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CONHARMONIC TRANSFORMATION AND CRITICAL RIEMANNIAN METRICS

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ABSTRACT. The conharmonic transformation is a conformal transformation which satisfies a specified differential equation. Such a transformation was defined by Y. Ishi and we generalize his results. In particular , we obtain a necessary and sufficient condition for the invariance of critical Riemannian metrics under the conharmonic transformation.

1. Introduction

It is well known that conformal transformation on the Riemannian manifold does not change the angle between two vectors at a point. But, in general, the harmonicity of functions, vectors and forms are not preserved by the conformal transformation. Related this fact, Y.Ishi [3] have studied conharmonic transformation. In this paper, we are to study the sufficient and necessary conditions of the invariance of the harmonicity for the geometric objects. Moreover, another properties of the conharmonic transformation are investigated. Finally, we are to study the conharmonic transformation which preserve the critical Riemannian metrics.

2. Conformal transformation

Let M be an n-dimensional Riemannian manifold with metric tensor g and ρ a positive function on M. Then $(M, \bar{g} = e^{2\rho} g)$ is conformally diffeomorphic to (M, g) and the conformal diffeomorphism $\phi : (M, g) \longrightarrow$

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$(M, \bar{g})$ is called a conformal transformation. When ρ is constant, ϕ is called a homothety. Let ∇(respectively $\bar{\nabla}$) denote the operator of covariant differentiation with respect to g (respectively $\bar{g}$) and let $\{{}^{h}_{ij}\}$ and $\{{}^{\bar{h}}_{ij}\}$ be Christoffel symbols formed by g and $\bar{g}$. Then we get ([2],[3],[6]).

$$\{{}^{\bar{h}}_{ij}\} = \{{}^{h}_{ij}\} + \rho_i \delta_j{}^h + \rho_j \delta_i{}^h - \rho^h g_{ji}, \tag{2.1}$$

$$\bar{R}_{kji}{}^h = R_{kji}{}^h + \rho_{ki}\delta_j{}^h - \rho_{ji}\delta_k{}^h + g_{ki}\rho_j{}^h - g_{ji}\rho_k{}^h, \tag{2.2}$$

$$\bar{S}_{ji} = S_{ji} - (n-2)\rho_{ji} - \rho_\alpha{}^\alpha g_{ji}, \tag{2.3}$$

$$\bar{K} = e^{-2\rho}(K - 2(n-1)\rho_\alpha{}^\alpha), \tag{2.4}$$

where $R_{kji}{}^h$, S_{ji} and K (respectively $\bar{R}_{kji}{}^h$,$\bar{S}_{ji}$ and $\bar{K}$) are Riemannian -Christoffel curvature tensor, Ricci curvature tensor and scalar curvature of the Riemannian metric g(respectively $\bar{g}$), and where we have put

$$\rho_{ji} = \nabla_j\ \rho_i - \rho_j \rho_i + \frac{1}{2}\|\rho_\alpha\|^2 g_{ji}, \tag{2.5}$$

$$\rho_k{}^k = g^{ji}\rho_{ji}, \tag{2.6}$$

$$\rho_i = \partial_i \rho. \tag{2.7}$$

3. Harmonic function and tensors

A harmonic function f is a function whose Laplacian

$$\Delta f = g^{ji}\nabla_j \nabla_i f$$

vanishes.

It is easily seen that

$$\bar{\Delta} f = \bar{g}^{ji}\bar{\nabla}_j \bar{\nabla}_i f = e^{-2\rho} g^{ji} \nabla_j \nabla_i f + (n-2)e^{-2\rho}\rho^h(\partial_h f).$$

Therefore, we can state

PROPOSITION 3.1. *Let f be a harmonic function on (M,g) with dim$M$ $= n > 2$. Then f is a hamonic function on $(M,\bar{g})$ if and only if $\rho^h(\partial_h f) = 0$.*

