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ON THE SPECTRAL GEOMETRY FOR THE JACOBI OPERATORS OF HARMONIC MAPS INTO A SASAKIAN OR COSYMPLETIC SPACE FORM

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ABSTRACT. When the target manifold is a Sasakian or cosympletic space form, we characterize invariant immersions, tangential anti-invariant immersions and normal anti-invariant immersions by the spectra of the Jacobi operator.

1. Introduction

The spectral geometry for the second order operators arising in Riemannian geometry has been studied by many authors. Among them, the spectral geometry for the Jacobi operator of the energy of a harmonic map was studied in [8,9] (for manifolds) and [6] (for Riemannian foliations), and for the Jacobi operator of the functional area was studied in [2,4,7]. The Jacobi operator of a harmonic map arises in the second variation formula of the energy of a harmonic map. This formula can be expressed in terms of an elliptic differential operator (called the *Jacobi operator*) defined on the space of cross sections of the induced bundle of the target manifold.

In this paper we shall study the spectral goemetry for the Jacobi operator of a harmonic map when the target manifold is a Sasakian or cosympletic space form.

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2. Preliminaries

Let (M, g) be an m-dimensional closed (*i.e.*, compact without boundary) Riemannian manifold with the metric g and (N, h) be an n-dimensional Riemannian manifold with the metric h. A smooth map $f : (M, g) \to (N, h)$ is said to be *harmonic* if it is a critical point of the energy functional E, which is defined by $E(f) := \int_M e(f) dv_g$, where the *energy density* $e(f)$ of f is defined to be $e(f) := \frac{1}{2}\sum_{i=1}^m h(f_* e_i, f_* e_i)$ (f_* is the differential of f, $\{e_1 \cdots e_m\}$ a local orthonomal frame field on M, and dv_g the volume element with respect to g). Let us consider the Jacobi operator J_f for a harmonic map f defined by $J_f V = \tilde{\Delta}_f V - R_f V$ for $V \in \Gamma(E)$ (the space of smooth sections of the induced bundle $f^*TN =: E$ of the tangent bundle TN), where $\tilde{\Delta}$ is the rough Laplacian associated to the induced connection $\tilde{\nabla}$ of E defined by $\tilde{\nabla}_X V := \nabla^h_{f_* X} V$ (for any tangent vector field X on M, ∇^h the Levi-Civita connection of the metric h), and $R_f V := \sum_{i=1}^m R_h(V, f_* e_i) f_* e_i$ (R_h is the Riemannian curvature tensor of (N, h)). In this paper, we take the convention $R_h(\tilde{X}, \tilde{Y}) := [\nabla^h_{\tilde{X}}, \nabla^h_{\tilde{Y}}] - \nabla^h_{[\tilde{X}, \tilde{Y}]}$, where $\tilde{X}$, $\tilde{Y}$ are tangent vector fields on N (for details, see [9,10]). Then J_f is self-adjoint, elliptic of second order and has a discrete spectrum as a conseqence of the compactness of M.

Consider the semigroup e^{-tJ_f} given by

$$e^{-tJ_f} V(x) = \int_M K(t, x, y, J_f) V(y) dv_g(y),$$

where $K(t, x, y, J_f) \in Hom(E_y, E_x)$ is the kernel function ($x, y \in M$, E_x is the fibre of E over x). Then we have asymptotic expansions for the L^2-trace

$$Tr(e^{-tJ_f}) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n a_n(J_f) \quad (t \downarrow 0^+), \tag{2.1}$$

where each $a_n(J_f)$ is the spectral invariant of J_f, which depends only on the discrete spectrum ;

$$Spec(J_f) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \cdots \uparrow +\infty\}.$$

Applying the Jacobi operator J_f of a harmonic map f to Gilkey's results in [3,p.327], we obtain

THEOREM [cf.9]. *For a harmonic map $f : (M, g) \longrightarrow (N, h)$*

$$a_0(J_f) = n \cdot Vol(M, g), \tag{2.2}$$

$$a_1(J_f) = \frac{n}{6} \int_M \tau_g dv_g + \int_M Tr(R_f) dv_g, \tag{2.3}$$

$$\begin{aligned} a_2(J_f) = &\frac{n}{360} \int_M (5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2) dv_g \\ &+ \frac{1}{360} \int_M [-30\|R^{\tilde{\nabla}}\|^2 + 60\tau_g Tr(R_f) \\ &+ 180 Tr({R_f}^2)] dv_g, \end{aligned} \tag{2.4}$$

where $R^{\tilde{\nabla}}$ is the curvature tensor of the connection $\tilde{\nabla}$ on E, which is defined by $R^{\tilde{\nabla}} := f^* R_h$, and R_g, ρ_g, τ_g are the curvature tensor, Ricci tensor, scalar curvature on M, respectively.

