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THE INDUCED AND INTRINSIC CONNECTIONS OF BERWALD TYPE IN A FINSLERIAN HYPERSURFACE

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ABSTRACT. The main purpose of the present paper is to derive the induced (Finsler) connections on the hypersurface from the Finsler connections of Berwald type (a Berwald h-recurrent connection and a $F\Gamma'$ connection) of a Finsler space and to seek the necessary and sufficient conditions that the induced connections coincide with the intrinsic connections. And we show the quantities and relations with respect to the respective induced connections. Finally we show some examples.

1. Introduction

The hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of an n-dimensional Finsler space $F^n = (M^n, L(x,y))$ with fundamental function $L(x,y)$ is an (n-1)-dimensional Finsler space with the *induced metric function* $\underline{L}(u,v)$ [2].

In the present paper we are concerned with the hypersurface F^{n-1} of a Finsler space F^n endowed with Finsler connections of Berwald type (a Berwald h-recurrent connection and a $F\Gamma'$ connection). By a similar method to [7] we shall derive the induced (Finsler) connections on the F^{n-1} from the given Finsler connections of F^n and seek the necessary and sufficient conditions that the induced connections coincide with the intrinsic connections. And we show the quantities and relations with respect to the respective induced connections. Finally we show some examples.

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Throughout the present paper we shall use the terminology and notations of [2] and especially the quotation from [2] is indicated by putting asterisk $(*)$.

2. The induced Berwald h-recurrent connection

As a Finsler connection which is valuable in the theory of conformal changes of metric we have a *Berwald h-recurrent connection* [3] $B^h\Gamma = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ of a Finsler space $F^n = (M^n, L(x,y))$ (B^h1 : h-recurrent, i.e. $L_{|i} = Ls_i$, where $s_i(x,y)$ is a covariant vector field, B^h2 : $(h)h$-torsion $T_j{}^i{}_k = 0$, B^h3 : deflection $D^i{}_k = 0$, B^h4 : $(v)hv$-torsion $P^i{}_{jk} = 0$, B^h5 : $(h)hv$-torsion $C_j{}^i{}_k = 0$).
We shall denote by $IB^h\Gamma$ the connection of a hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ induced from the Berwald h-recurrent connection $B^h\Gamma$ and indicate the quantities with respect to $IB^h\Gamma$ by putting "b^h" on them. Then $*(2.6)$ and (B^h5) show $\overset{b^h}{C}_\beta{}^\alpha{}_\gamma = 0$. From $*(3.12)$ and (B^h1) we have $\underline{L}_{|\alpha} = \underline{L}s_\alpha$. Next $*(2.18)$, (B^h3) and (B^h5) give $\overset{b^h}{D}{}^\alpha{}_\gamma = 0$. Further $*(2.16)$, (B^h2) and (B^h5) yield $\overset{b^h}{T}_\beta{}^\alpha{}_\gamma = 0$. From $*(2.20)$, (B^h4) and (B^h5) we get

$$\overset{b^h}{P}{}^\alpha{}_{\beta\gamma} = 2\overset{b^h}{H}_\beta M^\alpha_\gamma, \tag{2.1}$$

from which, taking account of $v^\alpha M_{\alpha\beta} = 0$ and $*(1.13)$, we have

$$\overset{b^h}{P}{}^\alpha{}_{\beta 0} = 0 \tag{2.2}$$

and

$$v^\delta(\dot\partial_\gamma \overset{b^h}{Q}{}^\alpha{}_{\delta\beta} - \dot\partial_\beta \overset{b^h}{Q}{}^\alpha{}_{\delta\gamma}) = 0, \tag{2.3}$$

where we put $\overset{b^h}{Q}{}^\alpha{}_{\delta\beta} = \overset{b^h}{P}{}^\alpha{}_{\delta\beta} - \overset{b^h}{P}{}^\alpha{}_{\beta\delta}$.

