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UNIFORM L^p-APPROXIMATION FOR THE SOLUTIONS OF FUNCTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this work is to obtain uniform L^p-approximation for the solutions of functional stochastic differential equations driven by continuous semimartingale.

1. Introduction

We are given an one-dimensional continuous semimartingale $\{Z_t, \mathcal{F}_t\}$ on a filtered, complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathrm{P})$ satisfying the usual hypotheses. For $\mathcal{F}_0$-measurable random variable ξ, we consider the following functional SDE driven by $\{Z_t\}$:

$$X_t = \xi + \int_0^t F(X)_s dZ_s. \tag{1.1}$$

Here F is a functional Lipschitz operator which is defined as follows. We denote C the class of adapted processes indexed by $[0, \infty)$ having continuous paths.

DEFINITION. An operator $F : C \to C$ is called functional Lipschitz if for any $X, Y \in C$ the following two conditions are satisfied:
(i) for any stopping time σ, $X^\sigma = Y^\sigma$ implies $F(X)^\sigma = F(Y)^\sigma$.
(ii) there exists an increasing process $K = \{K_t, t \geq 0\}$ such that

$$|F(X)_t - F(Y)_t| \leq K_t \sup_{s \leq t} |X_s - Y_s| \quad \text{a.s.}, \quad \text{each } t \geq 0,$$

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where for any $X \in C$, X^σ is definded to be $X_t^\sigma = X_{\sigma\wedge t}$.

Under this setting, it is well known that there exists a unique solution of equation (1.1) in C which is a semimartingale by using Picard iteration method. (e.g. Emery[1] or Protter[5], [6]) Although Picard iteration method is useful to prove the existence and uniqueness theorem, it is not efficient in numerical practice. For a classical Ito equation driven by Brownian motion, there are various numerical schemes including Euler-Maruyama method.([3], [4], [7]) In this work, we obtain uniform L^p-approximation to the solution of (1.1) with the order 1/2 of error by employing the analogue of Euler-Maruyama method. Moreover, this method seems to provide an alternative proof for existence and uniqueness theorem for (1.1).

To describe our main result, we assume that $Z_0 = 0$, and let $Z_t = M_t + B_t$, $M_0 = B_0 = 0$ be a decomposition of Z, where $\{M_t, \mathcal{F}_t\}$ is a continuous martingale and $\{B_t, \mathcal{F}_t\}$ is a continuous process of bounded variation with total variation $|B|_t$. Let

$$A_t = t + \langle M\rangle_t + |B|_t. \tag{1.2}$$

Assume that $F : C \to C$ is a functional Lipschitz satisfying the following conditions: there exists non-random constant β such that for any $X, Y \in C$ and $0 \le s, t \le T$,

$$A_T \le \beta \ a.s., \tag{1.3}$$

$$|F(X)_t - F(Y)_t| \le \beta |X_t - Y_t| \ a.s., \tag{1.4}$$

$$|F(X)_t - F(X)_s| \le \beta \left(|t-s|^{1/2} + |X_t - X_s|\right) \ a.s., \tag{1.5}$$

$$F(0) = 0. \tag{1.6}$$

Now for each n, and $0 \le k \le 2n$, we introduce stopping times $\sigma_k^{(n)}$, and define $\{\bar{X}_t^{(n)}\}$ as follows :

$$\begin{aligned} &\sigma_0^{(n)} = 0, \\ &\sigma_k^{(n)} = \inf\{t > \sigma_{k-1}^{(n)} ;\ A_t - A_{\sigma_{k-1}^{(n)}} > \frac{\beta}{n}\} \wedge T, \end{aligned} \tag{1.7}$$

$$\bar{X}_0^{(n)} = \xi,$$

$$\bar{X}_t^{(n)} = \bar{X}_{\sigma_k^{(n)}}^{(n)} + F(\bar{X}^{(n)\sigma_k^{(n)}})_{\sigma_k^{(n)}}(Z_t - Z_{\sigma_k^{(n)}}) \quad \text{for } \sigma_k^{(n)} < t \le \sigma_{k+1}^{(n)}. \tag{1.8}$$

Our main result is that if $p > 1$ and $E|\xi|^p < \infty$, then

$$E\left(\sup_{0\le t\le T} |X_t - \bar{X}_t^{(n)}|^p\right) = O\left((1/n)^{p/2}\right).$$

Finally we remark that generic constants throughout the work are denoted by the same letter C although they have different values from line to line.

