Comm. Korean Math. Soc. **12**(1997), No. 2, pp. 365-371

COVERING RADIUS, VOLUME COMPARISON AND SPHERE RIGIDITY

GABJIN YUN

ABSTRACT. We prove some relation of volume with the covering radius of Riemannian manifolds and reprove well-known sphere rigidity theorems by using it.

0. Introduction

Sphere rigidity problem is the classical and one of the main themes in differential geometry. So far there are so many theorems about sphere rigidity. In particular, after Gromov introduced the Gromov-Hausdorff distance between metric spaces ([5]), the theory of sphere rigidity has been developed so much and several new concepts came out, for example, radius, covering radius, packing radius, excess, etc. (see [3], [6], [8], [10], [9] and references are therein). In this note, we will get some relations of the covering radius with volume of a given Riemannian manifold with Ricci curvature condition and prove well-known sphere rigidity theorems by using simple argument.

Let X be a compact metric space. The k-th *covering radius* of X is defined by

$$Cov_k(X) = \inf_{p_1,\dots,p_k} r_k(p_1,\dots,p_k),$$

where $r_k(p_1,\dots,p_k) = r$ is the smallest number satisfying

$$\bar{B}(p_1,r) \cup \bar{B}(p_2,r) \cup \cdots \cup \bar{B}(p_k,r) = X.$$

In [3], he proved the following theorem.

Received November 15, 1996. Revised March 17, 1997.
1991 Mathematics Subject Classification: Primary 53C20.
Key words and phrases: radius, covering radius, diameter and volume.
Supported in part by The Natural Science Research Institute, Myong Ji University and BSRI-96-1434.

THEOREM 1 ([3]). *Let $n \geq 2$. Then there exists $\epsilon = \epsilon(n) > 0$ such that if (M, g) is a closed Riemannian n-manifold satisfying*

$$K_M \geq 1, \qquad Cov_q M \geq Cov_q S^n - \epsilon \quad \text{for} \quad q = n+1 \quad \text{or} \quad n+2,$$

then M is diffeomorphic to S^n.

Usually, the notion of k-th covering radius lies between volume and radius, the first covering radius. Recall that the radius of a compact metric space X is defined as

$$Rad(X) = \min_{p \in X} \max_{q \in X} d(p, q)$$

which is related with the diameter by

$$Rad(X) \leq diam(X) \leq 2Rad(X).$$

Note that $Rad(X)$ is the smallest positive number r so that X can be covered by the closed ball $\bar{B}(p, r) = \{x \in X : d(p, x) \leq r\}$ for some $p \in X$. For example, $Rad(S^n) = diam(S^n) = \pi$ and $Cov_k(S^n) = \frac{\pi}{2}, k = 2, \ldots, n+1$ with the standard round metric ([3]).

In section 1, we will prove a relation of covering radius with volume of a given Riemannian manifold and reprove the well-known results. In section 2, we will verify that for given $n \geq 2$ and $v > 0$, there exists $\epsilon = \epsilon(n, v) > 0$ such that if (M, g) is a closed Riemannian n-manifold satisfying

$$K_M \geq 1, \quad vol(M) \geq v \quad \text{and} \quad diam(M) \geq \pi - \epsilon,$$

then M is diffeomorphic to S^n. One can find this problem in [9]. Of course, it is also almost known due to T. Colding ([4]). However, we shall prove it here by using other method which is very simple and easy.

The problem without volume condition is still open and looks quite not easy to prove it. The main idea to prove this theorem is that for a closed Riemannian manifold M with $K_M \geq 1$, the lower bound of diameter of M gives the lower bound of the covering radius.

1. Volume and Covering Radius

In this section, we shall prove some relation of volume with covering radius and well-known diffeomorphism sphere theorems due to [8] and [10] as an application of Proposition 1. First, we have the following theorem.

THEOREM 2 ([3]). *Cov_k is continuous on the space of compact metric spaces relative to the Gromov-Hausdorff topology.*

This implies that for given $\epsilon > 0$, there exists a positive real number $\delta > 0$ such that if X and Y are compact metric spaces with $d_{GH}(X, Y) < \delta$, then $|Cov_k(X) - Cov_k(Y)| < \epsilon$.

Second, we need the following theorem which is due to T. Colding ([4]). Denote by ω_n the volume of S^n with the standard round metric.

