT (X,T). Then S is surjective if and only if $\mu = \mu S^{-1}$, where μ is the Parry measure of (X,T).*

PROOF. Suppose that S is not onto. Then $X \setminus S(X)$ is non–empty and open. From $(*)$, we see that $\mu(X \setminus S(X)) > 0$, and hence μ is not S–invariant.

Conversely, assume that S is onto. Let $\mathcal{A}$ be a Markov alphabet for T, and let $\mathcal{A}_n$ denote $\bigvee_{i=0}^{n} T^{-i}\mathcal{A}$. Then it is enough to show that $\mu(w) = \mu(S^{-1}(w))$ for all $w \in \bigcup_{n=0}^{\infty} \mathcal{A}_n$. Since S is continuous, there is a positive integer N such that $\mathcal{A}_N$ refines $S^{-1}\mathcal{A}$. Then, since $ST = TS$, $\mathcal{A}_{n+N}$ refines $S^{-1}\mathcal{A}_n$ for each $n = 0,1,2,\cdots$. Therefore for any $n = 0,1,2,\cdots$ and for each $w \in \mathcal{A}_{n+N}$, there exists $\Phi(w) \in \mathcal{A}_n$ such that $w \subset S^{-1}(\Phi(w))$. This defines a map Φ from $\bigcup_{n=N}^{\infty} \mathcal{A}_n$ to $\bigcup_{n=0}^{\infty} \mathcal{A}_n$. Define $C : \bigcup_{n=0}^{\infty} \mathcal{A}_n \to (0,\infty)$ as follows:

$$C(w)\mu(w) = \mu\left(S^{-1}(w)\right) \qquad (w \in \bigcup_{n=0}^{\infty} \mathcal{A}_n).$$

Since $S^{-1}(w)$ is the disjoint union of $\Phi^{-1}(w)$ for each $w \in \bigcup_{n=0}^{\infty} \mathcal{A}_n$, $(*)$ shows that the range of the function C is finite. Hence there are w_m, and w_M in $\bigcup_{n=0}^{\infty} \mathcal{A}_n$ such that for all $w \in \bigcup_{n=0}^{\infty} \mathcal{A}_n$ we have

$$C(w_m) \leq C(w) \leq C(w_M).$$

For $w \in \mathcal{A}_n$ define Pred (w) and Succ (w) as follows:

$$\text{Pred}\,(w) = \{a \in \mathcal{A} : a \cap T^{-1}w \neq \emptyset\},$$

and

$$\text{Succ}\,(w) = \{a \in \mathcal{A} : w \cap T^{-n-1}a \neq \emptyset\}.$$

Then for each $w \in \mathcal{A}_n$ we have

$$T^{-1}(w) = \bigcup_{a \in \text{Pred}\,(w)} a \cap T^{-1}(w) \quad \text{and} \quad w = \bigcup_{a \in \text{Succ}\,(w)} w \cap T^{-n-1}(a).$$

Since the above unions are disjoint,

$$\begin{aligned}
C(w_m)\mu(w_m) &= \mu\left(S^{-1}(w_m)\right) \\
&= \mu\left(T^{-1}S^{-1}(w_m)\right) \\
&= \mu\left(S^{-1}T^{-1}(w_m)\right) \\
&= \sum_{a \in \text{Pred}\,(w_m)} \mu\left(S^{-1}(a \cap T^{-1}w_m)\right) \\
&= \sum_{a \in \text{Pred}\,(w_m)} C(a \cap T^{-1}w_m)\mu(a \cap T^{-1}w_m) \\
&\geq \sum_{a \in \text{Pred}\,(w_m)} C(w_m)\mu(a \cap T^{-1}w_m) \\
&= C(w_m)\mu(T^{-1}w_m) \\
&= C(w_m)\mu(w_m).
\end{aligned}$$

Hence we have $C(w_m) = C(a \cap T^{-1}w_m)$ for all $a \in \text{Pred}\,(w_m)$. Similarly, if $w_M \in \mathcal{A}_n$, then $C(w_M) = C(w_M \cap T^{-n-1}a)$ for all $a \in \text{Succ}\,(w_M)$. Since T is transitive, we conclude that $C(w_m) = C(w_M)$, from which our assertion is obvious. □

Remark. Theorem 3.1 generalizes Proposition 2.1 of [3].