Here (2.2) and (2.3) are the conditions for $IB^h\Gamma$ to be a generalized Berwald P^1-connection of [2]. Thus we have

THEOREM 2.1. *The connection $IB^h\Gamma$ of a hypersurface F^{n-1} in a Finsler space F^n, induced from the Berwald h-recurrent connection $B^h\Gamma$ of F^n, is a generalized Berwald h-recurrent P^1-connection which is uniquely determined from the induced metric $\underline{L}(u,v)$ and a covariant vector field $s_\alpha(u,v)$ such that*

(IB^h1) *h-recurrent :* $\underline{L}_{|\alpha} = \underline{L}s_\alpha$,

(IB^h2) *The (h)h-torsion tensor* $\overset{b^h}{T}{}_\beta{}^\alpha{}_\gamma = 0$,

(IB^h3) *The deflection tensor* $\overset{b^h}{D}{}^\alpha{}_\gamma = 0$,

(IB^h4) *The (v)hv-torsion tensor* $\overset{b^h}{P}{}^\alpha{}_{\beta\gamma}$ *is given by(2.1),*

(IB^h5) *The (h)hv-torsion tensor* $\overset{b^h}{C}{}_\beta{}^\alpha{}_\gamma = 0$.

THEOREM 2.2. *The induced Berwald h-recurrent connection $IB^h\Gamma$ of F^{n-1} coincides with the intrinsic Berwald h-recurrent connection $B^h\underline{\Gamma}$ of F^{n-1} if and only if (1) a normal curvature vector $\overset{b^h}{H}{}_\beta = 0$ or (2) a Brown tensor $M_{\alpha\beta} = 0$.*

We shall find the quantities and relations with respect to $IB^h\Gamma = (\overset{b^h}{F}{}_\beta{}^\alpha{}_\gamma, \overset{b^h}{N}{}^\alpha{}_\gamma, 0)$ induced from $B^h\Gamma = (F_j{}^i{}_k, N^i{}_k, 0)$. From *(2.4) and *(2.9) we have

$$\text{(2.4)} \qquad \begin{aligned} \overset{b^h}{N}{}^\alpha{}_\gamma &= B_i^\alpha(B_{0\gamma}^i + N^i{}_k B_\gamma^k), \\ \overset{b^h}{F}{}_\beta{}^\alpha{}_\gamma &= B_i^\alpha(B_{\beta\gamma}^i + F_j{}^i{}_k B_{\beta\gamma}^{jk}). \end{aligned}$$

The normal curvature vector $\overset{b^h}{H}{}_\beta$ and the second fundamental h-tensor $\overset{b^h}{H}{}_{\beta\gamma}$ are respectively given by *(2.7) and *(2.10) :

$$\text{(2.5)} \qquad \overset{b^h}{H}{}_\beta = B_i(B_{0\beta}^i + N^i{}_j B^j{}_\beta)$$

and

$$\overset{b^h}{H}_{\beta\gamma} = B_i(B^i_{\beta\gamma} + F_j{}^i{}_k B^{jk}_{\beta\gamma}). \tag{2.6}$$

from which we have

$$\overset{b^h}{H}_{\beta\gamma} = \overset{b^h}{H}_{\gamma\beta}, \qquad \overset{b^h}{H}_{0\gamma} = \overset{b^h}{H}_{\gamma 0} = \overset{b^h}{H}_{\gamma}. \tag{2.7}$$

Finally *(2.21), ($B^h 4$) and ($B^h 5$) show

$$\dot{\partial}_\beta \overset{b^h}{H}_\gamma - \overset{b^h}{H}_{\beta\gamma} = M_\beta \overset{b^h}{H}_\gamma, \tag{2.8}$$

from which, by virtue of (2.7), we have

$$\dot{\partial}_\beta \overset{b^h}{H}_0 = 2\overset{b^h}{H}_\beta + M_\beta \overset{b^h}{H}_0. \tag{2.9}$$

