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ON THE SUPERSTABILITY OF SOME FUNCTIONAL INEQUALITIES WITH THE UNBOUNDED CAUCHY DIFFERENCE $f(x+y) - f(x)f(y)$

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ABSTRACT. Assume $H_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ are monotonically increasing (in both variables), homogeneous mappings for which $H_1(tu, tv) = t^p H_1(u, v)$ $(p > 0)$ and $H_2(tu, tv) = H_2(u, v)^{t^q}$ $(q \leq 1)$ hold for $t, u, v \geq 0$. Using an idea from the paper of Baker, Lawrence and Zorzitto [2], the superstability problems of the functional inequalities $\|f(x+y) - f(x)f(y)\| \leq H_i(\|x\|, \|y\|)$ shall be investigated.

1. Introduction and main results

Baker, Lawrence and Zorzitto [2] and Baker [1] proved the Hyers-Ulam stability of the functional equation

$$f(x+y) = f(x)f(y),$$

i.e., if the Cauchy difference $f(x+y) - f(x)f(y)$ of a complex-valued mapping f defined on a normed space is bounded for all x, y then either f is bounded or $f(x+y) = f(x)f(y)$ for all x, y. In particular, such a phenomenon for some functional equation is called superstability.

In this paper, we shall investigate the superstability problems for the case when the Cauchy difference $f(x+y) - f(x)f(y)$ is not bounded.

Throughout the paper, let X be a normed space over the complex numbers. Assume that $(Y, +, \cdot)$ is a field and $(Y, +, \|\cdot\|)$ is a normed space such that $\|y_1 y_2\| = \|y_1\| \|y_2\|$ for any $y_1, y_2 \in Y$, i.e., the norm on Y is multiplicative. Suppose $H_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ to be monotonically increasing (in both variables), homogeneous mappings for which $H_1(tu, tv) = t^p H_1(u, v)$ and $H_2(tu, tv) = H_2(u, v)^{t^q}$ hold for

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some $p > 0$, $q \leq 1$ and for all $t, u, v \in \mathbb{R}_+$. For $a, b, c > 0$, $au^p + bv^p$ and $cu^{p/2}v^{p/2}$ are examples for $H_1(u, v)$, and $\exp(au^q + bv^q)$ and $\exp(cu^{q/2}v^{q/2})$ are examples for $H_2(u, v)$. Let $f : X \to Y$ be a mapping such that

$$\|f(x+y) - f(x)f(y)\| \leq H_i(\|x\|, \|y\|) \tag{1}$$

for all $x, y \in X$. The following theorems are main results of this note.

THEOREM 1. *If f satisfies the functional inequality (1) for $i = 1$ then it holds either $\|f(x)\| = o(\|x\|^p)$ as $\|x\| \to \infty$ or else $f(x+y) = f(x)f(y)$ for every $x, y \in X$.*

THEOREM 2. *If f satisfies the functional inequality (1) for $i = 2$ then it holds either $\|f(x)\| \leq H_2(\|x\|, \|x\|) + 1$ for all $x \in X$ or else $f(x+y) = f(x)f(y)$ for every $x, y \in X$.*

2. Proofs of Theorems

PROOF OF THEOREM 1. By induction on n we first prove that

$$\|f(nx) - f(x)^n\| \leq \sum_{i=1}^{n-1} H_1(i\|x\|, \|x\|)\|f(x)\|^{n-i-1} \tag{2}$$

for all $n \geq 2$. In view of (1), it is trivial for $n = 2$. If we assume that (2) is true for some $n \geq 2$ then we get for $n+1$

$$\begin{aligned}
&\|f((n+1)x) - f(x)^{n+1}\| \leq \\
&\leq \|f((n+1)x) - f(nx)f(x)\| + \|f(x)\|\|f(nx) - f(x)^n\| \\
&\leq H_1(n\|x\|, \|x\|) + \sum_{i=1}^{n-1} H_1(i\|x\|, \|x\|)\, \|f(x)\|^{n-i} \\
&\leq \sum_{i=1}^{n} H_1(i\|x\|, \|x\|)\, \|f(x)\|^{n+1-i-1}
\end{aligned}$$

by using (1) and (2). By multiplying $\|f(x)^{-n}\| = \|f(x)\|^{-n}$ (remind that $f(x)^{-n}$ is the inverse element of $f(x)^n$ and $\|e\| = 1$ where e is the multiplicatively neutral element of Y) on both sides in (2), we get

$$\begin{aligned} \|f(nx)f(x)^{-n} - e\| &\le \sum_{i=1}^{n-1} H_1\left(i\|x\|, \|x\|\right) \|f(x)\|^{-i-1} \\ &\le \sum_{i=1}^{\infty} i^p H_1\left(\|x\|, \|x\|\right) \|f(x)\|^{-i-1} \\ &\le \left(H_1\left(\|x\|, \|x\|\right)/\|f(x)\|\right) \sum_{i=1}^{\infty} i^p/\|f(x)\|^i \end{aligned} \tag{3}$$

