Comm. Korean Math. Soc. **12**(1997), No. 2, pp. 427-456

WAVELETS ON THE UNIT INTERVAL WITH BOUNDARY TREATMENT

DAI-GYOUNG KIM

ABSTRACT. This paper concerns constructing wavelet bases on the unit interval, where a new boundary treatment is provided to overcome certain drawbacks of earlier constructions. Wavelet expansions on the unit interval usually suffer from artificial boundary effects and poor convergence at the boundaries. Many researchers have suggested a solution to the drawbacks. From a practical point of view, their solutions also have a common disadvantage. This paper provides a new solution using biorthogonality near the boundaries, that avoids the disadvantage while preserving their advantages.

1. Introduction

The standard wavelet theory has been developed and analyzed on $\mathbb{R}^d$. However, for many applications, it is natural to study wavelets on a domain such as an interval. For instance, a (wavelet) basis for certain function spaces on an interval is naturally required for the study of numerical methods for differential equations with boundary conditions (cf.[5]), and image compression (where the domain is a rectangle) (cf.[1], [8]). To be specific, this paper focuses on studying wavelets on the unit interval [0,1].

The first and simplest wavelet on [0,1] is the Haar function $H(x) := \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$. Since every dyadic dilation and translation $H_{k,j}(x) = 2^{k/2}H(2^k x - j)$, $k \in \mathbb{N}$, $j \in \mathbb{Z}$, of the Haar function is supported either in $[0,1]$ or in $\mathbb{R} \setminus [0,1]$, the family $\{\chi_{[0,1]}\} \cup \{H_{k,j} \mid k \in \mathbb{N},\ 0 \le j \le 2^k\}$ forms an orthonormal basis for $L_2([0,1])$. The Haar

Received August 31, 1996. Revised February 17, 1997.

1991 Mathematics Subject Classification: 65D15, 65T20, 68U10.

Key words and phrases: Wavelets, wavelet decompositions.

The author is supported by Korean Ministry of Education through Research Fund BSRI-96-2448 and by Hanyang University Research Fund.

wavelet is sometimes satisfactory in image compression applications. For instance, DeVore, Jawerth, and Lucier [8] have made a success of the compression of any image with spatial discontinuities in intensity. However, working on such applications as image compression (e.g., for motion picture) requiring high compression ratio, and as numerical solution to differential equations, we need to use smoother wavelet bases.

The expansion with smooth wavelets to a function on the interval usually suffers from artificial boundary effects (such as Gibbs' phenomenon), poor convergence at the boundaries, and redundant wavelet tails. Many researchers ([4], [15], [2]) have suggested a construction of (orthonormal) wavelet bases on $[0,1]$ to avoide the drawbacks. From a practical point of view, their constructions also have still a common disadvantage (that needs a precondition processor to reduce Gibbs' phenomenon). This paper particularly considers the constructions of [4] and [15] and discusses their advantages and disadvantages. In order to overcome the disadvantages while keeping the advantages, this paper suggests a new construction based on the discrete type of Whitney extension and biorthogonality near the boundaries.

This paper is organized as follows. In §2, wavelets and multiresolution analysis of $L_2(\mathbb{R})$ are summurized. Also, to gear up our construction, we briefly review from [3] biorthogonal wavelets. In §3, several examples of orthonomal wavelet bases on $[0,1]$ are described. In certain sense, [4] and [15] have handled the canonical case, so that we consider their constructions and point out their disadvantages. Futhermore, a new construction of wavelet bases on the interval is suggested to overcome their disadvantages. Finally, in the appendix, proofs of our propositions are collected.

2. Wavelets and Multiresolution Analysis on $\mathbb{R}$

Wavelets are usually constructed within the framework of multiresolution analysis, which was first introduced by Mallat [14] and Meyer. For wavelets constructed by other methods than multiresolution analysis, we refer to [10]. In this section, we review wavelet decompositions for $L_2(\mathbb{R})$ in the framework of multiresolution analysis. In addition, we briefly review from [3] biorthogonal wavelets for $L_2(\mathbb{R})$.

2.1. Multiresolution Analysis

Starting with a single function $\phi \in L_2(\mathbb{R})$, we form the shift invariant space

$$V := V_0 := \overline{\operatorname{span}\{\phi(\cdot - j) \mid j \in \mathbb{Z}\}}, \tag{2.1}$$

where the closure is in $L_2(\mathbb{R})$. By dyadic dilation, we also form a scale of the space

$$V_k := \{f(2^k \cdot) \mid f \in V\}, \quad k \in \mathbb{Z}. \tag{2.2}$$

A *multiresolution analysis* of $L_2(\mathbb{R})$ is defined as a sequence $(V_k)_{k\in\mathbb{Z}}$ of the closed subspaces of $L_2(\mathbb{R})$ with the following conditions:

$$\begin{aligned} &\text{(i)} \quad V_k \subset V_{k+1}, \\ &\text{(ii)} \quad \overline{\bigcup_{k\in\mathbb{Z}} V_k} = L_2(\mathbb{R}), \\ &\text{(iii)} \quad \bigcap_{k\in\mathbb{Z}} V_k = \{0\}, \\ &\text{(iv)} \quad \{\phi(\cdot - j)\}_{j\in\mathbb{Z}} \text{ forms an } L_2\text{-stable basis for } V_0. \end{aligned} \tag{2.3}$$

Here, the (iv) means that there exist positive constants C_1 and C_2 such that each $S \in V$ has a unique representation

$$\begin{aligned} &\text{(i)} \quad S = \sum_{j\in\mathbb{Z}} c_j \phi(\cdot - j), \\ &\text{(ii)} \quad C_1 \|S\|_{L_2(\mathbb{R})} \le \Big(\sum_{j\in\mathbb{Z}} |c_j|^2\Big) \le C_2 \|S\|_{L_2(\mathbb{R})}. \end{aligned}$$

The conditions (2.3)(ii), (iii) allow us to approximate a given function f in $L_2(\mathbb{R})$ by a function in each V_k. Let $\mathbf{P}_0$ be the orthogonal projector from $L_2(\mathbb{R})$ onto V_0. By dilation, let us define the orthogonal projector $\mathbf{P}_k$ from $L_2(\mathbb{R})$ onto V_k. It follows from (2.3)(ii), (iii) that each $f \in L_2(\mathbb{R})$ can be represented by the series

$$f = \sum_{k\in\mathbb{Z}} (\mathbf{P}_{k+1} f - \mathbf{P}_k f) = \sum_{k\in\mathbb{Z}} \mathbf{Q}_k f, \qquad \mathbf{Q}_k := \mathbf{P}_{k+1} - \mathbf{P}_k$$

because the partial sums, $\mathbf{P}_n f - \mathbf{P}_{-n} f$, of this series tend to f as $n \to \infty$.

The generator ϕ of V is called the *scaling function* of the multiresolution analysis. It follows from (2.3)(i), (iv) that ϕ satisfies a two scale difference equation called the *refinement equation*:

$$\phi(x) = \sum_{j \in \mathbb{Z}} a_j \phi(2x - j) \tag{2.4}$$

for a certain sequence $(a_j)_{j \in \mathbb{Z}} \in l_2(\mathbb{Z})$. This equation plays a crucial role in wavelet theory and applications. Several scaling functions can be constructed and characterized through the equation (2.4) with certain conditions on the coefficients (see [7] for example). For a sufficient condition on ϕ for multiresolution anaysis, we refer to Jia and Micchelli [11]. The main result of [6] is the existence of a scaling function with a compact support and arbitrary regularity, where their integer shifts form an orthonormal basis for V_0.

2.2. Wavelets

Let W_0 be the orthogonal complement of V_0 in V_1, that is, $V_1 = V_0 \oplus W_0$, where $\oplus$ denotes the orthogonal direct sum. Also, let W_k be the dyadic sacle of W_0. Then, it follows from (2.3) that

$$L_2(\mathbb{R}) = \bigoplus_{k \in \mathbb{Z}} W_k. \tag{2.5}$$

A generator function ψ of W_0 is called the *orthogonal wavelet* if the integer shifts $\psi(\cdot - j)$ form an orthonormal basis for W_0. In this case, the normalized and scaled functions $\psi_{k,j} := 2^{k/2}\psi(2^k \cdot - j)$ form an orthonormal system for $L_2(\mathbb{R})$ by (2.5).

Notice that, in general, the integer shifts of a generator of W^0 are not orthogonal. For instance, if the shifts of ϕ form an L_2-stable basis for V_0, then the $\psi_{k,j}$ form an L_2-stable basis for $L_2(\mathbb{R})$. Instead of global orthogonality, they always possess orthogonality between dyadic levels (cf. [11]),

$$\int_{\mathbb{R}} \psi_{k,j} \psi_{k',j'} \, dx = 0, \qquad k \neq k'.$$

Such a generator function ψ is called the *prewavelet*.

Prewavelets can be obtained by the formula (cf. [11], [14]):

$$\psi := \sum_{j\in\mathbb{Z}} (-1)^j \overline{\mu_{1-j}}\, \phi(2\,\cdot\, -j), \quad \text{with} \quad \mu_j := \int_{\mathbb{R}} \overline{\phi(y)}\phi(2y-j)\,dy.$$

By setting $\hat{\gamma} = \hat{\psi}/\sum_{\beta\in 2\pi\mathbb{Z}} \hat{\psi}(\,\cdot\, + \beta)\,\overline{\hat{\psi}(\,\cdot\, + \beta)}$, one sees that every f in $L_2(\mathbb{R})$ admits a wavelet decomposition

$$f = \sum_{k,j\in\mathbb{Z}} \langle f, \gamma_{k,j}\rangle\, \psi_{k,j}.$$

Also, there are different generalizations of orthogonal wavelets (e.g., biorthogonal wavelets).