3. The induced $F\Gamma'$ connection

We are concerned with a generalized Berwald P^1-connection of F^n with $P^i{}_{jk} = -A_j{}^i{}_k$, where $A_j{}^i{}_k = Lg_j{}^i{}_k(g_j{}^i{}_k$: Cartan's C-tensor). This connection is called a $F\Gamma'$ *connection* [8] $F\Gamma' = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ of $F^n(\Gamma'1$: L-metrical, $\Gamma'2$: $(h)h$-torsion $T_j{}^i{}_k = 0$, $\Gamma'3$: deflection $D^i{}_k = 0$, $\Gamma'4$: $(v)hv$-torsion $P^i{}_{jk} = -A_j{}^i{}_k$, $\Gamma'5$: $(h)hv$-torsion $C_j{}^i{}_k = 0$).

We shall denote by $IF\Gamma'$ the connection of F^{n-1} induced from this $F\Gamma'$ and indicate the quantities with respect to $IF\Gamma'$ by putting "γ'" on them. Then *(2.6) and ($\Gamma'5$) show $\overset{\gamma'}{C}_\beta{}^\alpha{}_\gamma = 0$. From *(3.12) and ($\Gamma'1$) we have $\underline{L}_{\overset{\gamma'}{|}\alpha} = 0$. Next *(2.18), ($\Gamma'3$) and ($\Gamma'5$) give $\overset{\gamma'}{D}{}^\alpha{}_\gamma = 0$. Further *(2.16), ($\Gamma'2$) and ($\Gamma'5$) yield $\overset{\gamma'}{T}_\beta{}^\alpha{}_\gamma = 0$. From *(2.20), ($\Gamma'4$), ($\Gamma'5$), *(1.2) and *(1.4) we get

$$\overset{\gamma'}{P}{}^\alpha{}_{\beta\gamma} = 2\overset{\gamma'}{H}_\beta M^\alpha_\gamma - A_\beta{}^\alpha{}_\gamma, \tag{3.1}$$

where $A_\beta{}^\alpha{}_\gamma = \underline{L}(u,v) g_\beta{}^\alpha{}_\gamma$, from which, taking account of $y^k g_j{}^i{}_k = 0$ and $v^\alpha \dot{\partial}_\beta \overset{\gamma'}{H}_\alpha - \overset{\gamma'}{H}_\beta = M_\beta \overset{\gamma'}{H}_0$, we have

$$\overset{\gamma'}{P}{}^\alpha{}_{\beta 0} = 0 \tag{3.2}$$

and

$$v^\delta (\dot{\partial}_\gamma \overset{\gamma'}{Q}{}^\alpha{}_{\delta\beta} - \dot{\partial}_\beta \overset{\gamma'}{Q}{}^\alpha{}_{\delta\gamma}) = 0, \tag{3.3}$$

where we put $\overset{\gamma'}{Q}{}^\alpha{}_{\delta\beta} = \overset{\gamma'}{P}{}^\alpha{}_{\delta\beta} - \overset{\gamma'}{P}{}^\alpha{}_{\beta\delta}$ by virtue of *(1.13).

Here (3.2) and (3.3) are the conditions for $IF\Gamma'$ to be a generalized Berwald P^1-connection. Thus we have

THEOREM 3.1. *The connection $IF\Gamma'$ of a hypersurface F^{n-1} in a Finsler space F^n, induced from the $F\Gamma'$ connection of F^n, is a generalized Berwald P^1-connection which is uniquely determined from the induced metric $\underline{L}(u, v)$ by the following five axioms :*

($I\Gamma'1$) $\underline{L}_{|\alpha} = 0$,

($I\Gamma'2$) *The (h)h-torsion tensor* $\overset{\gamma'}{T}_\beta{}^\alpha{}_\gamma = 0$,

($I\Gamma'3$) *The deflection tensor* $\overset{\gamma'}{D}{}^\alpha{}_\gamma = 0$,

($I\Gamma'4$) *The (v)hv−torsion tensor* $\overset{\gamma'}{P}{}^\alpha{}_{\beta\gamma}$ *is given by (3.1),*

($I\Gamma'5$) *The (h)hv-torsion tensor* $\overset{\gamma'}{C}_\beta{}^\alpha{}_\gamma = 0$.