A skew-symmetric tensor w which satisfies

$$\text{Rot}w = 0 \quad \text{and} \quad \text{Div}w = 0 \tag{3.1}$$

is called a harmonic tensor,that is

$$dw = 0 \quad \text{and} \quad \delta w = 0, \tag{3.2}$$

where d and δ are operators of exterior differential and codifferential with respect to ∇. Then we see that

$$dw = \bar{d}w, \tag{3.3}$$

and

$$\bar{\delta}w = \delta w + (n-2p)e^{-2\rho}\rho^\alpha w_{i_1\dots i_p}, \tag{3.4}$$

where $w_{i_1 i_2 \dots\dots i_p}$ is a component of a p-form w. Hence we have

PROPOSITION 3.2. *Let w be a harmonic p-form on (M,g) and dim M $\neq 2p$. Then w is a harmonic on $(M,\bar{g})$ if and only if $\rho^\alpha w_{\alpha i_2 \dots i_p} = 0$.*

4. Conharmonic transformation

From the proposition 3.1 , we see that a harmonic function is not in general transformed into a harmonic function by the conformal transformation. Y.Ishi [3] have studied the condition upon ρ in order that the function defined by

$$\bar{f} = e^{2\alpha\rho} f \tag{4.1}$$

may become a harmonic function to the Riemannian metric $\bar{g}$ for a suitable constant α. Then we obtain ([3]),

(4.2)

$$\bar{\Delta} f = e^{2(\alpha-1)\rho}[\Delta f + 2\alpha(\Delta\rho)f + (4\alpha+n-2)\rho^i(\partial_i f) + 2\alpha(2\alpha+n-2)\|\rho_i\|^2 f]$$

by use of (2.1). From which we have [3]

PROPOSITION 4.1. *Let f be a harmonic function on (M, g) and $\alpha = \frac{(2-n)}{4}$. Then the function $\bar{f}$ defined by (4.1) is harmonic for $\bar{g}$ if and only if*

$$\Delta\rho + \frac{(n-2)}{2}\|\rho_k\|^2 = 0, \tag{4.3}$$

where $n = \dim M > 2$.

If we consider proposition 3.1 and (4.2), then we can state

COROLLARY 4.2. *Let f be a harmonic function on M with respect to g and $\bar{g}$, and $n > 2$. Then $\bar{f}$ is a harmonic function for $\bar{g}$ if and only if $\Delta\rho + \frac{(n-2)}{2}\|\rho_k\|^2 = 0$.*

In this point of view, Y.Ishi [3] called a conharmonic transformation which is the conformal transformation $\phi : (M, g) \longrightarrow (M, \bar{g})$ satisfying (4.3). If we put

$$T_{kji}{}^h = R_{kji}{}^h + \frac{1}{(n-2)}[S_{ik}\delta_j^h + S_j{}^h g_{ik} - S_{ij}\delta_k^h - S_k{}^h g_{ij}], \tag{4.4}$$

then, by means of the equations (2.2)-(2.6) and (4.4), we have

PROPOSITION 4.3. *Let $\phi : (M, g) \longrightarrow (M, \bar{g})$ be a conformal transformation . Then the following conditions are equivalent :*

(1) *ϕ is a conharmonic transformation .*
(2) *$\rho_k{}^k = 0$.*
(3) *$K = e^{2\rho}\bar{K}$.*
(4) *$\bar{g}_{ij}\bar{K} = g_{ij}K$.*
(5) *$\bar{S}_{ij} = S_{ij} - (n-2)\rho_{ij}$.*
(6) *$T_{kji}{}^h$ is invariant under ϕ.*

If we consider (2.2) and (2.6), then we easily get

PROPOSITION 4.4. *Let $\phi : (M, g) \longrightarrow (M, \bar{g})$ be a conformal transformation with $\dim M > 2$. The Riemannian - Christoffel curvature tensor is invariant under ϕ if and only if $\rho_{ij} = 0$.*

In the case of an Einstein space , Y.Ishi [3] have showed that

PROPOSITION 4.5. *The Einstein space is transformed into an Einstein space by the conharmonic transformation if and only if* $\rho_{ij} = 0$.

As an extension of proposition 4.5, if we consider (2.2) and propositions 4.3 and 4.4 , then we have

PROPOSITION 4.6. *A necessary and sufficient condition for the invariance of the space of constant curvature by the conharmonic transformation is* $\rho_{ij} = 0$.