A simple example of wavelets is the Haar function derived from the characteristic function of $[0,1]$, $\phi = \chi_{[0,1]}$. This is a special case of compactly supported orthogonal wavelets. Compactly supported orthogonal wavelets with arbitrary regularity have been constructed by Daubechies [6]. The generator $\phi = {}_N\phi$ associated with given integer N in [6], satisfies the refinement equation

$$\phi(x) = \sqrt{2} \sum_{j=-N+1}^{N} h_j \phi(2x-j) \tag{2.6}$$

with $2N$ nonzero coefficients $(h_j)_{j=-N+1}^{N}$ satisfying certain conditions. Also, the wavelet $\psi = {}_N\psi$ associated N is defined as

$$\psi(x) = \sqrt{2} \sum_{j=-N+1}^{N} g_j \phi(2x-j) \tag{2.7}$$

with $g_j = (-1)^j h_{1-j}$.

An important property of Daubechies' wavelets is the vanishing moment condition:

$$\left.\frac{d^l}{d\xi^l} m_0\right|_{\xi=\pi} = 0, \qquad l = 0, \cdots, N-1, \tag{2.8}$$

where $m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=-N+1}^{N} h_n e^{-in\xi}$ is the trigonometric polynomial associated with $(h_j)_{j=-N+1}^{N}$. Since the functions $\phi(\cdot - j)$, $j \in \mathbb{Z}$, are linearly independent, (2.8) is equivalent to the fact that any polynomial of degree less than N can be written locally as a linear combination of the $\phi(\cdot - j)$ (cf. [9]). Thus, the spaces V_k locally contain the polynomials of degree less than N. Further, since ψ is orthogonal to all the $\phi(\cdot - j)$, ψ has the vanishing moments up to order $N - 1$; that is,

$$\int x^l \psi(x)\, dx = 0, \quad l = 0, \cdots, N-1. \tag{2.9}$$

Equivalently, the coefficients h_j satisfy the sum rule:

$$\sum_{j=-N+1}^{N} (-1)^j j^l h_j = 0, \quad l = 0, \cdots, N-1. \tag{2.10}$$

We shall implicitly use the sum rule for our construction in §3.3. Notice that $\operatorname{supp} \phi = \operatorname{supp} \psi = [-N+1, N]$. This is the minimal support under the constraint that m_0 has a zero of order N at $\xi = \pi$. The regularity of the ϕ and ψ increases linearly with N; indeed, Daubechies has shown that for large N, $\phi, \psi \in C^{\mu N}$ with $\mu \simeq 0.2$ (cf. [6], [7]).

2.3. Biorthogonal Wavelets

Biorthogonal wavelets initially start from a subband filtering scheme with exact reconstruction using synthesis filters different from analysis filters (cf. [3], [7]). Such a subband scheme with two filter coefficient sequences (h_n), $(\tilde{h}_n)$ under certain conditions gives two scaling functions ϕ, $\tilde{\phi}$, respectively that satisfy the biorthogonality condition

$$\langle \phi(\cdot - j), \tilde{\phi}(\cdot - n) \rangle = \delta_{j,n} \tag{2.11}$$

and the refinement equations

$$\phi(x) = \sum_n h_n \phi(2x - n), \quad \tilde{\phi}(x) = \sum_n \tilde{h}_n \tilde{\phi}(2x - n), \tag{2.12}$$

where $\delta_{j,n}$ is the Kronecker delta. These two filters (h_n), $(\tilde{h}_n)$ give two wavelets ψ, $\tilde{\psi}$ through the equations

$$\psi(x) = \sum_n g_n \phi(2x - n), \quad \tilde{\psi}(x) = \sum_n \tilde{g}_n \tilde{\phi}(2x - n),$$

where $g_n = (-1)^n \tilde{h}_{1-n}$, $\tilde{g}_n = (-1)^n h_{1-n}$. Then, by (2.11) the functions $\psi_{k,j}$, $\tilde{\psi}_{k',j'}$ satisfy the biorthogonality condition

$$\langle \psi_{k,j}, \tilde{\psi}_{k',j'} \rangle = \delta_{k,k'} \delta_{j,j'}.$$

The conditions (2.11), (2.12) (cf. [3]) allow us to have two sequences of nested subspaces $(V_k)_{k \in \mathbb{Z}}$, $(\tilde{V}_k)_{k \in \mathbb{Z}}$ of $L_2(\mathbb{R})$ such that

$$0 \to \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \to L_2(\mathbb{R}),$$
$$0 \to \cdots \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \cdots \to L_2(\mathbb{R}),$$

with $V_0 = \overline{\mathrm{span}\{\phi_{0,j} \mid j \in \mathbb{Z}\}}$, $\tilde{V}_0 = \overline{\mathrm{span}\{\tilde{\phi}_{0,j} \mid j \in \mathbb{Z}\}}$. In this case, W_k is a complement of V_k in V_{k+1}; that is, $V_{k+1} = V_k \dot{\oplus} W_k$, where $\dot{\oplus}$ denotes the (not necessarily orthogonal) direct sum of vector spaces. Moreover, W_k is generated by the dyadic dilation of ψ. The same is for $\widetilde{W}_k$. Instead of the orthogonal complement, we obtain $V_k \perp \widetilde{W}_k$, $\tilde{V}_k \perp W_k$.

As a consequence, we have two nonorthogonal decompositions of $L_2(\mathbb{R})$

$$L_2(\mathbb{R}) = \dot{\bigoplus_{k \in \mathbb{Z}}} W_k := \cdots \dot{\oplus} W_{-1} \dot{\oplus} W_0 \dot{\oplus} W_1 \dot{\oplus} \cdots,$$
$$L_2(\mathbb{R}) = \dot{\bigoplus_{k \in \mathbb{Z}}} \widetilde{W}_k := \cdots \dot{\oplus} \widetilde{W}_{-1} \dot{\oplus} \widetilde{W}_0 \dot{\oplus} \widetilde{W}_1 \dot{\oplus} \cdots.$$

In addition, we can write $f \in L_2(\mathbb{R})$ as

$$f = \sum_{k,j} \langle f, \tilde{\psi}_{k,j} \rangle \psi_{k,j} = \sum_{k,j} \langle f, \psi_{k,j} \rangle \tilde{\psi}_{k,j},$$

or for any integer k_0,

$$\begin{aligned} f &= \sum_j \langle f, \tilde{\phi}_{k_0,j} \rangle \phi_{k_0,j} + \sum_{k \geq k_0} \sum_j \langle f, \tilde{\psi}_{k,j} \rangle \psi_{k,j}, \\ &= \sum_j \langle f, \phi_{k_0,j} \rangle \tilde{\phi}_{k_0,j} + \sum_{k \geq k_0} \sum_j \langle f, \psi_{k,j} \rangle \tilde{\psi}_{k,j}. \end{aligned}$$

Moreover, the families $\{\psi_{k,j}\}_{k,j}$ and $\{\tilde{\psi}_{k,j}\}_{k,j}$ are dual L_2-stable bases for $L_2(\mathbb{R})$ (see [3] and [7], for detail). In this case, the functions ψ, $\tilde{\psi}$ are called *biorthogonal wavelets*. For various examples of biorthogonal wavelets, we refer to [7].

The biorthogonal wavelet decompositions have an advantage because there are flexibilities to choose the synthesis wavelet ψ and the analysis wavelet $\tilde{\psi}$. For instance, we can choose the wavelets ψ, $\tilde{\psi}$ to have different regularity properties and vanishing moments (cf. [7]). We shall use this advantage for our construction. Both the regularity of the synthesis wavelets and the vanishing moments of the analysis wavelets are important in applications (see Antonini *et al.* [1], for some experiments and discussions).

3. Wavelets on the Unit Interval

There are naive approaches to constructing wavelet decompositions on $[0,1]$ (cf. [4], [7]). For instance, given f defined on $[0,1]$, one can set $f(x) = 0$ outside $[0,1]$, and just apply the standard wavelet decompositions on $\mathbb{R}$. This approach has two major problems. The first problem comes from the artificial discontinuities in f at 0 and 1. Since a discontinuity is usually reflected by large wavelet coefficients at fine scales, the artificial discontinuities introduce extra wavelet coefficients near the edges even if f itself is very smooth on $[0,1]$. Therefore, the regularity of f can not be characterized by the wavelet coefficients unless f is compactly supported in $[0,1]$. The other problem is that this approach uses too many wavelets. At a fine scale, say k, the supports of $2^k + L - 1$ wavelets $\psi_{k,j}$ intersect $[0,1]$, where L is the support width of a given wavelet ψ. This fact is somewhat unsatisfactory in practical applications such as image processing, where images typically consist of $2^m \times 2^m$ pixels $(7 \leq m \leq 11)$.

To resolve these drawbacks, many authors [4], [15], and [2] have developed wavelets on $[0,1]$ that generate an analogous multiresolution analysis on $[0,1]$. In this section, we briefly describe the orthogonal wavelet bases developed by [15] and [4] for later reference.