2. Main result

We assume that (1.3)-(1.6) hold and $E|\xi|^p < \infty$ for a fixed $p > 1$. Recall that $\sigma_k^{(n)}$ and $\{\bar{X}_t^{(n)}\}$ are defined by (1.7)and (1.8). Before proving the main result, we need to prove preliminary lemmas. We first introduce stochastic integral inequality of Gronwall type.

LEMMA 1. *(Theorem 2.6.1 of Mao[2])*
Let $\{N_t,\ 0 \le t \le T\}$ be a nondecreasing continuous adapted process such that $N_0 = 0$ and $N_T \le K$ a.s., and let $\{Y_t,\ 0 \le t \le T\}$ be a nondecreasing progressively measurable process where K is a positive constant. If for any stopping time $\tau \le T$,

$$EY_\tau \le c + E\int_0^\tau Y_s dN_s$$

then $EY_T \le ce^K$.

LEMMA 2. *There exists a positive constant C such that the following hold :*
(a) $E\left(\sup_{0\le t\le T} |X_t|^p\right) \le C\beta^{p-1}E|\xi|^p e^{C\beta}$.
(b) $\sup_{0\le k\le 2n-1} E\left(\sup_{\sigma_k^{(n)}\le s\le\sigma_{k+1}^{(n)}} |X_s - X_{\sigma_k^{(n)}}|^p\right) \le CE|\xi|^p(1/n)^{p/2}$.

PROOF. (a) Note that for any stopping τ with $0 \le \tau \le T$,

$$E\left(\sup_{0\le s\le\tau}|X_s|^p\right) = E\left(\sup_{0\le s\le\tau}|\xi + \int_0^s F(X)_u dZ_u|^p\right)$$

$$\le 2^{p-1}E|\xi|^p + 2^{p-1}E\left(\sup_{0\le s\le\tau}|\int_0^s F(X)_u dZ_u|^p\right). \tag{2.1}$$

Using Hölder inequality and Burkholder-Davis-Gundy inequality with (1.3)-(1.6), we get

$$\begin{aligned} &E\left(\sup_{0\le s\le\tau}|\int_0^s F(X)_u dZ_u|^p\right)\\ &\le 2^{p-1}E\left(\sup_{0\le s\le\tau}|\int_0^s F(X)_u dM_u|^p\right)\\ &\quad + 2^{p-1}E\left(\sup_{0\le s\le\tau}|\int_0^s F(X)_u dB_u|^p\right) \qquad (2.2)\\ &\le CE\left|\int_0^\tau |F(X)_s|^2 dA_s\right|^{p/2} + CE\left|\int_0^\tau |F(X)_s| dA_s\right|^p\\ &\le C\beta^{p-1}E\int_0^\tau |X_s|^p dA_s \end{aligned}$$

for any stopping time τ with $0 \le \tau \le T$. Using Lemma 1 with (2.1) and (2.2), the assertion follows.

(b) As in the proof of (a), we get, for each $0 \le k \le 2n-1$,

$$\begin{aligned} &E\left(\sup_{\sigma_k^{(n)}\le s\le\sigma_{k+1}^{(n)}}|X_s - X_{\sigma_k^{(n)}}|^p\right) = E\left(\sup_{\sigma_k^{(n)}\le s\le\sigma_{k+1}^{(n)}}|\int_{\sigma_k^{(n)}}^s F(X)_u dZ_u|^p\right)\\ &\le C(1/n)^{(p-2)/2}E\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}} |X_s|^p dA_s\\ &\le C(1/n)^{p/2}E\left(\sup_{0\le t\le T}|X_t|^p\right)\\ &\le C(1/n)^{p/2}E|\xi|^p. \end{aligned}$$

□

LEMMA 3. $\sup_{0\le k\le 2n} E|X_{\sigma_k^{(n)}} - \bar{X}^{(n)}_{\sigma_k^{(n)}}|^p = O\left((1/n)^{p/2}\right)$.