THEOREM 3 ([4]). *For given $\epsilon > 0$, there exists a $\delta > 0$ such that if M is a closed Riemannian n-manifold satisfying $Ric(M) \geq n-1, vol(M) \geq \omega_n - \delta$, then $d_{GH}(M, S^n) < \epsilon$.*

The following proposition shows that if the volume of a given Riemannian manifold M is almost equal to the volume of the round sphere S^n, then the covering radius of M is also almost equal to the covering radius of S^n.

PROPOSITION 1. *Let (M, g) be a closed Riemannian n-manifold with $Ric(M) \geq n-1$. Then, for any $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that if $vol(M) \geq \omega_n - \delta$, then $Cov_{n+1}(M) \geq Cov_{n+1}(S^n) - \epsilon$.*

PROOF. It follows immediately from Theorem 2 and Theorem 3. □

COROLLARY 1 ([8]). *For $n \geq 2$, there exists $\delta = \delta(n) > 0$ such that if (M, g) is a closed Riemannian n-manifold satisfying*

$$K_M \geq 1, \qquad vol(M) \geq vol(S^n) - \delta, \tag{1}$$

then M is diffeomorphic to S^n.

PROOF. Choose a positive small number $\epsilon > 0$ so that Theorem 1 holds. By Proposition 1 above, there exists $\delta = \delta(n, \epsilon) > 0$ such that the condition (1) implies

$$Cov_{n+1}(M) \geq Cov_{n+1}(S^n) - \epsilon.$$

Applying Theorem 1, we get the result. □

Next, we will show that under the assumption of sectional curvature ≥ 1, manifolds with the radius almost π also has the property that the $(n+1)$-th covering radius is almost equal to that of S^n. And then we will prove a differentiable sphere theorem with radius condition due to Shiohama-Yamaguchi ([10]).

For a metric space (X, d) with a metric d, we define the *distance of length* d_l on X by

$$d_l(x, y) = \inf\{l(\gamma) : \gamma(0) = x, \gamma(1) = y\},$$

where $l(\gamma)$ denotes the length of the curve γ. There is no reason, in general, that $d = d_l$ holds for a given metric space (X, d). In fact, the topologies induced by d and d_l may be different (cf. [5]). A metric space (X, d) is called a *length space* (or *inner metric space*) if it is locally compact and $d = d_l$. An Alexandrov space is a finite topological dimensional complete length space with a lower curvature bound in distance comparison sense (See, for example, [2]).

PROPOSITION 2. *For given $\epsilon > 0$ and n, there is $\delta = \delta(\epsilon, n) > 0$ such that if (M, g) is a closed Riemannian n-manifold satisfying*

$$K_M \geq 1, \qquad Rad(M) \geq \pi - \delta,$$

then $Cov_{n+1}M \geq Cov_{n+1}S^n - \epsilon = \pi/2 - \epsilon$.

PROOF. Suppose Proposition 2 is not true. Then for some $\epsilon_o > 0$, we can choose a sequence of Riemannian n-manifolds (M_i, g_i) satisfying $K_{M_i} \geq 1$, $Rad(M_i) \geq \pi - 1/i$, but $Cov_{n+1}M_i < \pi/2 - \epsilon_o$. By the Gromov's precompactness theorem ([5]), (M_i, g_i) subconverges to an Alexandrov space (X, d) with $curv \geq 1$. In particular, curvature and radius conditions imply that M_i does not collapse, i.e., $vol(M_i, g_i) \geq v(n)$

for some constant $v > 0$, depending only on the dimension n(see [7]). Since Cov_k is continuous with respect to the Gromov-Hausdorff topology ([Theorem 2]), we have

$$Cov_{n+1}X = \lim_{i\to\infty} Cov_{n+1}M_i \leq \frac{\pi}{2} - \epsilon_o. \tag{2}$$

In particular, since $Cov_1 X = Rad(X)$, we get

$$Rad(X) = Cov_1 X = \lim_{i\to\infty} Cov_1 M_i = \lim_{i\to\infty} Rad(M_i) \geq \pi.$$

On the other hand, $curv \geq 1$ implies ([2]) that $diam(X) \leq \pi$ and so $Rad(X) \leq diam(X) \leq \pi$. Hence, $Rad(X) = \pi = diam(X)$. Thus, it is easy to see ([3], [6], [2], [11]) that X is isometric to S^n. In particular, $Cov_{n+1}X = Cov_{n+1}S^n = \pi/2$, which is a contradiction to (2). □

COROLLARY 2 ([10]). *For given $n \geq 2$, there exists $\epsilon = \epsilon(n) > 0$ such that if (M, g) is a closed Riemannian n-manifold satisfying*

$$K_M \geq 1, \qquad Rad(M) \geq \pi - \epsilon,$$

then M is diffeomorphic to S^n.