Since $L_2([0,1])$ is not invariant under translations and dilations, there would be some changes in the definition of multiresolution analysis for $L_2([0,1])$. The conditions (2.1), (2.2), and (2.3)(iii), (iv) of multiresolution analysis are evidently relaxed. One starts from an initial closed subspace $V_{k_0}^{\Omega}$ of $L_2([0,1])$ for a certain level $k_0 \geq 0$ and investigates closed subspaces V_k^{Ω} only for $k \geq k_0$ such that

$$V_k^{\Omega} \subset V_{k+1}^{\Omega} \quad \text{and} \quad \overline{\bigcup_{k \geq k_0} V_k^{\Omega}} = L_2([0,1]),$$

where the closure is in $L_2([0,1])$. To obtain an analogous condition to (2.3)(iv), one develops an L_2-stable basis for the spaces V_k^{Ω} consisting of a finite set of suitable dyadic translations $\phi_{k,j}$ of original scaling functions ϕ on $\mathbb{R}$ and a finite set of some special functions adapted to (and localized at) the edges. Once such subspaces V_k^{Ω} and their bases are found, the orthogonal decomposition of $L_2([0,1])$ is obtained; that is,

$$L_2([0,1]) = V_{k_0}^{\Omega} \oplus \left(\bigoplus_{k \geq k_0} W_k^{\Omega} \right), \quad V_{k+1}^{\Omega} = V_k^{\Omega} \ominus W_k^{\Omega}, \qquad k \geq k_0.$$

Here W_k^{Ω} is also generated by suitable dyadic translations $\psi_{k,j}$ of original wavelets ψ on $\mathbb{R}$ and some special functions adapted to the edges.

3.1. Meyer's Construction

The approach above was first used by Meyer [15] with Daubechies' scaling function $\phi = {}_N\phi$ and the wavelet $\psi = {}_N\psi$. By restricting the V_k to $[0,1]$, one obtained the closed spaces

$$V_k^{\Omega} = \{ g \in L_2([0,1]) \mid g = f|_{[0,1]}, \quad f \in V_k \}, \qquad k \geq 0$$

that form a multiresolution analysis of $L_2([0,1])$ (cf. [15], [7]), where $f|_{\Omega}$ is the restriction of f on a set Ω. With this setting, one [15] proved that the restriction of the functions $\phi_{k,j}$ to $[0,1]$ gives a basis for V_k^Ω, and then orthonormalized those by the Gram-Schmidt process (cf. [7]). The main advantage of Meyer's construction is that the resulting wavelets are unconditional bases for the Zygmund space $C^\gamma([0,1])$ for $\gamma < s$, where s is the regularity index of the standard wavelet.

However, Meyer's construction has a couple of disadvantages in a practical point of view. Among the restrictions of the $\phi_{k,j}$ to $[0,1]$, there are some that have small tails. Hence, the collection of the restrictions $\phi_{k,j}\big|_{[0,1]}$, although a basis, is almost linearly dependent. As a result, the matrix associated with a change of basis from $\phi_{k,j}\big|_{[0,1]}$ to the corresponding orthonormal basis, is ill conditioned for $N > 3$ (the condition number is indeed very large when N is large) (cf. [4]). This fact results in some difficulties in the computation of the adapted filter coefficients near the edges. Further, since $\operatorname{supp}{}_N\phi = [-N+1, N]$, we have $\dim(V_k^\Omega) = 2^k + 2N - 2$ and $\dim(W_k^\Omega) = 2^k$. This imbalance between V_k^Ω and W_k^Ω is inconvenient in applications.

3.2. Cohen et al.'s Construction

Cohen *et al.* [4] have developed a different construction that overcomes the problems above. Their main idea is to adapt the $\phi_{k,j}$ near the edges so that polynomials of certain degree in $[0,1]$ can be reproduced. For a brief description of their construction, we consider a dyadic level fine enough so that

$$\operatorname{supp}\phi_{k,N-1} = [0, 2^{-k}(2N-1)] \subset [0, 1/2].$$

Let k_0 be the smallest such k (or $2^k \geq 2N$). Then, the functions $\phi_{k,j}$, $k \geq k_0$ that are nonzero around 0 and 1, do not interact with each other, so that special functions at the edges can be defined independently.

First note that Daubechies' wavelets $\psi = {}_N\psi$ satisfy the vanishing moment condition (2.9) up to the order $N-1$. This is due to the fact that the scaling function ϕ satisfies the Strang-Fix condition [9]

$$\text{(3.1)} \qquad \begin{aligned} &\text{(i)} \quad \widehat{\phi}(0) = 1, \quad \widehat{\phi}(2\pi j) = 0, \quad j \in \mathbb{Z}\setminus\{0\} \\ &\text{(ii)} \quad D^r\widehat{\phi}(2\pi j) = 0, \quad j \in \mathbb{Z}\setminus\{0\}, \quad |r| < N. \end{aligned}$$

Then the spaces V_k contain the space $\mathcal{P}_N$ of polynomials of total degree less than N. In particular, each monomial x^ν, $0 \le \nu \le N-1$ admits the representation

$$x^\nu = \sum_n \langle x^\nu, \phi(\cdot - n)\rangle \phi(x-n).$$

To modify the property above on the interval $[0,\infty)$, one employs the following N edge functions defined on the interval: for $0 \le \nu \le N-1$,

$$\varphi_\nu^L(x) = \sum_{n=\nu}^{2N-2} \binom{n}{\nu} \phi(x+n-N+1).$$

These edge functions φ_ν^L have $\operatorname{supp}\varphi_\nu^L = [0, 2N-1-\nu]$ and satisfy the following properties (cf. [4]):

(3.2)
(i) φ_ν^L are linearly independent,
(ii) φ_ν^L are orthogonal to the $\phi_{0,j}$, $j \ge N$,
(iii) $\{\varphi_\nu^L\}_{\nu=0}^{N-1} \cup \{\phi_{0,j}\}_{j=N}^{\infty}$ reproduces the space $\mathcal{P}_N\big|_{[0,\infty)}$,
(iv) there exist constants $a_{\nu,l}$, $b_{\nu,l}$ such that

$$\varphi_\nu^L(x) = \sum_{l=0}^{\nu} a_{\nu,l}\, \varphi_l^L(2x) + \sum_{l=N}^{3N-2-2\nu} b_{\nu,l} \phi(2x-l).$$

Then, since $\overline{\bigcup_{k\in\mathbb{Z}} \operatorname{span}\{\phi_{k,j} \mid j \ge N\}} = L_2([0,\infty))$, by the equation (3.2) (iv), the spaces

$$V_k^L = \overline{\operatorname{span}\big[\{\varphi_\nu^L(2^k\,\cdot) \mid 0 \le \nu \le N-1\} \cup \{\phi_{k,j} \mid j \ge N\}\big]}$$

form nested subspaces in $L_2([0,\infty))$ such that $V_k^L \to L_2([0,\infty))$ as $k \to \infty$.

Applying the Gram-Schmidt process to the φ_ν^L, $0 \le \nu \le N-1$ (starting from $N-1$ and working down to 0 to keep their minimal supports), one obtains orthonormal functions ϕ_j^L, $j = 0, \cdots, N-1$, with $\operatorname{supp}\phi_j^L = [0, N+j]$. If $\phi_{k,j}^L(x) := 2^{k/2}\phi_j^L(2^k x)$ are defined, then the family $\{\phi_{k,j} \mid j \geq N\} \cup \{\phi^L_{k,j} \mid 0 \leq j \leq N-1\}$ forms an orthonormal basis for the V^L_k. In addition, the functions $\phi^L_{k,j}$ satisfy the properties (3.2)(iii), (iv) (that are preserved through the Gram-Schmidt process).

For the case of the interval $(-\infty, 0]$, the construction on $[0,\infty)$ can be analogously repeated with the reflected coefficients $h^{\#}_n = h_{-n+1}$. This gives the functions $\phi^R_{k,j}(x) = 2^{k/2}\phi^R_j(2^k x)$, $j = -1, \cdots, -N$ with $\operatorname{supp}\phi^R_j = [j-N+1, 0]$, where $\phi^R_j(x) = (\phi^{\#})^L_{-1-j}(-x)$, with $h^{\#}_n = h_{-n+1}$.

Now, on the interval $[0,1]$, for $k \geq k_0$ one chooses adapted scaling functions $\phi^0_{k,m}(x) = \phi^L_{k,m}(x)$, $m = 0, \cdots, N-1$, at the edge 0 and $\phi^1_{k,l}(x) = \phi^R_{k,l-2^k}(x-1)$, $l = 2^k - N, \cdots, 2^k - 1$, at the edge 1. Then, together with the $\phi_{k,j}$, $j = N, \cdots, 2^k - N + 1$, the adapted scaling functions constitute an orthonormal basis for the 2^k-dimensional space V^Ω_k that contains the space $\mathcal{P}_N\big|_{[0,1]}$.