DEFINITION 3.2. An endomorphism S on a subshift (X,T) is called *positively expansive* provided that there is a $\delta > 0$ such that for all $x, y \in X$ if $d(S^i(x), S^i(y)) < \delta$ for $i = 0, 1, 2, \cdots$, then $x = y$. In this case, δ is called an *expansive constant* of S.

Suppose that S is a positively expansive endomorphism on a transitive SFT (X,T) with expansive constant δ. Then S is bounded to one, so that (X,T) and $(S(X),T)$ have the same topological entropy. Since (X,T) is transitive, it follows that $X = S(X)$, i.e. S is onto. Let $\mathcal{A}$ be an alphabet for (X,T). Then there is a positive integer N such that for all $w \in \bigvee_{i=0}^{N} T^{-i}\mathcal{A}$ the diameter of w is less than δ, so that $\bigvee_{i=0}^{N} T^{-i}\mathcal{A}$ is an alphabet for the system (X,S), i.e. (X,S) is a subshift. Moreover Theorem 3.1 implies that the Parry measure μ of (X,T) is S-invariant, and consequently the measure theoretic entropy of S with respect to μ does not exceed the topological entropy of S. In Section 4, we will show that if S is a positively expansive endomorphism on a mixing SFT (X,T) then the Parry measures of (X,T) and (X,S) coincide.

LEMMA 3.3. *Suppose that S is a positively expansive endomorphism on (X,T) with expansive constant δ. Let $\mathcal{A}$ be an alphabet for (X,T) such that the diameter of each element in $\mathcal{A}$ is less than δ. Then for each positive integer M there is a positive integer N such that*

$$\bigvee_{i=0}^{M} T^{-i}\mathcal{A} \preccurlyeq \mathcal{A} \vee S^{-1}\left(\bigvee_{i=0}^{N} T^{-i}\mathcal{A}\right),$$

where $\mathcal{A} \preccurlyeq \mathcal{B}$ means that $\mathcal{B}$ refines $\mathcal{A}$.

PROOF. It is enough to prove the assertion when $M = 1$. A compactness argument shows that there is a positive integer K such that if $d(S^i(x), S^i(y)) < \delta$ for all $i = 0, \cdots, K$ then x and y lie in the same element of $\mathcal{A} \vee T^{-1}\mathcal{A}$. Since S is continuous, there is an $\eta > 0$ such that if $d(S(x), S(y)) < \eta$ then $d(S^i(x), S^i(y)) < \delta$ for all $i = 1, \cdots, K$. Choose a positive integer N such that for all $w \in \bigvee_{i=0}^{N} T^{-i}\mathcal{A}$, the diameter of w is less than η. This completes the proof. □

THEOREM 3.4. *Let (X,T) be a transitive SFT and S be a positively expansive endomorphism on (X,T). Then (X,S) is an SFT.*

PROOF. By Lemma 2.1, it suffices to show that S is open. Let $\mathcal{A}$ be a Markov alphabet for (X,T) such that the diameter of each element in $\mathcal{A}$ is less than an expansive constant of S. Then it is enough to show that for all $a \in \mathcal{A}$, $S(a)$ is open. To get a contradiction, assume that $S(a_0)$ is not open for some $a_0 \in \mathcal{A}$. Then there is a point $x_0 \in a_0$ such that $S(x_0)$ is not an interior point of $S(a_0)$. From Lemma 3.3, there is a positive integer N such that $\mathcal{A} \vee S^{-1}(\bigvee_{i=0}^{N} T^{-i}\mathcal{A})$ refines $\mathcal{A} \vee T^{-1}\mathcal{A}$.