If we assume that M is compact and applying the Green's theorem , then we have

PROPOSITION 4.7. *Let* M *be a compact Riemannian manifold and* $\phi : (M, g) \longrightarrow (M, \bar{g})$ *be a conharmonic transformation . Then* ϕ *is a homothety.*

PROOF. By the Green's theorem,

$$\int_M (\nabla_i \rho^i) d\sigma = 0 .$$

for the volume element $d\sigma$. Then the equation (4.3) reduces $\rho_k = 0$, that is ρ is constant. □

In [6], the following fact was suggested as an exercise.

LEMMA 4.8. *The homothety on the compact manifold is an isometry.*

If we consider proposition 4.7 and lemma 4.8, then we get

PROPOSITION 4.9. *The conharmonic transformation on the compact manifold is an isometry.*

5. Critical Riemannian metrics

M.Berger[1] and Y.Muto[5] have studied the critical Riemannian metrics of the Riemannian functionals

$$A(g) = \int_M K dV_g \tag{5.1}$$

and

$$B(g) = \int_M K^2 dV_g \tag{5.2}$$

on the compact manifold M and g is a Riemannian metric on M satisfying $\int_M dV_g = 1$, where dV_g is the volume element measured by g. Although critical Riemannian metrics were first defined on a compact manifold, it is easy to generalize the definition when M is not compact. It is well known that g is a critical Riemannian metric for the function A defined by (5.1) if and only if g is an Einstein metric. By this fact and proposition 4.5 , we can state

PROPOSITION 5.1. *The critical Riemannian metric of a function A is preserved by the conharmonic transformation if and only if $\rho_{ij} = 0$.*

One of the present authors [4] proved

LEMMA 5.2. *The Riemannian metric g on M is a critical Riemannian metric for the function B defined by (5.2) if and only if*

$$n\nabla_j\nabla_i K - nKS_{ij} - (\Delta K)g_{ij} + K^2 g_{ij} = 0, \tag{5.3}$$

where ΔK is the Laplacian of K.

Assume that g is the critical Riemannian metric of the function B and if we consider the object on the left hand side of (5.3) with respect to $\bar{g} = e^{2\rho}g$, then we get

$$\bar{\nabla}_i\bar{K} = e^{-2\rho}(\partial_i K - 2\rho_i K), \tag{5.4}$$

$$\begin{aligned}\bar{\nabla}_j\bar{\nabla}_i\bar{K} = e^{-2\rho}[&-2(\nabla_j\rho_i)K + \nabla_j\nabla_i K - 3\rho_i(\partial_j K) - 3\rho_j(\partial_i K)\\ &+ 8\rho_j\rho_i K - 2\|\rho_k\|^2 K g_{ji} + \rho^h(\partial_h K)g_{ji}],\end{aligned} \tag{5.5}$$

$$\bar{\Delta}\bar{K} = e^{-4\rho}[-2K\Delta\rho + (n-6)\rho^h(\partial_h K) + \Delta K - 2(n-4)\|\rho_k\|^2 K], \tag{5.6}$$

$$\bar{K}\bar{S}_{ji} = e^{-2\rho}[KS_{ji} - (n-2)\rho_{ji}K], \tag{5.7}$$

$$\bar{K}^2\bar{g}_{ji} = e^{-2\rho}K^2 g_{ji}. \tag{5.8}$$

Henceforth we obtain

$$\begin{aligned}\bar{\nabla}_j\bar{\nabla}_i\bar{K} - n\bar{K}\bar{S}_{ji} - \bar{\Delta}\bar{K}\bar{g}_{ji} + \bar{K}^2\bar{g}_{ji} \\ = n(n-4)K\rho_{ji} + 6n\rho_j\rho_i K - 6\|\rho_k\|^2 K g_{ji} \\ - 3n\rho_i(\partial_j K) - 3n\rho_j(\partial_i K) + 6\rho^h(\partial_h K)g_{ji}.\end{aligned}$$

Therefore we have

PROPOSITION 5.3. *Let g be a critical Riemannian metric for the function B and let $\phi : (M, g) \longrightarrow (M, \bar{g})$ be a conharmonic transformation. Then $\bar{g}$ is a critical Riemannian metric for B if and only if*

$$\begin{aligned}n(n-4)K\rho_{ji} + 6n\rho_j\rho_i K - 6\|\rho_k\|^2 K g_{ji} \\ - 3n\rho_i(\partial_j K) - 3n\rho_j(\partial_i K) + 6\rho^h(\partial_h K)g_{ji} = 0.\end{aligned}$$

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