By the same way as the construction of the scaling functions on the interval, one chooses adapted wavelets $\psi^0_m(x) = \psi^L_{k,m}$, $m = 0, \cdots, N-1$, at the edge 0 and $\psi^1_l(x) = \psi^R_{k,l-2^k}(x-1)$, $l = 2^k - N, \cdots, 2^k - 1$, at the edge 1. Furthermore, together with the $\psi_{k,j}$, $j = N, \cdots, 2^k - N + 1$, the adapted wavelets at the edges constitute an orthonormal basis for the space W^Ω_k. Then, the family

$$\{\Phi_{k_0,j}\}_{j\in\Lambda_k} \cup \left(\bigcup_{k\geq k_0} \{\Psi_{k,j}\}_{j\in\Lambda_k}\right)$$

is an orthonormal basis for $L_2([0,1])$, where

$$\Phi_{k_0,j} = \begin{cases} \phi^0_{k_0,j} & \text{for } j = 0, \cdots, N-1, \\ \phi_{k_0,j} & \text{for } j = N, \cdots, 2^{k_0} - N - 1, \\ \phi^1_{k_0,j} & \text{for } j = 2^{k_0} - N, \cdots, 2^k - 1, \end{cases}$$

$$\Psi_{k,j} = \begin{cases} \psi^0_{k,j} & \text{for } j = 0, \cdots, N-1, \\ \psi_{k,j} & \text{for } j = N, \cdots, 2^k - N - 1, \\ \psi^1_{k,j} & \text{for } j = 2^k - N \cdots, 2^k - 1, \end{cases}$$

and $\Lambda_k = \{0, \cdots, 2^k - 1\}$. In other words, any $f \in L_2([0,1])$ admits a wavelet decomposition of the form

$$f = \sum_{j \in \Lambda_{k_0}} \langle f, \Phi_{k_0,j} \rangle \, \Phi_{k_0,j} + \sum_{k \geq k_0} \sum_{j \in \Lambda_k} \langle f, \Psi_{k,j} \rangle \, \Psi_{k,j}.$$

Also, this wavelet basis is an unconditional basis for the space $C^\gamma([0,1])$ for $\gamma < s$, where s is the regularity index of the standard wavelet on $\mathbb{R}$; that is,

$$f \in C^\gamma([0,1]) \quad \text{if and only if} \quad |\langle f, \Psi_{k,j} \rangle| \leq C 2^{-k(\gamma+1/2)}$$

where C is independent of $k \geq k_0$ and $j \in \Lambda_k$ (cf. [4]).

3.3. Wavelets on the unit interval with boundary treatment

As reported in [4], there have not been yet found a construction of orthonormal wavelets on $[0,1]$ whose fast algorithm leads simple polynomial sequences with a certain degree (such as $\{1, \ldots, 1\}$ or $\{1, 2, 3, 4, \ldots\}$) both to zeros by its high pass filter (associated with wavelet) and to themselves by its low pass filter (associated with scaling function). In applications, such a fast algorithm is required. For instance, in view of image and surface compression applications, the fast algorithm corresponding to a wavelet on $\mathbb{R}$ is itself a lossless compression algorithm. This enables us to design a lossy compression algorithm with higher compression ratio. Then, working on $[0,1]$, we wish to construct a lossless compression algorithm preserving the same compression ratio at least for the polynomial sequence data as that of the standard algorithms on $\mathbb{R}$. In this sense, Cohen *et al.*'s construction is unsatisfactory, because their algorithm does not assign the constant sequence $\{1, \ldots, 1\}$ to zeros (near the edges) by its high filter (cf. [4]). The same problem arises for higher order polynomial sequences near the edges. In practice, this drawback develops a kind of Gibbs' phenomenon near the edges (cf. [13]). To overcome this disadvantage, one [4] adds preconditioning transforms to the analysis and synthesis steps near the edges. However, these introduce extra steps.

The objective of this section is to provide a construction of wavelet bases on $[0,1]$ that generate a good fast algorithm avoiding the disadvantage above and characterize smoothness spaces such as the Zygmund space $C^\gamma([0,1])$, $\gamma > 0$ or the Besov space $B_q^\alpha(L_p[0,1])$. To develop our construction, we employ Daubechies' scaling function $\phi = {}_N\phi$. Here, we relax the orthogonality near the edges and require only the biorthogonality condition near the edges to obtain such a fast algorithm instead. Of course, our construction can be applied equally well to the standard biorthogonal wavelets on $\mathbb{R}$. The basic idea behind our construction is to design the filter coefficients near the edges (associated with the scaling function and wavelet) to keep the sum rule (2.10). This idea is different from that of Cohen *et al.* [4].

Let $\phi = {}_N\phi$, $\psi = {}_N\psi$ and set k_0 to be the positive smallest integer such that $2^{k_0} \geq 2N$. Recall that the ϕ satisfies the Strang-Fix condition (3.1) of order N. For our construction, we then choose the polynomial $P_r(x)$ of degree r such that for each $k \in \mathbb{Z}$,

$$P_r(x) = \sum_{n\in\mathbb{Z}} n^r \phi(2^k x - n), \qquad r = 0, \cdots, N-1, \tag{3.3}$$

where $P_0 = 1$.

To modify the property (3.3) on $[0,1]$, we consider the restriction of (3.3) to $[0,1]$; that is, for $k \geq k_0$

$$P_r(x)\big|_{[0,1]} = \left(\sum_{n\in\Lambda_k^L} + \sum_{n\in\Lambda_k^I} + \sum_{n\in\Lambda_k^R} \right) n^r \phi(2^k x - n)\big|_{[0,1]},$$

where

$$\begin{aligned}
\Lambda_k^L &= \{n \in \mathbb{Z} \mid -N+1 \leq n \leq N-1\},\\
\Lambda_k^I &= \{n \in \mathbb{Z} \mid N \leq n \leq 2^k - N + 1\},\\
\Lambda_k^R &= \{n \in \mathbb{Z} \mid 2^k - N \leq n \leq 2^k + N - 2\}.
\end{aligned}$$

Since $2^{k_0} \geq 2N$, the supports of the $\phi(2^k \cdot -n)$, $n \in \Lambda_k^L$, Λ_k^R, do not intersect each other. Also, the supports of the $\phi(2^k \cdot -n)$, $n \in \Lambda_k^I$, are all contained in $[0,1]$. We then keep the $\phi(2^k \cdot -n)$, $n \in \Lambda_k^I$ for interior functions; and adapt the $\phi(2^k \cdot -n)$, $n \in \Lambda_k^L$, Λ_k^R for left edge functions ϕ_ν^L and right edge functions ϕ_ν^R, $\nu = 0, \cdots, N-1$.

Let us focus only on the construction of N left edge functions near 0, so that we can work on the half interval $[0,\infty)$. N right edge functions near 1 can be analogously considered on the half interval $(-\infty,0]$. We then define the N edge functions compactly supported on $[0,\infty)$ by

$$\phi_\nu^L(x) = \sum_{l=-N+1}^{-1} \xi_{\nu,l}\phi(x-l)\big|_{[0,\infty)} + \phi(x-\nu)\big|_{[0,\infty)}, \quad 0 \le \nu \le N-1.$$

Here for each $l = -N+1, \dots, -1$, $\mathbf{x}i_l := (\xi_{\nu,l})_{\nu=0}^{N-1}$ is the solution of the system

$$\mathbf{A}\mathbf{x} = \mathbf{b}_l,$$

where $\mathbf{A}$ is a $N \times N$ Vandermonde matrix with entries $a_{i,j} = (i-1)^{j-1}$, $1 \le i,j \le N$, and $\mathbf{b}_l = (b_{i,l})_i$ with $b_{i,l} = l^{i-1}$, $1 \le i \le N$. Then, by a simple computation, we obtain

$$(3.4)\quad P_r(x)\big|_{[0,\infty)} = \sum_{n=0}^{N-1} n^r \phi_n^L(x) + \sum_{n \ge N} n^r \phi(x-n), \quad r = 0, \cdots, N-1,$$

for the polynomial P_r in (3.3). Moreover, the N edge functions ϕ_ν^L satisfy the following properties:

PROPOSITION 3.1. *The edge functions ϕ_ν^L, $\nu = 0, \cdots, N-1$, are linearly independent. The family $\{\phi_\nu^L\}_{\nu=0}^{N-1} \cup \{\phi(\cdot - j)\}_{j=N}^{\infty}$ generates the space $\mathcal{P}_N\big|_{[0,\infty)}$. Further, there exist constants $H_{\nu,l}^L$, $h_{\nu,j}^L$ such that*

$$(3.5)\qquad \phi_\nu^L(x) = \sqrt{2}\sum_{l=0}^{N-1} H_{\nu,l}^L \phi_l^L(2x) + \sqrt{2}\sum_{j=N}^{N+2\nu} h_{\nu,j}^L \phi(2x-j).$$

PROOF. For a proof, see §A.1 of Appendix. □

In [13], we present the filter coefficients $H_{\nu,l}^L$, $h_{\nu,j}^L$ for the case $N = 2, 3$.