Let $S(x_0) \in u_0 \in \bigvee_{i=0}^{N-1} T^{-i}\mathcal{A}$. Then $a_0 \cap S^{-1}(u_0) \neq \emptyset$. Since $S(x_0)$ is not an interior point of $S(a_0)$, and since $\mathcal{A}$ is Markov for T, there is a $v_0 \in \bigvee_{i=0}^{N} T^{-i}\mathcal{A}$ such that $v_0 \neq \emptyset$, $v_0 \subset u_0$, and $a_0 \cap S^{-1}(v_0) = \emptyset$. Since S is onto, there is an $a_1 \in \mathcal{A}$ such that $a_1 \cap S^{-1}(v_0) \neq \emptyset$. It is obvious that $a_0 \neq a_1$. Since T is transitive, we can find a positive integer K such that

$$\left(a_1 \cap S^{-1}(v_0)\right) \cap T^{-K}\left(a_0 \cap S^{-1}(u_0)\right) \neq \emptyset.$$

Take a point x_1 in the above nonempty set, and let u_1 and w_1 be such that

$$S(x_1) \in u_1 \in \bigvee_{i=0}^{N+K-1} T^{-i}\mathcal{A}, \qquad x_1 \in w_1 \in \bigvee_{i=0}^{K} T^{-i}\mathcal{A}.$$

Now consider v_1 given by

$$v_1 = u_1 \cap (T^{-K}v_0) \in \bigvee_{i=0}^{N+K} T^{-i}\mathcal{A}.$$

Since $\mathcal{A}$ is Markov for T, and since $N \geq 1$, we have $v_1 \neq \emptyset$. Since $v_1 \subset v_0$, it is clear that $a_0 \cap S^{-1}(v_1) = \emptyset$. On the other hand, we have

$$a_1 \cap S^{-1}(v_1) \subset a_1 \cap S^{-1}(u_1) \subset w_1,$$

because

$$\bigvee_{i=0}^{K} T^{-i}\mathcal{A} \preccurlyeq \mathcal{A} \vee S^{-1}\left(\bigvee_{i=0}^{N+K-1} T^{-i}\mathcal{A}\right).$$

Hence we must have $a_1 \cap S^{-1}(v_1) = \emptyset$. (If $x \in a_1 \cap S^{-1}(v_1)$, then $x \in w_1$, so that $T^K(x) \in a_0 \cap S^{-1}(v_0) = \emptyset$.) Since S is onto, there is an $a_2 \in \mathcal{A}$, $a_2 \neq a_i$ for $i = 0, 1$, such that $a_2 \cap S^{-1}(v_1) \neq \emptyset$.

Continuing this way, we eventually obtain a nonempty neighborhood v such that $a \cap S^{-1}(v) = \emptyset$ for all $a \in \mathcal{A}$, which is a contradiction. $\square$

REMARK. This theorem is a generalization of Theorem 3.3 in [3]. It is not too hard to prove that if S is a positively expansive endomorphism on a transitive two–sided SFT (X, T), then (X, S) is a one–sided SFT, while we deal with only those on a one–sided SFT in Theorem 3.4.

4. Endomorphisms on mixing SFTs

Let $\mathcal{A}_0$ be a Markov alphabet for T which is also an alphabet for S. Then Theorem 3.4 implies that there is a positive integer N such that $\mathcal{A} = \bigvee_{i=0}^{N} S^{-i}\mathcal{A}_0$ is a Markov alphabet for S. Since any alphabet which refines a Markov alphabet is again Markov, we see that $\mathcal{A}$ is a common Markov alphabet for both S and T.