3. The calculation of spectral invariants

Let (ϕ, ξ, η, h) be an almost contact metric structure on a smooth manifold N. This means that

$$\begin{aligned} &\phi^2 = -I + \xi \otimes \eta, \quad && \phi(\xi) = \eta \circ \phi = 0, \\ &\eta(\xi) = 1, && h(\phi\tilde{X}, \tilde{Y}) = -h(\tilde{X}, \phi\tilde{Y}), \\ &\eta(\tilde{X}) = h(\tilde{X}, \xi), && \end{aligned} \tag{3.1}$$

where ϕ is a tensor field of type (1,1), ξ a vector field, η a 1-form, I the identity transformation, h a Riemannian metric and $\tilde{X}$, $\tilde{Y}$ vector fields on N [cf.1,5,11]. Define a 2-form Φ on N by $\Phi(\tilde{X}, \tilde{Y}) := h(\tilde{X}, \phi\tilde{Y})$ for any vector fields $\tilde{X}$, $\tilde{Y}$ on N.

If $N(\tilde{X}, \tilde{Y}) + 2d\eta \otimes \xi = 0$, where N is defined by

$$N(\tilde{X}, \tilde{Y}) := [\phi\tilde{X}, \phi\tilde{Y}] + \phi^2[\tilde{X}, \tilde{Y}] - \phi[\tilde{X}, \phi\tilde{Y}] - \phi[\phi\tilde{X}, \tilde{Y}],$$

then the almost contact metric structure (ϕ, ξ, η, h) is said to be *normal*. If $\Phi = d\eta$, the almost contact metric structure (ϕ, ξ, η, h) is called a *contact metric structure*.

$\mathcal{N} = (N, \phi, \xi, \eta, h)$ is called a *Sasakian manifold* if a smooth manifold N admits a normal contact metric structure (ϕ, ξ, η, h). $\mathcal{N} = (N, \phi, \xi, \eta, h)$ is called a *cosympletic manifold* if a smooth manifold N admits a normal almost contact metric structure (ϕ, ξ, η, h) such that Φ is closed and $d\eta = 0$.

From now on $\mathcal{N} = (N, \phi, \xi, \eta, h)$ will denote either a Sasakian manifold or a cosympletic manifold unless otherwise stated.

COROLLARY 1. *Let f, f' be harmonic maps of compact Riemannian manifold (M, g) into an η-Einstein manifold $\mathcal{N} = (N, \phi, \xi, \eta, h)$ i.e., the Ricci tensor ρ_h of N is of the form ; $\rho_h = \lambda h + \mu\eta \otimes \eta$, where $\lambda (\neq 0)$ and μ are some constants. If $Spec(J_f) = Spec(J_{f'})$ and the structure vector field ξ is normal to $f(M)$ and $f'(M)$, then $E(f) = E(f')$.*

PROOF. From (2.3), we get

$$2\lambda E(f) + \mu \int_M \|f^*\eta\|^2 dv_g = 2\lambda E(f') + \mu \int_M \|f'^*\eta\|^2 dv_g.$$

But $\|f^*\eta\|^2 = 0 = \|f'^*\eta\|^2$ because of the normality of the structure vector field ξ, which completes the proof. □

On $\mathcal{N} = (N, \phi, \xi, \eta, h)$ we call a sectional curvature

$$k = \frac{h(R_h(\tilde{X}, \phi\tilde{X})\phi\tilde{X}, \tilde{X})}{h(\tilde{X}, \tilde{X})h(\phi\tilde{X}, \phi\tilde{X})}$$

determined by two orthogonal vectors $\tilde{X}$ and $\phi\tilde{X}$ (which are orthogonal to ξ) the *ϕ-sectional curvature* with respect to $\tilde{X}$ of N. If the ϕ-sectional curvature is always constant with respect to any vector at every point of the manifold $\mathcal{N}$, then we call $\mathcal{N} = (N, \phi, \xi, \eta, h)$ a *manifold of constant ϕ-sectional curvature k*, or a *Sasakian space form* (a *cosympletic space form resp.*) when $\mathcal{N}$ is a Sasakian manifold (a cosympletic manifold resp.)