By dilation, let us define $\phi_{k,\nu}^L(x) := 2^{k/2}\phi_\nu^L(2^k x)$, $\nu = 0, \cdots, N-1$ and the space V_k^L by

$$V_k^L := \overline{\operatorname{span}\big[\{\phi_{k,\nu}^L \mid 0 \le \nu \le N-1\} \cup \{\phi_{k,j} \mid j \ge N\}\big]}.$$

Then by Proposition 3.1, the $\{\phi_{k,j}\}_{j\geq N} \cup \{\phi^L_{k,\nu}\}_{\nu=0}^{N-1}$ forms a basis for the V_k^L. Moreover, since $\overline{\bigcup_k \operatorname{span}\{\phi_{k,j} \mid j \geq N\}} = L_2([0,\infty))$, the spaces V_k^L, $k \in \mathbb{Z}$ form a nested sequence in $L_2([0,\infty))$ such that

$$0 \to \cdots \subset V_{-1}^L \subset V_0^L \subset V_1^L \subset \cdots \to L_2([0,\infty)).$$

We next develop the dual functions corresponding to the $\phi^L_{k,\nu}$, $0 \leq \nu \leq N-1$, and $\phi_{k,j}$, $j \geq N$. First note that the interior functions $\phi(\cdot - j)$, $j \geq N-1$ form orthogonal functions with support inside $[0,\infty)$ and are orthogonal to the $\phi(\cdot - l)\big|_{[0,\infty)}$, $l < N-1$. Then we may choose their dual functions as $\tilde{\phi}_j = \phi(\cdot - j)$, for $j \geq N$. Also, since

$$\phi^L_{N-1}(x) = \sum_{l=-N+1}^{-1} \xi_{N-1,l}\, \phi(x-l)\big|_{[0,\infty)} + \phi(x-N+1),$$

we may choose $\tilde{\phi}^L_{N-1} = \phi(\cdot - N+1)$. It remains to develop dual functions to the ϕ^L_ν, for $\nu = 0, \cdots, N-2$. We define functions $\tilde{\phi}^L_\nu$ by

$$\tilde{\phi}^L_\nu(x) = \sum_{l=N-1}^{3N-4} \eta_{l,\nu}\phi(2x-l),$$

where for each $\nu = 0, \cdots, N-2$, $\eta_\nu := (\eta_{l,\nu})_{l=N-1}^{3N-4}$ is the solution of the following system:

$$\mathbf{Bx} = \mathbf{c}_\nu.$$

Here, $\mathbf{c}_\nu = (c_{i,\nu})_{i=1}^{2N-2}$ with $c_{i,\nu} = \delta_{i,\nu+1}$ and $\mathbf{B}$ is a $(2N-2) \times (2N-2)$ block matrix generated by the following $N \times 2$ matrix:

$$\mathbf{L} = \frac{\sqrt{2}}{2} \begin{pmatrix} h_{N-1} & h_N \\ h_{N-3} & h_{N-2} \\ \vdots & \vdots \\ h_{-N+1} & h_{-N+2} \end{pmatrix}$$

so that for $1 \leq i, j \leq 2N-2$,

$$\mathbf{B}_{i,j} = \begin{cases} \mathbf{L} & \text{if } \lceil j/2 \rceil \leq i \leq \lceil j/2 \rceil + (N+1), \\ 0 & \text{otherwise.} \end{cases}$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. For instance, when $N = 2$

$$\mathbf{B} = \frac{\sqrt{2}}{2}\begin{pmatrix} h_1 & h_2 \\ h_{-1} & h_0 \end{pmatrix};$$

when $N = 3$

$$\mathbf{B} = \frac{\sqrt{2}}{2}\begin{pmatrix} h_2 & h_3 & 0 & 0 \\ h_0 & h_1 & h_2 & h_3 \\ h_{-2} & h_{-1} & h_0 & h_1 \\ 0 & 0 & h_{-2} & h_{-1} \end{pmatrix},$$

and so on.

We need to show that the $\mathbf{B}$ is non-singular. It is sufficient to show that the column vectors $\mathbf{L}_1$, $\mathbf{L}_2$ of $\mathbf{L}$ are linearly independent. We recall that the trigonometric function $m_0(\xi) = \frac{1}{\sqrt{2}}\sum_{n=-N+1}^{N} h_n e^{-in\xi}$ has a zero of order N at $\xi = \pi$ (see (2.8)); equivalently, the coefficients h_n satisfy the sum rule (2.10):

$$\sum_{n=-N+1}^{N} (-1)^n n^s h_n = 0, \qquad s = 0, 1, \cdots, N-1.$$

This implies that for $N \geq 2$ (N=1 gives the Haar wavelet case),

(3.6)

$$\text{(i)} \quad \sum_{i=0}^{N-1} h_{N-2i+1} = \sum_{i=0}^{N-1} h_{N-2i}, \qquad \text{for} \quad s = 0,$$

$$\text{(ii)} \quad \sum_{i=0}^{N-1}\left[(-1)^{N-1}(N-2i-1)h_{N-2i+1} + (-1)^N (N-2i)h_{N-2i}\right] = 0, \qquad \text{for} \quad s = 1.$$

Now, assume that $\mathbf{L}_1 = \lambda \mathbf{L}_2$ for some scalar λ. Then $h_{N-2i+1} = \lambda h_{N-2i}$ for $i = 0, \cdots, N-1$. So, we have $\lambda = 1$ from (3.6)(i), and then

$$\sum_{i=0}^{N-1} h_{N-2i+1} = \sum_{i=0}^{N-1} h_{N-2i} = 0$$

from (3.6)(ii). On the other hand, since $m_0(0) = 1$, we obtain

$$\sqrt{2} = \sum_{n=-N+1}^{N} h_n = \sum_{i=0}^{N-1} h_{N-2i+1} + \sum_{i=0}^{N-1} h_{N-2i}.$$

This is a contradiction.

Thus, the $\tilde{\phi}_\nu^L$, $\nu = 0, \cdots, N-2$ are well defined. Moreover, the following proposition shows that the $\tilde{\phi}_\nu^L$ are dual functions to the ϕ_ν^L.

PROPOSITION 3.2. *The functions $\tilde{\phi}_\nu^L$, $\nu = 0, \cdots, N-1$, are linearly independent. Also, the $\tilde{\phi}_\nu^L$ satisfy the following biorthogonality:*

$$\text{(3.7)}\qquad \begin{aligned} &\text{(i)}\quad \langle \phi_l^L, \tilde{\phi}_\nu^L \rangle = \delta_{l,\nu} \qquad \textit{for}\quad l = 0, \cdots, N-1,\\ &\text{(ii)}\quad \langle \phi(\cdot - l), \tilde{\phi}_\nu^L \rangle = 0 = \langle \tilde{\phi}(\cdot - l), \phi_\nu^L \rangle \qquad \textit{for}\quad l \geq N. \end{aligned}$$

PROOF. For a proof, see §A.2 of Appendix. □

By dilations, we define $\tilde{\phi}_{k,\nu}^L(x) = 2^{k/2}\tilde{\phi}_\nu^L(2^k x)$, $\nu = 0, \cdots, N-1$, and the space $\widetilde{V}_k^L$ by

$$\widetilde{V}_k^L := \overline{\operatorname{span}\left[\{\tilde{\phi}_{k,\nu}^L \mid 0 \leq \nu \leq N-1\} \cup \{\phi_{k,j} \mid j \geq N\}\right]}.$$

Then, by Proposition 3.2, together with the $\phi_{k,j}$, $j \geq N$, the $\tilde{\phi}_{k,\nu}^L$ form a basis for the $\widetilde{V}_k^L$. Moreover, since $\overline{\bigcup_k \operatorname{span}\{\phi_{k,j} \mid j \geq N\}} = L_2([0,\infty))$ and $\tilde{\phi}_{k,\nu}^L \in \widetilde{V}_{k+1}^L$, the $\widetilde{V}_k^L$, $k \in \mathbb{Z}$ form a nested sequence in $L_2([0,\infty))$ such that

$$0 \rightarrow \cdots \subset \widetilde{V}_{-1}^L \subset \widetilde{V}_0^L \subset \widetilde{V}_1^L \subset \cdots \rightarrow L_2([0,\infty)).$$

So far, we have obtained two families $\{\phi_\nu^L\}_{\nu=0}^{N-1} \cup \{\phi_j\}_{j\geq N}$ and $\{\tilde{\phi}_\nu^L\}_{\nu=0}^{N-1} \cup \{\phi_j\}_{j\geq N}$ that generate two nested sequences of 2^k-dimensional spaces V_k^L, $\widetilde{V}_k^L$, $(k \geq k_0)$ converging to $L_2([0,\infty))$, respectively.

We shell next consider the complement space W_k^L of V_k^L in V_{k+1}^L (same for the $\widetilde{W}_k$); and find biorthogonal wavelets. Let us first define $2N$ functions as follows: for each $\nu = 0, \cdots, N-1$,

$$q_\nu(x) := \sqrt{2}\,\phi_\nu^L(2x) - \sum_{l=0}^{N-1} \langle \sqrt{2}\phi_\nu^L(2\,\cdot\,), \tilde{\phi}_l^L \rangle \, \phi_l^L(x),$$

$$\tilde{q}_\nu(x) := \begin{cases} \sqrt{2}\,\tilde{\phi}_\nu^L(2x) - \sum_{l=0}^{N-2} \langle \sqrt{2}\tilde{\phi}_\nu^L(2\,\cdot\,), \phi_l^L \rangle \, \tilde{\phi}_l^L(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \nu = 0, \cdots, N-2, \\ \tilde{\psi}(\,\cdot\, - N - 1) \qquad \text{for } \nu = N-1. \end{cases}$$

Then, from the relations (2.6), (2.7), and (3.7) it follows that for $\nu = 0, \cdots, N-1$ and $j \geq N$,

$$\text{(3.8)} \qquad \begin{aligned} &\text{(i)} \quad \langle q_\nu, \tilde{\phi}(\,\cdot\, - j) \rangle = 0 = \langle \tilde{q}_\nu, \phi(\,\cdot\, - j) \rangle, \\ &\text{(ii)} \quad \langle q_\nu, \tilde{\psi}(\,\cdot\, - j) \rangle = 0 = \langle \tilde{q}_\nu, \psi(\,\cdot\, - j) \rangle. \end{aligned}$$