for all $n \ge 2$. Assume that $\|f(x)\| \ne o\left(\|x\|^p\right)$ as $\|x\| \to \infty$, i.e., there exist some $c > 0$ and a sequence (x_k) in X such that $\|x_k\| \to \infty$ as $k \to \infty$ and $\|f(x_k)\| \ge c\|x_k\|^p > 1$ for sufficiently large k. We can then let the series $\sum_i i^p/\|f(x_k)\|^i$ converge to a value $< c/2$ by taking k sufficiently large, since $\|x_k\|^p \to \infty$ as $k \to \infty$. Hence, it follows from (3) and the above consideration that

$$\|f(nx_k)f(x_k)^{-n} - e\| < 1/2 \tag{4}$$

for some sufficiently large k and any $n \ge 2$. Since the fact $\|f(x_k)\| > 1$ implies

$$n^p H_1\left(\|x_k\|, \|x_k\|\right) = o\left(\|f(x_k)\|^n\right) \quad \text{as} \quad n \to \infty,$$

we can easily show

$$n^p H_1\left(\|x_k\|, \|x_k\|\right) = o\left(\|f(nx_k)\|\right) \text{ as } n \to \infty \tag{5}$$

by (4). Now let $x, y \in X$ be arbitrarily given. If k is sufficiently large then we have

$$\begin{aligned} &\|f(nx_k)\|\|f(x+y) - f(x)f(y)\| \\ &\quad \le \|f(x+y)f(nx_k) - f(x+y+nx_k)\| \\ &\qquad + \|f(x+y+nx_k) - f(x)f(y+nx_k)\| \\ &\qquad + \|f(x)\|\|f(y+nx_k) - f(y)f(nx_k)\| \\ &\quad \le H_1\left(\|x+y\|, n\|x_k\|\right) \\ &\qquad + H_1\left(\|x\|, \|y+nx_k\|\right) + \|f(x)\|H_1\left(\|y\|, n\|x_k\|\right) \\ &\quad \le CH_1\left(n\|x_k\|, n\|x_k\|\right) \\ &\quad \le Cn^p H_1\left(\|x_k\|, \|x_k\|\right) \end{aligned} \tag{6}$$

for some $C > 0$ and all sufficiently large n. It then follows from (5) and (6) that $f(x+y) = f(x)f(y)$. $\square$

PROOF OF THEOREM 2. Assume that there exists an $x_0 \in X$ such that

$$\|f(x_0)\| > H_2(\|x_0\|, \|x_0\|) + 1.$$

As in the proof of Theorem 1 we can verify that for all $n \geq 2$

$$\begin{aligned}
&\|f(nx_0)f(x_0)^{-n} - e\| \\
&\qquad \leq \sum_{i=1}^{n-1} H_2(i\|x_0\|, \|x_0\|)\, \|f(x_0)\|^{-i-1} \\
&\qquad \leq \sum_{i=1}^{\infty} H_2(\|x_0\|, \|x_0\|)^{i^q}\, \|f(x_0)\|^{-i-1} \\
&\qquad \leq \|f(x_0)\|^{-1} \sum_{i=1}^{\infty} \left(H_2(\|x_0\|, \|x_0\|) / \|f(x_0)\|\right)^i \\
&\qquad \leq H_2(\|x_0\|, \|x_0\|) \,/\, \left(\|f(x_0)\|^2 - \|f(x_0)\| H_2(\|x_0\|, \|x_0\|)\right) \\
&\qquad \leq c < 1
\end{aligned} \tag{7}$$

by the hypothesis. As in the proof of Theorem 1, on account of (7) and the hypothesis, we get

$$H_2(\|x_0\|, \|x_0\|)^n = o(\|f(nx_0)\|) \quad \text{as} \quad n \to \infty. \tag{8}$$

Now let $x, y \in X$ be arbitrarily given. By (1) we have

$$\begin{aligned}
&\|f(nx_0)\| \|f(x+y) - f(x)f(y)\| \\
&\qquad \leq \|f(x+y)f(nx_0) - f(x+y+nx_0)\| \\
&\qquad\quad + \|f(x+y+nx_0) - f(x)f(y+nx_0)\| \\
&\qquad\quad + \|f(x)\| \|f(y+nx_0) - f(y)f(nx_0)\| \\
&\qquad \leq H_2(\|x+y\|, n\|x_0\|) \\
&\qquad\quad + H_2(\|x\|, \|y+nx_0\|) + \|f(x)\| H_2(\|y\|, n\|x_0\|) \\
&\qquad \leq C H_2(n\|x_0\|, n\|x_0\|) \\
&\qquad \leq C H_2(\|x_0\|, \|x_0\|)^n
\end{aligned} \tag{9}$$

for some $C>0$ and all sufficiently large n. Finally, by (8) and (9), we conclude that $f(x+y)=f(x)f(y)$. □

REMARK. More precisely, we can replace $H_2(\|x\|,\|x\|)+1$ in Theorem 2 by

$$\frac{1}{2}\left(H_2(\|x\|,\|x\|)+\sqrt{H_2(\|x\|,\|x\|)^2+4H_2(\|x\|,\|x\|)}\right).$$

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References

[1] J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80** (1980), 411–416.

[2] J. Baker, J. Lawrence and F. Zorzitto, *The stability of the equation* $f(x+y)=f(x)f(y)$, Proc. Amer. Math. Soc. **74** (1979), 242–246.

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