THEOREM 3.2. *The induced connection $IF\Gamma'$ of F^{n-1} coincides with the intrinsic connection $F\underline{\Gamma}'$ of F^{n-1} if and only if (1) a normal curvature vector $\overset{\gamma'}{H}_\beta = 0$ or (2) a Brown tensor $M_{\alpha\beta} = 0$.*

We shall find the quantities and relations with respect to $IF\Gamma' = (\overset{\gamma'}{F}_\beta{}^\alpha\gamma, \overset{\gamma'}{N}{}^\alpha\gamma, 0)$ induced from $F\Gamma' = (G_j{}^i{}_k + A_j{}^i{}_k, G^i{}_k, 0)$ by virtue of Theorem 1 of Aikou and Hashiguchi [1], where $B\Gamma = (G_j{}^i{}_k, G^i{}_k, 0)$ is a Berwald connections [6].

From $^*(2.4)$, $^*(1.2)$, $^*(1.4)$ and $^*(2.9)$ we have

$$\overset{\gamma'}{N}{}^{\alpha}\gamma = \overset{b}{N}{}^{\alpha}\gamma, \tag{3.4}$$
$$\overset{\gamma'}{F}{}_{\beta}{}^{\alpha}\gamma = \overset{b}{F}{}_{\beta}{}^{\alpha}\gamma + A_{\beta}{}^{\alpha}\gamma,$$

where $IB\Gamma = (\overset{b}{F}{}_{\beta}{}^{\alpha}\gamma, \overset{b}{N}{}^{\alpha}\gamma, 0)$ is an induced Berwald connections.
The normal curvature vector $\overset{\gamma'}{H}{}_{\beta}$ and the second fundamental h-tensor $\overset{\gamma'}{H}{}_{\beta\gamma}$ are respectively given by $^*(2.7)$, $^*(1.2)$, $^*(1.9)$ and $^*(2.10)$:

$$\overset{\gamma'}{H}{}_{\beta} = \overset{b}{H}{}_{\beta} \tag{3.5}$$

and

$$\overset{\gamma'}{H}{}_{\beta\gamma} = \overset{b}{H}{}_{\beta\gamma} + \underline{L}M_{\beta\gamma}, \tag{3.6}$$

where $\overset{b}{H}{}_{\beta}$ and $\overset{b}{H}{}_{\beta\gamma}$ are the normal curvature vector and the second fundamental h-tensor with respect to $IB\Gamma$ respectively, from which we have

$$\overset{\gamma'}{H}{}_{\beta\gamma} = \overset{\gamma'}{H}{}_{\gamma\beta}, \qquad \overset{\gamma'}{H}{}_{0\gamma} = \overset{\gamma'}{H}{}_{\gamma 0} = \overset{b}{H}{}_{\gamma}. \tag{3.7}$$

Finally $^*(2.21)$, $(\Gamma'4)$ and $(\Gamma'5)$ show

$$\dot{\partial}_{\beta}\overset{\gamma'}{H}{}_{\gamma} - \overset{\gamma'}{H}{}_{\beta\gamma} = M_{\beta}\overset{b}{H}{}_{\gamma} - \underline{L}M_{\beta\gamma}, \tag{3.8}$$

from which, by virtue of (3.5) and (3.7), we have

$$\dot{\partial}_{\beta}\overset{\gamma'}{H}{}_{0} = 2\overset{b}{H}{}_{\beta} + M_{\beta}\overset{b}{H}{}_{0}. \tag{3.9}$$

4. Some examples

M.Matsumoto and S.Hojo [5] have proved that C-reducible Finsler spaces [4] are confined solely to the Randers and Kropina spaces among the Finsler spaces with an (α, β)-metric $L(\alpha, \beta)$.