It has been shown [cf.5,11] that on $\mathcal{N} = (N, \phi, \xi, \eta, h)$ with constant

ϕ-sectional curvature k,

$$\begin{aligned} h(R(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}) &= \alpha\{h(\tilde{Y},\tilde{Z})h(\tilde{X},\tilde{W}) - h(\tilde{X},\tilde{Z})h(\tilde{Y},\tilde{W})\} \\ &+ \beta\{\eta(\tilde{X})\eta(\tilde{Z})h(\tilde{Y},\tilde{W}) + \eta(\tilde{Y})\eta(\tilde{W})h(\tilde{X},\tilde{Z}) \\ &- \eta(\tilde{X})\eta(\tilde{W})h(\tilde{Z},\tilde{Y}) - \eta(\tilde{Z})\eta(\tilde{Y})h(\tilde{X},\tilde{W}) \\ &+ \Phi(\tilde{X},\tilde{Z})\Phi(\tilde{W},\tilde{Y}) - \Phi(\tilde{X},\tilde{W})\Phi(\tilde{Z},\tilde{Y}) \\ &- 2\Phi(\tilde{X},\tilde{Y})\Phi(\tilde{Z},\tilde{W})\}, \end{aligned} \tag{3.2}$$

where $\alpha = \frac{k+3}{4}, \beta = \frac{k-1}{4}$ in the Sasakian case and $\alpha = \beta = \frac{k}{4}$ in the cosympletic case.

Throughout this paper, $\mathcal{N}(k)$ will denote a $(2n+1)$-dimensional Sasakian space form or cosympletic space form with constant ϕ-sectional curvature k unless otherwise stated. Obviously, $\mathcal{N}(k)$ is an η-Einstein manifold.

For a harmonic map $f : (M^m, g) \longrightarrow \mathcal{N}(k)$ we obtain from (3.1) and (3.2)

$$\begin{aligned} Tr(R_f) &= \sum_{i=1}^{m}\sum_{a=1}^{2n+1} h(R_h(v_a, f_*e_i)f_*e_i, v_a) \\ &= 4(\alpha n + \beta)e(f) - 2\beta(n+1)\|f^*\eta\|^2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} Tr(R_f{}^2) &= \sum_{i,j=1}^{m}\sum_{a=1}^{2n+1} h(R_h(v_a, f_*e_i)f_*e_i, R_h(v_a, f_*e_j)f_*e_j) \\ &= \{(2n-1)\alpha^2 + 4\alpha\beta + \beta^2\}(trf^*h)^2 \\ &+ (\alpha^2 + 9\beta^2)\|f^*h\|^2 - 6\alpha\beta\|f^*\Phi\|^2 \\ &+ (-4\alpha\beta - 16\beta^2)\sum_{i,j=1}^{m} \eta(f_*e_i)\eta(f_*e_j)h((f_*e_i, f_*e_j) \\ &+ 2(n+7)\beta^2\|f^*\eta\|^4, \end{aligned} \tag{3.4}$$

(3.5)

$$\begin{aligned}\|R^{\tilde{\nabla}}\|^2 &= \sum_{i,j=1}^{m}\sum_{a,b=1}^{2n+1} h(R_h(f_*e_i, f_*e_j)v_a, v_b)h(R_h(f_*e_i, f_*e_j)v_a, v_b)\\ &= -2(\alpha^2+\beta^2)\|f^*h\|^2 + 8\alpha\beta\sum_{i,j=1}^{m}\eta(f_*e_i)\eta(f_*e_j)h(f_*e_i, f_*e_j)\\ &+ 2(\alpha^2+\beta^2)(trf^*h)^2 - 8\alpha\beta(trf^*h)\|f^*\eta\|^2\\ &+ \{12\alpha\beta + 8\beta^2(n+1)\}\|f^*\Phi\|^2,\end{aligned}$$

where $\|f^*\eta\|^2 := \sum_{i=1}^{m}\eta(f_*e_i)\eta(f_*e_i)$, $\|f^*\Phi\|^2 := \sum_{i,j=1}^{m} h(f_*e_i, \phi f_*e_j)^2$, $\|f^*h\|^2 := \sum_{i,j=1}^{m} h(f_*e_i, f_*e_j)^2$, $\{e_i : i = 1, \cdots, m\}$ is a local orthonormal frame field on M, and $\{v_a : a = 1, \cdots, 2n+1\}$ is a local orthonormal frame field on $\mathcal{N}(k)$.