PROOF. It follows immedieately from Proposition 2 and Theorem 1. □

2. Non-collapsing manifolds with diameter pinching

In this section we will show some sphere rigidity theorem by using covering radius. Almost the same proof as in Proposition 2 shows the following proposition which *Rad* is just replaced by diameter in non-collapsing case.

PROPOSITION 3. *For given $\epsilon > 0$ and $n \geq 2, v > 0$, there is $\delta = \delta(\epsilon, n, v) > 0$ such that if (M, g) is a closed Riemannian n-manifold satisfying*

$$K_M \geq 1, \quad vol(M) \geq v \quad \text{and} \quad diam(M) \geq \pi - \delta,$$

then $Cov_{n+1}M \geq Cov_{n+1}S^n - \epsilon = \pi/2 - \epsilon$.

PROOF. Suppose Proposition 3 is not true. Then for some $\epsilon_o > 0$, we can choose a sequence of Riemannian n-manifolds (M_i, g_i) satisfying $K_{M_i} \geq 1, vol(M) \geq v$ and $diam(M_i) \geq \pi - 1/i$, but $Cov_{n+1}M_i < \pi/2 - \epsilon_o$. By the Gromov's precompactness theorem ([5]), (M_i, g_i) subconverges to an Alexandrov space (X, d) with $curv \geq 1$. Since Cov_{n+1} is continuous with respect to the Gromov-Hausdorff topology, we have

$$Cov_{n+1}X = \lim_{i \to \infty} Cov_{n+1}M_i \leq \frac{\pi}{2} - \epsilon_o. \tag{3}$$

Also it is easy to see that $diam(X) = \pi$. Thus, X is isometric to S^n again. In particular, $Cov_{n+1}X = Cov_{n+1}S^n = \pi/2$, which is a contradiction to (3). □

Applying Proposition 3, we can prove the following theorem which is one of the main theorems in this note.

THEOREM 4. *For given $n \geq 2, v > 0$, there exists $\epsilon = \epsilon(n, v) > 0$ such that if (M, g) is a closed Riemannian n-manifold satisfying*

$$K_M \geq 1, \quad vol(M) \geq v \quad \text{and} \quad diam(M) \geq \pi - \epsilon, \tag{4}$$

then M is diffeomorphic to S^n.

PROOF. By Proposition 3 above, (4) implies that $Cov_{n+1}M \geq Cov_{n+1} S^n - \delta = \pi/2 - \delta$ for some small number δ. Then applying Theorem 1, we get the conclusion. □

ACKNOWLEDGEMENT. The author would like to thank M. Cho and J-W. Yim for discussing this topic.

References

[1] R. L. Bishop and R. Crittenden, *Geometry of Manifolds*, Academic press, New York, 1964.

[2] Y. Burago, M. Gromov and G. Perelman, *Alexandrov spaces with curvature bounded below*, English transl. in Russian Math. surveys **47** (1992), 1-58, Uspekhi Math. Nauk. **47** (1992), 3-51.

[3] M. Cho, *Sphere theorems in Alexandrov spaces via new metric invariant*, preprint.

[4] T. Colding, *Convergence and Positive Ricci curvature*, will appear in Invent. Math. (1996).

[5] M. Gromov, (rédige par J. Lafontaine et P. Pansu), *Structure métrique pour les variété riemanniennes*, Cedic/Fernand Nathan, Paris, 1982.

[6] K. Grove and P. Petersen, *On the excess of metric spaces and manifolds*, preprint.

[7] ———, *Volume comparison à la Alexandrov*, Acta Math. **169** (1992), 131-151.

[8] Y. Otsu, K. Shiohama and T. Yamaguchi, *A new version of differentiable sphere theorem*, Invent. Math. **98** (1989), 219-228.

[9] K. Shiohama, *Recent developments in sphere theorems*, Proceedings of Symposia in Pure Math., part 3 **54** (1993), 551-576.

[10] K. Shiohama and T. Yamaguchi, *Positively curved manifolds with restricted diameters*, Geometry of manifolds (K. Shiohama, ed.), Perspect. Math., **8** (1989), 345-350.

[11] J-Y. Wu, *Hausdorff convergence and sphere theorems*, Proceedings of Symposia in Pure Math., part 3 **54** (1993), 685-692.

Department of Mathematics
Myong Ji University
Yongin 449-728, Korea
E-mail: gabjin@math.snu.ac.kr
gabjin@wh.myongji.ac.kr