We showed in [7] that the $(h)hv$-torsion tensor $C_{\alpha\beta\gamma}$ and the Brown tensor $M_{\alpha\beta}$ of hypersurfaces in a Randers space and a Kropina space are respectively given by

$$\text{(4.1)} \qquad \overset{R}{C}_{\alpha\beta\gamma} = (h_{\alpha\beta}L_\gamma + h_{\beta\gamma}L_\alpha + h_{\gamma\alpha}L_\beta)/2\underline{L}, \qquad \overset{R}{M}_{\alpha\beta} = b_i B^i h_{\alpha\beta}/2\underline{L},$$

where $L_\alpha = b_\alpha - \mu l_\alpha$, $\mu = \beta/\alpha$ (a Riemannian metric $\alpha = a_{ij}(x)y^i y^j$, a differential 1-form $\beta = b_i(x)y^i$) and $h_{\alpha\beta}$ is the angular metric tensor and

$$\text{(4.2)} \qquad \overset{K}{C}_{\alpha\beta\gamma} = (h_{\alpha\beta}m_\gamma + h_{\beta\gamma}m_\alpha + h_{\gamma\alpha}m_\beta)/2\underline{L}, \qquad \overset{K}{M}_{\alpha\beta} = -\tau b_i B^i h_{\alpha\beta}/2\underline{L},$$

where $m_\alpha = l_\alpha - \tau b_\alpha$, $\tau = \alpha^2/\beta^2$.

Let $F^n = (M^n, L = \alpha + \beta)$ be a Randers space. From (4.1) the hypersurface F^{n-1} of F^n is C-reducible and the $(v)hv$-torsion tensors $\overset{b^h}{P}{}^\alpha{}_{\beta\gamma}$ and $\overset{\gamma'}{P}{}^\alpha{}_{\beta\gamma}$ of F^{n-1} are respectively given by

$$\text{(4.3)} \qquad \overset{b^h}{P}{}^\alpha{}_{\beta\gamma} = 2\overset{b^h}{R}_\beta h^\alpha_\gamma$$

where $\overset{b^h}{R}_\beta = b_i B^i \overset{b^h}{H}_\beta/2\underline{L}$ and

$$\text{(4.4)} \qquad \overset{\gamma'}{P}{}^\alpha{}_{\beta\gamma} = 2\overset{b}{R}_\beta h^\alpha_\gamma - A_\beta{}^\alpha{}_\gamma,$$

where $\overset{b}{R}_\beta = b_i B^i \overset{b}{H}_\beta/2\underline{L}$.

Next, let $F^n = (M^n, L = \alpha^2/\beta)$ be a Kropina space. From (4.2) the hypersurface F^{n-1} of F^n is C-reducible and the $(v)hv$-torsion tensors $\overset{b^h}{P}{}^{\alpha}{}_{\beta\gamma}$ and $\overset{\gamma'}{P}{}^{\alpha}{}_{\beta\gamma}$ of F^{n-1} are respectively given by

$$\overset{b^h}{P}{}^{\alpha}{}_{\beta\gamma} = 2\overset{b^h}{K}_{\beta} h^{\alpha}_{\gamma}, \tag{4.5}$$

where $\overset{b^h}{K}_{\beta} = -\tau b_i B^i \overset{b^h}{H}_{\beta}/2\underline{L}$
and

$$\overset{\gamma'}{P}{}^{\alpha}{}_{\beta\gamma} = 2\overset{b}{K}_{\beta} h^{\alpha}_{\gamma} - A_{\beta}{}^{\alpha}{}_{\gamma}, \tag{4.6}$$

where $\overset{b}{K}_{\beta} = -\tau b_i B^i \overset{b}{H}_{\beta}/2\underline{L}$. Thus we have

THEOREM 4.1. *Any hypersurface F^{n-1} of a Randers space is C-reducible. The $IB^h\Gamma$ and $IF\Gamma'$ of F^{n-1} satisfy (4.3) and (4.4) respectively.*

THEOREM 4.2. *Any hypersurface F^{n-1} of a Kropina space is C-reducible. The $IB^h\Gamma$ and $IF\Gamma'$ of F^{n-1} satisfy (4.5) and (4.6) respectively.*

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