Thus substituting (3.3) $\sim$ (3.5) into (2.2) $\sim$ (2.4), we get

THEOREM 2. *For a harmonic map $f : (M, g) \longrightarrow \mathcal{N}(k)$ of an m-dimensional compact Riemannian manifold (M, g) into a $(2n+1)$-dimensional Sasakian or cosympletic space form $\mathcal{N}(k)$. Then the coefficients $a_0(J_f)$, $a_1(J_f)$ and $a_2(J_f)$ of the asymptotic expansion for the Jacobi operator J_f are respectively given by*

$$a_0(J_f) = (2n+1)Vol(M, g). \tag{3.6}$$

$$\begin{aligned}a_1(J_f) = \frac{(2n+1)}{6}\int_M \tau_g dv_g - 2\beta(n+1)\int_M \|f^*\eta\|^2 dv_g\\ + 4(\alpha n + \beta)E(f),\end{aligned} \tag{3.7}$$

(3.8)

$$\begin{aligned}a_2(J_f) = \frac{2n+1}{360}\int_M [5{\tau_g}^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2]dv_g\\ + \frac{1}{12}\int_M [8(\alpha^2+7\beta^2)\|f^*h\|^2 - 32(\alpha\beta+3\beta^2)\sum_{i,j=1}^{m}\eta(f_*e_i)\eta(f_*e_j)\times\\ h(f_*e_i, f_*e_j) + 16\{(3n-2)\alpha^2 + 6\alpha\beta + \beta^2\}e(f)^2 + 16\alpha\beta\|f^*\eta\|^2 e(f)\end{aligned}$$

$$- 8\{6\alpha\beta + \beta^2(n+1)\}\|f^*\Phi\|^2 + 12(n+7)\beta^2\|f^*\eta\|^4]dv_g$$
$$+ \frac{2}{3}\int_M (\alpha n + \beta)\tau_g e(f) dv_g - \frac{1}{3}\int_M \beta(n+1)\|f^*\eta\|^2 \tau_g dv_g.$$

4. Isometric minimal immersions

Let $\mathcal{N} = (N, \phi, \xi, \eta, h)$ be a $(2n+1)$-dimensional Sasakian or cosympletic manifold and $f : (M, g) \longrightarrow \mathcal{N}$ be an isometric immersion of a Riemannian manifold (M, g) into $\mathcal{N}$. f is called an *invariant immersion* if $\phi(f_*TM) \subset f_*TM$ and ξ is tangent to $f(M)$ everywhere on M. If f is an invariant immersion, then the immersion f is minimal([cf.11]). f is called an *tangential*(*normal* resp.)*anti-invariant immersion* if $\phi(f_*TM) \perp f_*TM$ and ξ is tangent(normal resp.)to $f(M)$ everywhere on M. It was known in a Sasakian manifold $\mathcal{N} = (N, \phi, \xi, \eta, h)$ ([cf.11]) that if the structure vector field ξ is normal to $f(M)$ (*i.e.*, f is a *C-totally real immersion*), then the immersion f is normal anti-invariant.

PROPOSITION 3. *Let f and f' be isometric minimal immersions of compact Riemannian manifolds (M, g) and (M', g') into an η-Einstein manifold, respectively. Assume that $Spec(J_f) = Spec(J_{f'})$ and the structure vector field ξ is normal (or tangent) to $f(M)$ and $f'(M')$. Then we have*

(i) $dim(M) = dim(M')$,
(ii) $Vol(M, g) = Vol(M', g')$,
(iii) $\int_M \tau_g dv_g = \int_{M'} \tau_{g'} dv_{g'}$.

PROOF. (i) follows from the asymptotic expansion (2.1), (ii) $\sim$ (iii) from (2.2), (2.3) and Corollary 1. □

The following Propositions 4 and 5 are due to the structure equation of Gauss and (iii) of Proposition 3.