PROOF. To simplify the notation, we fix n and denote $\sigma_k^{(n)}$ and $\bar{X}^{(n)}_{\sigma_k^{(n)}}$ by σ_k and $\bar{X}_{\sigma_k}$ respectively. We first consider the case when p is an even integer. Observe that for $0 \le k \le 2n-1$,

$$|X_{\sigma_{k+1}} - \bar{X}_{\sigma_{k+1}}|^p = \sum_{m=0}^{p} \binom{p}{m} (X_{\sigma_k} - \bar{X}_{\sigma_k})^{p-m} \left(\int_{\sigma_k}^{\sigma_{k+1}} \left(F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}\right) dZ_s\right)^m .$$

As in the proof of Lemma 2-(a), we get the following estimates:

$$\begin{aligned} &|E(X_{\sigma_k} - \bar{X}_{\sigma_k})^{p-1} \int_{\sigma_k}^{\sigma_{k+1}} \left(F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}\right) dZ_s| \\ &\le C(1/n)^{1-\frac{1}{p}} \left(E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p\right)^{1-\frac{1}{p}} \\ &\cdot \left(E\int_{\sigma_k}^{\sigma_{k+1}} |F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}|^p dA_s\right)^{1/p}, \end{aligned} \tag{2.3}$$

and for $2 \le m \le p$,

$$\begin{aligned} &E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^{p-m} | \int_{\sigma_k}^{\sigma_{k+1}} \left(F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}\right) dZ_s|^m \\ &\le C(1/n)^{\frac{m}{2}-\frac{m}{p}} \left(E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p\right)^{1-\frac{m}{p}} \\ &\cdot \left(E\int_{\sigma_k}^{\sigma_{k+1}} |F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}|^p dA_s\right)^{m/p}. \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\begin{aligned} E|X_{\sigma_{k+1}} - \bar{X}_{\sigma_{k+1}}|^p \le\ & E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p + C(1/n)^{1-1/p} \left(E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p\right)^{1-1/p} \\ &\cdot \left(E\int_{\sigma_k}^{\sigma_{k+1}} |F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}|^p dA_s\right)^{1/p} \\ &+ C\sum_{m=2}^{p} (1/n)^{\frac{m}{2}-\frac{m}{p}} \left(E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p\right)^{1-m/p} \\ &\cdot \left(E\int_{\sigma_k}^{\sigma_{k+1}} |F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}|^p dA_s\right)^{m/p}. \end{aligned} \tag{2.5}$$

Using (1.3)-(1.5) and Lemma 2-(b), we get

$$
\begin{aligned}
&E\int_{\sigma_k}^{\sigma_{k+1}} |F(X)_s - F(\bar{X}^{\sigma_k})_{\sigma_k}|^p dA_s \\
&\le 2^{p-1}E\int_{\sigma_k}^{\sigma_{k+1}} |F(X)_{\sigma_k} - F(\bar{X}^{\sigma_k})_{\sigma_k}|^p dA_s \\
&\quad + 2^{p-1}E\int_{\sigma_k}^{\sigma_{k+1}} |F(X)_s - F(X)_{\sigma_k}|^p dA_s \\
&\le CE\int_{\sigma_k}^{\sigma_{k+1}} |X_{\sigma_k} - \bar{X}_{\sigma_k}|^p dA_s \\
&\quad + CE\int_{\sigma_k}^{\sigma_{k+1}} \Big(|s-\sigma_k|^{1/2} + |X_s - X_{\sigma_k}|\Big)^p dA_s \\
&\le C(1/n)E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p + C(1/n)^{1+p/2}.
\end{aligned}
\tag{2.6}
$$

Putting (2.6) into (2.5), we obtain that for $0 \le k \le 2n-1$,

$$
\begin{aligned}
E|X_{\sigma_{k+1}} - \bar{X}_{\sigma_{k+1}}|^p \le & E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p + C(1/n)E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p \\
& + C(1/n)^{1+1/2}\left(E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p\right)^{1-1/p} \\
& + C\sum_{m=2}^{p}(1/n)^m\left(E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p\right)^{1-m/p}.
\end{aligned}
$$

Let $\epsilon_k = E|X_{\sigma_k} - \bar{X}_{\sigma_k}|^p$. Then

$$
\epsilon_{k+1} \le \epsilon_k + C\epsilon_k(1/n) + C\epsilon_k^{1-1/p}(1/n)^{1+1/2} + C\sum_{m=2}^{p}\epsilon_k^{1-m/p}(1/n)^m.
$$