Moreover, these $2N$ functions q_ν, $\tilde{q}_\nu$ satisfy the following properties:

PROPOSITION 3.3.

$$\text{(3.9)} \qquad \begin{aligned} &\text{(i)} \quad \langle q_\nu, \tilde{\phi}_\mu^L \rangle = 0 = \langle \tilde{q}_\nu, \phi_\mu^L \rangle, \qquad 0 \leq \nu, \mu \leq N-1, \\ &\text{(ii)} \quad \langle q_\nu, \tilde{q}_\mu \rangle = \delta_{\nu,\mu}, \qquad 0 \leq \nu, \mu \leq N-2, \\ &\text{(iii)} \quad \langle q_{N-1}, \tilde{q}_{N-1} \rangle = (-1)^{-N+1} h_N. \end{aligned}$$

PROOF. For a proof, see §A.3 of Appendix. □

Using the q_ν, $\tilde{q}_\nu$, we may obtain biorthogonal wavelets on $[0, \infty)$ by setting

$$\tilde{\psi}_\nu^L = \tilde{q}_\nu,$$

$$\psi_\nu^L = \begin{cases} q_\nu \qquad \text{for } 0 \leq \nu \leq N-2, \\ \dfrac{1}{\langle q_{N-1}, \tilde{q}_{N-1} \rangle} \left(q_{N-1} - \sum_{l=0}^{N-2} \langle q_{N-1}, \tilde{q}_l \rangle q_l \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \nu = N-1. \end{cases}$$

By simple manipulations, the ψ_ν^L, $\tilde{\psi}_\nu^L$ satisfy (3.8), (3.9)(i), and

$$\langle \psi_\nu^L, \tilde{\psi}_\mu^L \rangle = \delta_{\nu,\mu}, \quad 0 \le \nu, \mu \le N-1.$$

So, the ψ_ν^L are linearly independent in W_0^L (same for the $\tilde{\psi}_\nu^L$) by (3.9)(ii). Thus, together with the $\{\psi_{0,j}\}_{j \ge N}$, the ψ_ν^L constitute a basis for the W_0^L (same for the $\tilde{\psi}_\nu^L$, $\widetilde{W}_0^L$). Furthermore, by the refinement equation (3.5), the ψ_ν^L can be written as linear combinations of ϕ_ν^L and $\phi_{0,j}$; that is, there exist constants $G_{\nu,l}^L$, $g_{\nu,j}^L$ such that

$$\psi_\nu^L(x) = \sqrt{2} \sum_{l=0}^{N-1} G_{\nu,l}^L \phi_l^L(2x) + \sqrt{2} \sum_{j=N}^{N+2\nu} g_{\nu,j}^L \phi(2x-j)$$

(the same is true for the $\tilde{\psi}_\nu^L$ with $\widetilde{G}_{\nu,l}^L$, $\tilde{g}_{\nu,j}^L$). In [13], we present the filter coefficients $G_{\nu,l}^L$, $g_{\nu,j}^L$ for the case $N = 2, 3$.

By dilation, we define $\psi_{k,\nu}^L(x) = 2^{k/2}\psi_\nu^L(2^k x)$ (same for the $\tilde{\psi}_{k,\nu}^L$). Then the family $\{\psi_{k,\nu}^L \mid 0 \le \nu \le N-1\} \cup \{\psi_{k,j} \mid j \ge N\}$ constitutes a basis for the W_k^L (same for the $\widetilde{W}_k^L$). By (3.8), (3.9), and dilation, we then have the relations among the spaces V_k^L, W_k^L, $\widetilde{V}_k^L$, $\widetilde{W}_k^L$ as follows:

$$V_k^L \perp \widetilde{W}_k^L, \quad \widetilde{V}_k^L \perp W_k^L.$$

The constructions of right edge functions near the 1 can be done on $(-\infty, 0]$ by translation. Also, the same constructions on $[0, \infty)$ can be analogously repeated with the reflected coefficients $h_n^{\#} = h_{1-n}$. This gives us right edge functions

$$\phi_\mu^R(x) = (\phi^{\#})_{-1-\mu}^L(-x), \quad \mu = -1, \cdots, -N$$

where $\phi^{\#}$ is the reflected function of ϕ with respect to $\frac{1}{2}$. The same is true for the ψ_μ^R, $\tilde{\phi}_\mu^R$, $\tilde{\psi}_\mu^R$ (note that most properties about ϕ are invariant under reflection).

Let us now return to the work on the interval $[0,1]$. For fixed $k \ge k_0$, we choose adapted functions $\phi_{k,\nu}^0 := \phi_{k,\nu}^L$, $\nu = 0, \cdots, N-1$ at the edge 0, and $\phi^1_{k,\mu} := \phi^R_{k,\mu-2^k}(\cdot - 1)$, $\mu = 2^k - N, \cdots, 2^k - 1$ at the edge 1 (same for the $\psi^0_{k,\nu}$, $\psi^1_{k,\mu}$). Then, together with the $\phi_{k,j}$, $j = N, \cdots, 2^k - N + 1$, the adapted functions at the edges constitute a basis for the 2^k-dimensional space V_k^Ω defined by

$$V_k^\Omega := \mathrm{span}\big(\{\phi^L_{k,\nu}\}_{\nu=0}^{N-1} \cup \{\phi_{k,j}\}_{j=N}^{2^k-N+1} \cup \{\phi^R_{k,\mu}\}_{\mu=2^k-N}^{2^k-1}\big)$$

(same for the W_k^Ω). By the same way, we obtain $\tilde{\phi}^0_{k,\nu}$, $\tilde{\phi}^1_{k,\mu}$ and the 2^k-dimensional space $\widetilde{V}_k^\Omega$ (same for the $\tilde{\psi}^0_{k,\nu}$, $\tilde{\psi}^1_{k,\mu}$, $\widetilde{W}_k^\Omega$).

As a consequence, two sequences of 2^k-dimensional spaces V_k^Ω and W_k^Ω are obtained so that

$$\begin{aligned} &V_{k_0}^\Omega \subset V_{k_0+1}^\Omega \subset \cdots \to L_2([0,1]), \\ &V_{k+1}^\Omega = V_k^\Omega \dot{\oplus} W_k^\Omega, \quad \text{and} \quad W_k^\Omega \perp \widetilde{V}_k^\Omega \end{aligned}$$

for $k \geq k_0$ (same for the $\widetilde{V}_k^\Omega$, $\widetilde{W}_k^\Omega$). Therefore, any $f \in L_2([0,1])$ admits wavelet decompositions of the form

$$f = \sum_{j\in\Lambda_{k_0}} \langle f, \widetilde{\Phi}_{k_0,j}\rangle\, \Phi_{k_0,j} + \sum_{k\geq k_0}\sum_{j\in\Lambda_k} \langle f, \widetilde{\Psi}_{k,j}\rangle\, \Psi_{k,j} \tag{3.10}$$

where

$$\Phi_{k_0,j} = \begin{cases} \phi^0_{k_0,j} & \text{for } j = 0, \cdots, N-1, \\ \phi_{k_0,j} & \text{for } j = N, \cdots, 2^{k_0} - N - 1, \\ \phi^1_{k_0,j} & \text{for } j = 2^{k_0} - N, \cdots, 2^k - 1, \end{cases}$$

$$\Psi_{k,j} = \begin{cases} \psi^0_{k,j} & \text{for } j = 0, \cdots, N-1, \\ \psi_{k,j} & \text{for } j = N, \cdots, 2^k - N - 1, \\ \psi^1_{k,j} & \text{for } j = 2^k - N \cdots, 2^k - 1, \end{cases}$$

and $\Lambda_k = \{0, \cdots, 2^k - 1\}$ (same for the $\widetilde{\Phi}_{k_0,j}$, $\widetilde{\Psi}_{k,j}$).

The (3.10) provides the characterization of the space $C^\gamma([0,1])$, $\gamma > 0$ with the family $\{\Phi_{k_0,j}\}_{j\in\Lambda_k} \cup \big(\bigcup_{k\geq k_0}\{\Psi_{k,j}\}_{j\in\Lambda_k}\big)$, that is,

$$f \in C^\gamma([0,1]) \quad \text{if and only if} \quad |\langle f, \widetilde{\Psi}_{k,j}\rangle| \leq C 2^{-k(\gamma+1/2)} \tag{3.11}$$

where C is independent of $k \geq k_0$ and $j \in \Lambda_k$ (cf. [12]). To show the above, we need the following proposition:

PROPOSITION 3.4. *If $\phi \in C^s$, then the $\Psi_{k,j}$, $k \geq k_0$, $j \in \Lambda_k$ satisfy the following:*

$$|\Psi_{k,j}|_{W^r_\infty([0,1])} \leq C 2^{kr} 2^{k/2}, \qquad 0 \leq r \leq \lfloor s \rfloor \tag{3.12}$$

with a constant C independent of k and j. Also, the $\widetilde{\Psi}_{k,j}$ satisfy the following:

$$\begin{aligned} &\text{(i)} \quad \|\widetilde{\Psi}_{k,j}\|_{L_\infty} \leq C 2^{k/2}, \\ &\text{(ii)} \quad \int_{\mathbb{R}} x^\gamma \, \widetilde{\Psi}_{k,j} \, dx = 0, \qquad |\gamma| < N \end{aligned} \tag{3.13}$$

with a constant C independent of k and j.