THEOREM 4.1. *Let (X, T) be a mixing SFT and S be a positively expansive endomorphism on (X, T). Then (X, S) is a mixing SFT.*

PROOF. Let $\mathcal{A}$ be a common Markov alphabet for both S and T. Then we must show that there is a positive integer K such that for all $k \geq K$ and for all $a, b \in \mathcal{A}$

$$a \cap S^{-k}(b) \neq \emptyset.$$

Since (X, T) is mixing, there is a positive integer L such that for all $a, b \in \mathcal{A}$

$$a \cap T^{-L}(b) \neq \emptyset.$$

Let K be so large that for all $k \geq K$, $\bigvee_{i=0}^{k} S^{-i}\mathcal{A}$ refines $T^{-L}\mathcal{A}$.

Now assume that $a, b \in \mathcal{A}$, and $k \geq K$. Take a point $x \in T^{-L}(a)$, and let u be such that

$$x \in u \in \bigvee_{i=0}^{k} S^{-i}\mathcal{A}.$$

Since $\bigvee_{i=0}^{k} S^{-i}\mathcal{A}$ refines $T^{-L}\mathcal{A}$, it follows that

$$u \subset T^{-L}(a).$$

Let $c = S^k(u) \in \mathcal{A}$, and take a point y from the nonempty neighborhood $c \cap T^{-L}(b)$. Since $\mathcal{A}$ is Markov for S, there is a uniquely determined point $z \in X$ such that

$$S^k(z) = y \qquad \text{and} \qquad z \in u.$$

Then it is easy to see the following:

$$T^L(z) \in a \cap S^{-k}(b),$$

from which we obtain the desired result. □

THEOREM 4.2. *Let (X, T) be a mixing SFT and S be a positively expansive endomorphism on (X, T). Then the Parry measures of (X, T) and (X, S) are identical.*

PROOF. Let $\mathcal{A}_0$ be a common Markov alphabet for S and T. Let m and n be so large that $\bigvee_{i=0}^{m} S^{-i}\mathcal{A}_0$ refines $T^{-1}\mathcal{A}_0$ and $\bigvee_{j=0}^{n} T^{-j}\mathcal{A}_0$ refines $S^{-1}\mathcal{A}_0$. Set

$$\mathcal{A} = \bigvee_{i=0}^{m} \bigvee_{j=0}^{n} S^{-i}T^{-j}\mathcal{A}_0.$$

Then $\mathcal{A}$ is again a common Markov alphabet for S and T, which refines $\bigvee_{i=0}^{m+1} S^{-i}\mathcal{A}_0$ and $\bigvee_{j=0}^{n+1} T^{-j}\mathcal{A}_0$. Hence we have the following.

(a) For all $a, b, c \in \mathcal{A}$ if $a \cap S^{-1}(b) \neq \emptyset$ and $b \cap T^{-1}(c) \neq \emptyset$, then there is a unique $d \in \mathcal{A}$ such that $a \cap T^{-1}(d) \neq \emptyset$ and $d \cap S^{-1}(c) \neq \emptyset$.

(b) For all $a, b, c \in \mathcal{A}$ if $a \cap T^{-1}(b) \neq \emptyset$ and $b \cap S^{-1}(c) \neq \emptyset$, then there is a unique $d \in \mathcal{A}$ such that $a \cap S^{-1}(d) \neq \emptyset$ and $d \cap T^{-1}(c) \neq \emptyset$.

Let M and N denote the transition matrices of the systems $(X, S, \mathcal{A})$ and $(X, T, \mathcal{A})$, respectively. Then (a) and (b) imply that $MN = NM$. Since M and N are primitive matrices, it follows that if V is a left(right) Perron eigenvector of M, then V is also a left(right) Perron eigenvector of N. This means that for all $a \in \mathcal{A}$ we have $\mu_S(a) = \mu_T(a)$, where μ_S and μ_T are the Parry measures of S and T, respectively. Since

$$\bigcup_{m,n} \left(\bigvee_{i=0}^{m} \bigvee_{j=0}^{n} S^{-i}T^{-j}\mathcal{A}_0 \right)$$

is a basis for the topology of X, the proof is complete. □

REMARK. Theorems 4.1 and 4.2 are generalizations of Theorems 3.8 and 3.9 in [3], respectively.