Let $\eta_{k+1} = \eta_k + C\eta_k(1/n) + C\eta_k^{1-1/p}(1/n)^{1+1/2} + C\sum_{m=2}^{p}\eta_k^{1-m/p}(1/n)^m$, $\eta_0 = 0$. Since $\epsilon_k \le \eta_k$, it suffices to show that $\eta_k \le O(1/n)^{p/2}$. Since η_k is monotonically increasing, there exists a k_0 such that $\eta_{k_0} \le (1/n)^{p/2}$, and $\eta_k > (1/n)^{p/2}$ for $k > k_0$. Then for $k > k_0$,

$$
\begin{aligned}
\eta_{k+1} =& \eta_k + C\eta_k(1/n) + C\eta_k^{1-1/p}(1/n)^{1+1/2} + C\sum_{m=2}^{p}\eta_k^{1-m/p}(1/n)^m \\
\le& \eta_k + C\eta_k(1/n) + C\sum_{m=2}^{p}\eta_k(1/n)^{m/2} \\
\le& \eta_k(1 + C/n).
\end{aligned}
$$

Hence for any even integer $p > 1$,

$$\eta_k \leq \eta_{k_0}(1 + C/n)^n \leq C\eta_{k_0} \leq C(1/n)^{p/2}.$$

The proof is completed for general $p > 1$, if we write $p = k + \alpha$ where k is an even integer and $0 < \alpha < 2$ and use Hölder inequality. □

Now we are ready to present the proof of the main result.

THEOREM. *If* $p > 1$ *and* $E|\xi|^p < \infty$, *then*

$$E\left(\sup_{0\leq t\leq T}|X_t - \bar{X}_t^{(n)}|^p\right) = O\left((1/n)^{p/2}\right).$$

PROOF. Let $\phi(s) = \sum_{k=0}^{2n-1} F(\bar{X}^{(n)})_{\sigma_k^{(n)}}\chi_{\{\sigma_k^{(n)} < s \leq \sigma_{k+1}^{(n)}\}}$ and $\phi(0) = 0$. Note that for $\sigma_k^{(n)} \leq t \leq \sigma_{k+1}^{(n)}$,

$$\bar{X}_t^{(n)} = \xi + \int_0^t \phi(s)dZ_s.$$

As in the proof of Lemma 2-(a), we have

$$\begin{aligned} E\sup_{0\leq t\leq T}|\bar{X}_t^{(n)} - X_t|^p =& E\sup_{0\leq t\leq T}\left|\int_0^t(\phi(s) - F(X)_s)dZ_s\right|^p \\ \leq& CE\int_0^T|\phi(t) - F(X)_t|^p dA_t \\ =& CE\sum_{k=0}^{2n-1}\int_{\sigma_k^{(n)}}^{\sigma_{k+1}^{(n)}}|\phi(t) - F(X)_t|^p dA_t \\ \leq& C\sum_{k=0}^{2n-1}(1/n)E\left(\sup_{\sigma_k^{(n)}<t\leq\sigma_{k+1}^{(n)}}|\phi(t) - F(X)_t|^p\right). \end{aligned}$$

Using (1.3)-(1.5), Lemma 3 and Lemma 2-(b), we get

$$E\left(\sup_{\sigma_k^{(n)}<t\leq\sigma_{k+1}^{(n)}}|\phi(t)-F(X)_t|^p\right)=E\left(\sup_{\sigma_k^{(n)}<t\leq\sigma_{k+1}^{(n)}}|F(\bar{X}^{(n)})_{\sigma_k^{(n)}}-F(X)_t|^p\right)$$
$$\leq 2^{p-1}\beta^pE|\bar{X}^{(n)}_{\sigma_k^{(n)}}-X_{\sigma_k^{(n)}}|^p$$
$$+2^{p-1}\beta^pE\sup_{\sigma_k^{(n)}\leq t\leq\sigma_{k+1}^{(n)}}\left(|t-\sigma_k^{(n)}|^{1/2}+|X_t-X_{\sigma_k^{(n)}}|\right)^p$$
$$\leq C(1/n)^{p/2}$$

which completes the proof. □

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