PROOF. For a proof, see §A.4 of Appendix. □

The proposition above gives a sufficient condition for the characterization (3.11). More generally, for a suitable range of α, the Besov space $B^\alpha_q(L_p[0,1])$ can be characterized through the wavelet bases pair satisfying (3.12) and (3.13) (cf. [12]).

Appendix

A.1. Proof of Proposition 3.1

Since $\operatorname{supp} \phi^L_\nu = [0, N+\nu]$, it is clear that the ϕ^L_ν are linearly independent. Also, since the equation (3.4) holds for the polynomial P_r of (3.3), together with the $\phi(\,\cdot\, - j)$, $j \geq N$, the ϕ^L_ν, $\nu = 0, \ldots, N-1$ generate all polynomials up to degree $N-1$.

To complete the proposition, we shall derive the equation (3.5) for functions φ_ν, $\nu = 0, \cdots, N-1$, defined by

$$\varphi_\nu(x) = \sum_{n=-N+1}^{N-1} n^\nu \phi(x-n)\big|_{[0,\infty)}.$$

Once such an equation is established for the φ_ν, the equation (3.5) for the ϕ^L_ν can be obtained through the non-singular matrix $\mathbf{A}$, because

$$\begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_{N-1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \phi^L_0 \\ \phi^L_1 \\ \vdots \\ \phi^L_{N-1} \end{pmatrix} \tag{A.1.1}$$

Let us now show that there exist constants $H_{\nu,l}$, $h_{\nu,j}$ such that for $\nu = 0, \cdots, N-1$,

$$\varphi_\nu(x) = \sqrt{2}\sum_{l=0}^{\nu} H_{\nu,l}\varphi_l(2x) + \sqrt{2}\sum_{j=N}^{3N-2} h_{\nu,j}\phi(2x-j). \tag{A.1.2}$$

For fixed ν, by using the refinement equation (2.6), we can rewrite the φ_ν as follows:

$$\begin{aligned}
\varphi_\nu(x) &= \sqrt{2}\sum_{n=-N+1}^{N-1} n^\nu \sum_{m=-N+1}^{N} h_m\phi(2x-2n-m)\\
&= \sqrt{2}\sum_{l=-N+1}^{3N-2}\phi(2x-l)\sum_{n=-N+1}^{N-1} n^\nu h_{l-2n}\\
&= \sqrt{2}\sum_{l=-N+1}^{N-1}\phi(2x-l)\sum_{n=-N+1}^{N-1} n^\nu h_{l-2n}\\
&\qquad + \sqrt{2}\sum_{j=N}^{3N-2}\phi(2x-j)\sum_{n=-N+1}^{N-1} n^\nu h_{j-2n}\\
&=: \mathrm{I}_1 + \mathrm{I}_2,
\end{aligned}$$

where for the second equality, we have used the facts that $\phi(2x-l)\big|_{[0,1]} = 0$ for $l \le N$ and $h_n = 0$ for $n < -N+1$ or $n > N$. Note that the term I_2 is the second term of (A.1.2) with $h_{\nu,j} = \sum_{n=-N+1}^{N-1} n^\nu h_{j-2n}$. We then need to express I_1 in terms of the $\varphi_l(2x)$, $l = 0, \cdots, N-1$.

From the sum rule (2.10), we obtain the following: for any integer l

$$\begin{aligned}
0 = \sum_m (-1)^m m^\nu h_{m-l} &= (-1)^l \sum_n (-1)^n (n+l)^\nu h_n\\
&= (-1)^l \sum_n (-1)^n h_n \left(\sum_{s=0}^{\nu}\binom{\nu}{s} l^{\nu-s} n^s\right)\\
&= (-1)^l \sum_{s=0}^{\nu}\binom{\nu}{s} l^{\nu-s}\left(\sum_n (-1)^n n^s h_n\right).
\end{aligned}$$

Dividing the sum by even and odd terms, we have

$$\begin{aligned} C_{\nu,l} := & \sum_n (2n)^\nu h_{2n-l} \\ = & \frac{1}{2} \sum_m m^\nu h_{m-l} \\ = & \frac{1}{2} \sum_{s=0}^{\nu} \binom{\nu}{s} l^s \sum_n n^{\nu-s} h_n. \end{aligned}$$

Using this fact, we derive the following:

$$\begin{aligned} \sum_n n^\nu h_{l-2n} = & (-2)^\nu \sum_n (2n)^\nu h_{2n+l} \\ = & (-2)^\nu C_{\nu,-l} \\ = & \frac{(-2)^\nu}{2} \sum_{s=0}^{\nu} \binom{\nu}{s} (-l)^s \sum_n n^{\nu-s} h_n. \end{aligned}$$

Therefore, by substituting the above into I_1, we can rewrite I_1 as

$$\mathrm{I}_1 = \sqrt{2} \sum_{s=0}^{\nu} H_{\nu,s} \varphi_l(2x)$$

with $H_{\nu,s} = (-1)^{\nu+s} 2^{-\nu-1} \binom{\nu}{s} \sum_{n=-N+1}^{N} n^{\nu-s} h_n$.

It remains to show the relation (A.1.1). Notice that

$$\begin{pmatrix} \phi_0^L \\ \phi_1^L \\ \vdots \\ \phi_{N-1}^L \end{pmatrix} = \left[\mathbf{x}i_{-N+1}, \cdots, \mathbf{x}i_{-1}, \mathbf{e}_1, \cdots, \mathbf{e}_N \right] \begin{pmatrix} \phi_{0,-N+1} \\ \vdots \\ \phi_{0,0} \\ \vdots \\ \phi_{0,N-1} \end{pmatrix},$$

where $\mathbf{e}_i := (\delta_{i,j})_{j=1}^N$ denote the unit coordinate vectors in $\mathbb{R}^N$. Since

$\mathbf{x}i_l = \mathbf{A}^{-1}\mathbf{b}_l$, we then have

$$\mathbf{A}\begin{pmatrix}\phi_0^L\\ \phi_1^L\\ \vdots\\ \phi_{N-1}^L\end{pmatrix} = \left[\mathbf{b}_{-N+1},\cdots,\mathbf{b}_{-1},\mathbf{A}_1,\cdots,\mathbf{A}_N\right]\begin{pmatrix}\phi_{0,-N+1}\\ \vdots\\ \phi_{0,0}\\ \vdots\\ \phi_{0,N-1}\end{pmatrix} = \begin{pmatrix}\varphi_0\\ \varphi_1\\ \vdots\\ \varphi_{N-1}\end{pmatrix},$$

where the $\mathbf{A}_i$ are the column vectors of the $\mathbf{A}$.

Hence, we obtain the equation (3.5) for ϕ_ν^L as follows:

$$\begin{pmatrix}\phi_0^L\\ \phi_1^L\\ \vdots\\ \phi_{N-1}^L\end{pmatrix} = \mathbf{A}^{-1}\left[\mathbf{H},\mathbf{h}\right]\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\phi_{1,0}^L\\ \vdots\\ \phi_{1,N-1}^L\\ \phi_{1,N}\\ \vdots\\ \phi_{1,3N-2}\end{pmatrix},$$

where $\mathbf{H} = (H_{\nu,l})_{\nu,l=0}^{N-1}$ ($H_{\nu,l} = 0$ for $l \geq \nu$); $\mathbf{h} = (h_{\nu,j})_{\nu=0,j=N}^{N-1,3N-2}$; and $\mathbf{I}$ is the $2N-1 \times 2N-1$ identity matrix. This completes the proof. □

A.2. Proof of Proposition 3.2

Let us first show the (3.7)(ii). Since $\tilde{\phi}_{N-1}^L = \phi(\cdot - N+1)$, the (3.7)(ii) clearly follows when $\nu = N-1$. We then fix $0 \leq \nu \leq N-2$. For each $l \geq N$, by the refinement equation (2.6), we have

$$\begin{aligned}\phi(x-l) &= \sqrt{2}\sum_{n=-N+1}^{N} h_n\phi(2x-2l-n)\\ &= \sqrt{2}\sum_{j=-N+1+2l}^{N+2l} h_{j-2l}\phi(2x-j).\end{aligned}$$

Thus, for given ν, the left hand side of (3.7)(ii) can be calculated as follows:

(A.2.1)
$$\langle \phi(x-l), \tilde{\phi}_\nu^L \rangle = \sqrt{2} \Big\langle \sum_{j=-N+1+2l}^{N+2l} h_{j-2l}\phi(2\,\cdot\,-j), \sum_{m=N-1}^{3N-4} \eta_{m,\nu}\phi(2\,\cdot\,-m) \Big\rangle$$

$$= \frac{\sqrt{2}}{2} \sum_{m=\max(-N+1+2l,N-1)}^{\min(3N-4,N+2l)} h_{m-2l}\,\eta_{m,\nu}$$

$$\& = \frac{\sqrt{2}}{2} \sum_{m=-N+1+2l}^{3N-4} h_{m-2l}\,\eta_{m,\nu}$$

$$= \frac{\sqrt{2}}{2} \sum_{n=-N+1}^{3N-4-2l} h_n \eta_{n+2l,\nu}.$$

Here for the third equality, we have reduced the range $l \ge N$ to $N \le l \le 2N-3$ because if $l \ge 2N-2$, then $2l+n \ge 3N-3$ for $-N+1 \le n \le N$. In fact, $\langle \phi(\cdot - 2l - n), \phi(2\,\cdot\,-m) \rangle = 0$, for $m \le 3N-4$. This implies $\langle \phi(x-l), \tilde{\phi}_\nu^L \rangle = 0$ for $l \ge 2N-2$. Notice that the last sum in (A.2.1) is nothing but the $(l+1)$-th component of the $\mathbf{B}\eta_\nu$; so that

$$\begin{aligned} \langle \phi(x-l), \tilde{\phi}_\nu^L \rangle &= (\mathbf{B}\eta_\nu)_{l+1} \\ &= (\mathbf{c}_\nu)_{l+1} = c_{l+1,\nu} = \delta_{l+1,\nu+1}. \end{aligned}$$

Thus, the left hand side of (3.7)(ii) is zero for $l \ge N$. The relation $0 = \langle \tilde{\phi}(\cdot - l), \phi_\nu^L \rangle$ is clear from the definition of ϕ_ν^L.