PROPOSITION 4. *Let f, f' be invariant immersions of compact Riemannian manifolds (M, g) and (M', g') into a Sasakian or cosympletic space form $\mathcal{N}(k)$ respectively. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is a totally geodesic immersion, then so is f'.*

PROPOSITION 5. *Let f, f' be tangential or normal anti-invariant, minimal immersions of compact Riemannian manifolds (M, g), (M', g') into a Sasakian or cosympletic space form $\mathcal{N}(k)$ respectively. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is a totally geodesic immersion, then so is f'.*

LEMMA 6. *Let f, f' be isometric minimal immersions of compact Riemannian manifolds (M, g) into a Sasakian or cosympletic space form $\mathcal{N}(k)$ ($k \neq 1$ for the Sasakian case). Assume that $Spec(J_f) = Spec(J_{f'})$. Then ξ is tangent(normal resp.) to $f(M)$ if and only if ξ is tangent(normal resp.) to $f'(M)$.*

PROOF. Since f and f' are isometric immersions, $e(f) = \frac{1}{2}dim(M) = e(f')$. It is clear from (3.6) and (3.7) that $\int_M \|f^*\eta\|^2 dv_g = \int_M \|f'^*\eta\|^2 dv_g$. From which, the tangency and normality of ξ are respectively preserved. □

THEOREM 7. *Let f, f' be isometric minimal immersions of a compact Riemannian manifold (M, g) into a Sasakian or cosympletic space form $\mathcal{N}(k)$ with constant ϕ-sectional curvature k ($k \neq 1$ for the Sasakian case). Assume that $Spec(J_f) = Spec(J_{f'})$. Then*

(a) *if f is an invariant immersion, then so is f',*

(b) *if f is a tangential anti-invariant immersion, then so is f',*

(c) *if f is a normal anti-invariant immersion, then so is f'.*

PROOF. To begin with we prove (a) and (b). Since f, f' are isometric immersions, we have

$$e(f) = e(f') = \frac{1}{2}dim(M), \quad \|f^*h\|^2 = \|f'^*h\|^2 = dim(M).$$

Moreover

$$\sum_{i,j=1}^{m} \eta(f_*e_i)\eta(f_*e_j)h(f_*e_i, f_*e_j) = \|f^*\eta\|^2 = 1$$

$$= \|f'^*\eta\|^2 = \sum_{i,j=1}^{m} \eta(f'_*e_i)\eta(f'_*e_j)h(f'_*e_i, f'_*e_j)$$

because of Lemma 6, where $\{e_i : i = 1, \cdots, m\}$ is a local orthonormal frame field on M. Thus (3.8) implies that

$$\int_M \|f^*\Phi\|^2 dv_g = \int_M \|f'^*\Phi\|^2 dv_g. \tag{4.1}$$

On the the other hand,

$$\begin{aligned} \|f^*\Phi\|^2 &= \sum_{i,j=1}^{m} h(\phi f_* e_i, f_* e_j) h(\phi f_* e_i, f_* e_j) \\ &= \sum_{i,j=1}^{m} h(P\phi f_* e_i, f_* e_j) h(P\phi f_* e_i, f_* e_j) \\ &= \sum_{i=1}^{m} h(P\phi f_* e_i, P\phi f_* e_i), \end{aligned} \tag{4.2}$$

where P is the orthogonal projection of $T_{f(x)}M$ onto $f_* T_x M$ with respect to the metric h. Hence we obtain the inequality

$$0 \leq \|f^*\Phi\|^2 \leq \sum_{i=1}^{m} h(\phi f_* e_i, \phi f_* e_i) = m - 1, \tag{4.3}$$

if ξ is tangent to $f(M)$.

Under the assumption that ξ is tangent to $f(M)$, the following statements hold ;

(i) $\phi(f_*TM) \subset f_*TM$ iff $\int_M \|f^*\Phi\|^2 dv_g = (m-1)Vol(M, g)$,

(ii) $\phi(f_*TM) \perp f_*TM$ iff $\int_M \|f^*\Phi\|^2 dv_g = 0$.

Then (a) follows from (i) and (4.1) $\sim$ (4.3), and (b) from (ii) and (4.1). Finally (c) follows from Lemma 6 and (4.1), since (4.1) still holds. Hence we complete the proof. □

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