5. Examples

On seeing Theorem 4.1, we ask a natural question: is a positively expansive endomorphism on a transitive SFT transitive? The answer is negative as we see in the following two examples.

EXAMPLE 5.1. (One–sided case) Let (X,T) be a transitive one-sided SFT whose transition matrix M has period $p > 1$. Let $S = T^p$. Clearly S is a positively expansive endomorphism on (X,T), but (X,S) is not transitive since the transition matrix M^p is reducible.

EXAMPLE 5.2. (Two–sided case) Fix two positive integers p and q. Let P denote the set of residue classes modulo p and Q the set of residue classes modulo q. Let X be the collection of bi-infinite sequences $\langle a_i\rangle_{i\in\mathbb{Z}}$ where either $a_{2i} \in P$ and $a_{2i+1} \in Q$ for each $i \in \mathbb{Z}$ or $a_{2i} \in Q$ and $a_{2i+1} \in P$ for each $i \in \mathbb{Z}$. Let T be the shift map on X. Then (X,T) is a transitive two-sided SFT whose transition matrix is $\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$. We define an endomorphism $S : X \to X$ by

$$S(\langle a_i\rangle_{i\in\mathbb{Z}})_i = \begin{cases} a_{i-2} + a_{i+2} \mod p & \text{if } a_i \in P, \\ a_{i-2} + a_{i+2} \mod q & \text{if } a_i \in Q. \end{cases}$$

It is easy to see that S is positively expansive and that (X,S) is not transitive. In fact, the transition matrix of (X,S) is $\begin{pmatrix} p^2q^2 & 0 \\ 0 & p^2q^2 \end{pmatrix}$.

One of the main results in [3] asserts that if S is a positively expansive endomorphism on a full N-shift then S is shift equivalent to a full K-shift where K and N have the same prime factors. For the converse part, they constructed a positively expansive endomorphism on the full N–shift, which is conjugate to the full K–shift, for each integer $K \leq N$ with same prime factors as N. In the following example we see that the converse is true even for $K > N$.

EXAMPLE 5.3. For any positive integer n, let $X[n]$ denote the full n-shift with the shift map $T[n]$. Fix a positive integer N with prime factorization

$$N = {p_1}^{n_1}{p_2}^{n_2}\cdots{p_t}^{n_t}.$$

Then clearly $(X[N], T[N])$ is conjugate to

$$(X[p_1], T[p_1]^{n_1}) \times \cdots \times (X[p_t], T[p_t]^{n_t}).$$

Suppose that another integer $K = {p_1}^{k_1} \ldots {p_t}^{k_t}$, $k_i \geq 0$, is given. Consider the endomorphism $S = T[p_1]^{k_1} \times \cdots \times T[p_t]^{k_t}$ on $(X[p_1], T[p_1]^{n_1}) \times \cdots \times (X[p_t], T[p_t]^{n_t})$. Then S is positively expansive if and only if $k_i \geq 1$ for all $i = 1, \cdots, t$. Obviously S is conjugate to the full K-shift.

References

[1] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer Verlag, New York, 1975.

[2] M. Boyle, D. Fiebig and U.-R. Fiebig, *A dimension group for local homeomorphisms and endomorphisms of onesided shifts of finite type*, preprint (1996).

[3] F. Blanchard and A. Maass, *Dynamical properties of expansive one-sided cellular automata*, Israel Journal of Mathematics (to appear).

[4] M. Nasu, *Maps in symbolic dynamics*, Lecture Notes of the 10th KAIST Mathematics Workshop (Geon Ho Choe, ed.), Taejon, 1995.

[5] W. Parry, *Symbolic dynamics and transformations of the unit interval*, Trans. A.M.S. **122** (1966), 368–378.

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