Let us next show the (3.7)(i). Notice from the refinement equation (2.6) that $\langle \phi(\cdot - j)\big|_{[0,\infty)}, \phi(2\,\cdot\,-m) \rangle = 0$ for $j \le -1$ and $N-1 \le m \le 3N-4$. Then, the (3.7)(i) can be reduced to the following relation:

(A.2.2)
$$\langle \phi(x-l)\big|_{[0,\infty)}, \tilde{\phi}_\nu^L \rangle = \delta_{l,\nu}, \qquad l = 0, \cdots, N-1.$$

Since $\tilde{\phi}^L_{N-1} = \phi(\cdot - N + 1)$ and $\operatorname{supp}\phi(\cdot - N + 1) \subset [0, \infty)$, the (A.2.2) is true for $\nu = N - 1$. We then fix $0 \leq \nu \leq N - 2$. For $l = 0, \cdots, N - 1$, the left hand side of (A.2.2) can be rewritten as

$$\begin{aligned}
\langle \phi(x-l)\big|_{[0,\infty)}, \tilde{\phi}^L_\nu \rangle &= \sqrt{2}\Big\langle \sum_{j=-N+1+2l}^{N+2l} h_{j-2l}\phi(2\,\cdot\,-j)\big|_{[0,\infty)}, \\
&\qquad \sum_{m=N-1}^{3N-4} \eta_{m,\nu}\phi(2\,\cdot\,-m)\Big\rangle \\
&= \frac{\sqrt{2}}{2} \sum_{m=\max(-N+1+2l,N-1)}^{\min(3N-4,N+2l)} h_{m-2l}\,\eta_{m,\nu} \qquad \text{(A.2.3)}\\
&= \frac{\sqrt{2}}{2} \sum_{m=N-1}^{\min(3N-4,N+2l)} h_{m-2l}\,\eta_{m,\nu} \\
&= \frac{\sqrt{2}}{2} \sum_{n=N-1-2l}^{\min(3N-4-2l,N)} h_n\,\eta_{n+2l,\nu}.
\end{aligned}$$

For $l = N-1$, the last sum of (A.2.3) is equal to $\frac{\sqrt{2}}{2}\sum_{-N+1}^{N-2} h_n\,\eta_{n+2N-2,\nu}$, that is just the N-th component of the $\mathbf{B}\eta_\nu$. So, we have for $0 \leq \nu \leq N - 2$,

$$\begin{aligned}
\langle \phi(x - N + 1), \tilde{\phi}^L_\nu \rangle &= (\mathbf{B}\eta_\nu)_N \\
&= (\mathbf{c}_\nu)_N = c_{N,\nu} = \delta_{N,\nu+1} = 0.
\end{aligned}$$

For $0 \leq l \leq N - 2$, the last sum of (A.2.3) is equal to

$$\frac{\sqrt{2}}{2} \sum_{n=N-1-2l}^{N} h_n\,\eta_{n+2N-2,\nu},$$

that is again nothing but the $(l + 1)$-th component of the $\mathbf{B}\eta_\nu$. So, we have for $0 \leq \nu \leq N - 2$,

$$\langle \phi(x - l), \tilde{\phi}^L_\nu \rangle = \delta_{l+1,\nu+1}.$$

This completes the proof. □

A.3. Proof of Proposition 3.3

The relation (3.9)(i) is immediate from the (3.7)(i).

To show the relation (3.9)(ii), (iii), we notice that for $0 \le s \le N-2$,

$$\begin{aligned}\langle \sqrt{2}\,\phi_\nu^L(2\,\cdot\,), \tilde{\phi}_s^L\rangle &= \Big\langle \sqrt{2}\sum_{l=-N+1}^{-1} \xi_{\nu,l}\phi(2\,\cdot\,-l) + \sqrt{2}\,\phi(2\,\cdot\,-\nu), \\ &\qquad\qquad \sum_{l=N-1}^{3N-4} \eta_{l,s}\phi(2\,\cdot\,-l)\Big\rangle \\ &= \begin{cases} 0 & \text{for} \quad 0 \le \nu \le N-2, \\ \frac{\sqrt{2}}{2}\eta_{N-1,s} & \text{for} \quad \nu = N-1. \end{cases}\end{aligned}$$

Also, we notice that for $s = N-1$,

$$\begin{aligned}\langle \sqrt{2}\,\phi_\nu^L(2\,\cdot\,), \tilde{\phi}(\,\cdot\,-\nu)\rangle &= \langle \sqrt{2}\,\phi(2\,\cdot\,-N+1), \phi(\,\cdot\,-N+1)\rangle \\ &= \begin{cases} 0 & \text{for} \quad 0 \le \nu \le N-2, \\ h_{-N+1} & \text{for} \quad \nu = N-1. \end{cases}\end{aligned}$$

From the observation, we can rewrite the q_ν as

$$q_\nu(x) = \begin{cases} \sqrt{2}\,\phi_\nu^L(2x), & 0 \le \nu \le N-2, \\ \sqrt{2}\,\phi_{N-1}^L(2x) - \frac{\sqrt{2}}{2}\sum_{s=0}^{N-2}\eta_{N-1,s}\phi_s^L(x) - h_{-N+1}\phi_{N-1}^L, & \\ \qquad\qquad \nu = N-1. & \end{cases}$$

Thus, we have for $0 \le \nu, \mu \le N-2$,

$$\begin{aligned}\langle q_\nu, \tilde{q}_\mu\rangle &= \langle \sqrt{2}\,\phi_\nu^L(2\,\cdot\,), \sqrt{2}\,\tilde{\phi}_\mu^L(2\,\cdot\,)\rangle \\ &\qquad - \sum_{s=0}^{N-1}\langle \sqrt{2}\,\tilde{\phi}_\mu^L(2\,\cdot\,), \phi_s^L\rangle\langle \sqrt{2}\,\phi_\nu^L(2\,\cdot\,), \tilde{\phi}_s^L\rangle \\ &= \langle \sqrt{2}\phi_\nu^L(2\,\cdot\,), \sqrt{2}\tilde{\phi}_\mu^L(2\,\cdot\,)\rangle \\ &= \delta_{\nu,\mu},\end{aligned}$$

that proves the (3.9)(ii).

Finally, when $\nu = N-1$, we have

$$\begin{aligned}\langle q_{N-1}, \tilde{q}_{N-1}\rangle &= \langle q_{N-1}, \tilde{\psi}(\,\cdot\, - N+1)\rangle \\ &= \Big\langle \sqrt{2}\,\phi^L_{N-1}(2\,\cdot\,) - \frac{\sqrt{2}}{2}\sum_{s=0}^{N-2} \eta_{N-1,s}\phi^L_s(\,\cdot\,) - h_{-N+1}\phi^L_{N-1}, \\ &\qquad\qquad \tilde{\psi}(\,\cdot\, - N+1)\Big\rangle \\ &= \langle \sqrt{2}\,\phi^L_{N-1}(2\,\cdot\,), \tilde{\psi}(\,\cdot\, - N+1)\rangle \\ &= (-1)^{-N+1} h_N\end{aligned}$$

where the (3.9)(i) is employed for the third equality. Thus, we obtain (3.9)(iii). This completes the proof. □

A.4. Proof of Proposition 3.4

It is sufficient to prove this proposition for the edge functions. From the definitions of ϕ^L_ν and ψ^L_ν, and the refinment equation (3.5), we notice that each $\psi^0_{k,j}$ is a finite linear combination of $\phi_{k+1,j}\big|_{[0,1]}$, $j = -N+1, \cdots, 3N-2$ with coefficients independent of k, j (the coefficients may depend on the matrix $\mathbf{A}$ and the coefficients of the refinement equations (3.5)). So, we obtain for $0 \le r \le \lfloor s \rfloor$,

$$\begin{aligned}|\psi^0_{k,j}|_{W^r_\infty([0,1])} &\le C \sum_{j=-N+1}^{3N-2} |\phi_{k+1,j}|_{W^r_\infty} \\ &\le C 2^{(k+1)/2} 2^{(k+1)r} \|\phi^{(r)}\|_{L_\infty} \\ &\le C 2^{k/2} 2^{kr},\end{aligned}$$

where the constant C is independent of k, j (the same is true for the $\psi^1_{k,j}$).

By the same way, the inequality (3.13)(i) can be proved. Also, the (3.13)(ii) follows clearly from that $\mathcal{P}_N\big|_{[0,1]} \subset V^\Omega_k$, $k \ge k_0$, and $V^\Omega_k \perp \widetilde{W}^\Omega_k$.
□

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Department of Mathematics
Hanyang University
Ansan 425-791, Korea
E-mail: kim@math1.hanyang.ac.kr
kim@banach.hanyang.ac.kr