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A NONLINEAR GALERKIN METHOD FOR THE BURGERS EQUATION

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ABSTRACT. A nonlinear Galerkin method for the Burgers equation is considered. Due to the lack of the divergence free condition, the nonlinear term is treated differently compared to that of the Navier-Stokes equations. Strong convergence results are proved for the nonlinear Galerkin method.

1. Introduction

Nonlinear Galerkin methods have been developed for nonlinear partial differential equations such as the Navier-Stokes equations describing the motion of incompressible viscous fluid flow from the computational point of view [2,4,5,7]. The methods focus mainly on treatment of the exchange of energy between the low and high mode components of the flow. It is computationally inefficient to allocate as much computing resources to compute the small scale component carrying little energy as we do with the large scale component of the flow.

The usual Galerkin methods project the governing system to a finite dimensional one and put the computing efforts equally on the finite dimensional system regardless of low or high frequency modes. On the other hand, the nonlinear Galerkin methods decouple the system into the low frequency modes and the high frequency ones, and treat them separately. Nonlinear Galerkin methods compute an approximate solution on the high frequency mode space and plug it in the dynamics

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represented by low frequency modes. By doing so, the two separated systems (low and high frequency systems) have much less computing dimensions than those of Galerkin methods, and, accordingly, save much computing times for solving the given system of equations. Specifically, Jauberteau, Rosier and Temam [4] saved 30-40 % computing time for the two-dimensional Navier-Stokes equations with space periodic boundary conditions compared with the classical Galerkin approaches.

The Burgers equation is a one-dimensional simple model describing fluid flows such as traffic flows, supersonic flow about airfoils, acoustic transmission and turbulence in hydrodynamic flows [1,3]. In this paper, a nonlinear Galerkin approach for the Burgers equation with Dirichlet boundary conditions is considered. However, the equation with general boundary conditions can be treated by similar analyses using superpositions. Recently, Shen and Temam [7] analyzed error bounds for a version of nonlinear Galerkin methods for the Burgers equation using Chebyshev and Legendre polynomials. In this paper, we employ sine functions to analyze strong convergence properties of the nonlinear Galerkin method. Since the divergence free condition satisfied by the Navier-Stokes equations does not applied to the Burgers equation, we need to treat the nonlinear term differently. We obtain stronger convergence results than those from standard convergence analysis.

Section 2 describes the governing equation and the nonlinear Galerkin method employed in this paper. The convergence property of the method is proved in Section 3. Throughout this paper, all notations are standard [8].

2. A nonlinear Galerkin approach

We consider the following Burgers equation with Dirichlet boundary conditions.

$$
\begin{aligned}
&\frac{\partial}{\partial t}u(t,x) + u(t,x)\frac{\partial}{\partial x}u(t,x) = \epsilon\frac{\partial^2}{\partial x^2}u(t,x), \quad t>0, \quad x\in\Omega,\\
&u(0,x) = u_0(x), \quad x\in\Omega, \\
&u(t,0) = u(t,1) = 0, \quad t>0,
\end{aligned} \tag{2.1}
$$

where $\epsilon > 0$ and $\Omega = (0,1)$. To change the system (2.1) into a dynamical system, let $u(t)(\cdot) = u(t,\cdot)$, $u(0)(\cdot) = u(0,\cdot)$, $H = L^2(\Omega)$, and $V =$

$H_0^1(\Omega)$. Define an operator A in H by

$$A\phi = -\frac{d^2}{dx^2}\phi, \quad \phi \in \mathcal{D}(A), \tag{2.2}$$

where $\mathcal{D}(A) = H^2(\Omega) \cap V \subset V \subset H$. Then it is easy to see that A is a linear unbounded self-adjoint operator in H with domain $\mathcal{D}(A)$. It is also easy to see that A is positive and closed, and that A^{-1} is compact (see, for example, [6]). By the first Poincaré inequality [9], we take the norm in V as $|\cdot|_V = |A^{\frac{1}{2}} \cdot|_H$. Since A^{-1} is compact and self-adjoint, there exists an orthonormal basis $\{\phi_j\}$ of H consisting of eigenvectors of A. Specifically, let

$$\begin{aligned} \lambda_j &= j^2\pi^2, \\ \phi_j(x) &= \sqrt{2}\sin j\pi x, \quad j = 1, 2, \cdots . \end{aligned} \tag{2.3}$$

Then $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$ is a set of eigenpairs of A and $\{\phi_j\}$ forms an orthonormal basis of H.

To treat the nonlinear term in (2.1), define a bilinear operator B from $V \times V$ into V^* by

$$B(\phi, \psi) = \phi\psi', \quad \phi, \psi \in V, \tag{2.4}$$

where $' = \frac{d}{dx}$ and V^* is the strong dual space of V with respect to the H-norm. Then (2.1) becomes the following dynamical system

$$\begin{aligned} \frac{d}{dt}u(t) - \epsilon Au(t) + B(u(t), u(t)) &= 0, \\ u(0) &= u_0. \end{aligned} \tag{2.5}$$

To write the system (2.5) as a variational formulation, define a bilinear form $a(\cdot,\cdot) : V \times V \to \mathbf{R}$ by

$$a(\phi, \psi) = \int_0^1 \phi'\psi'\, dx, \quad \phi, \psi \in V, \tag{2.6}$$

and a trilinear form $b(\cdot,\cdot,\cdot) : V \times V \times V \to \mathbf{R}$ by

$$b(\phi, \psi, \xi) = \frac{1}{3}\int_0^1 ((\phi\psi)' + \phi\psi')\, \xi\, dx, \quad \phi, \psi, \xi \in V. \tag{2.7}$$

Then the system (2.5) (equivalently (2.1)) can be written as

$$
\begin{aligned}
&< \frac{d}{dt}u, \phi > +\epsilon a(u, \phi) + b(u, u, \phi) = 0, \quad \phi \in V, \\
&u(0) = u_0.
\end{aligned}
\tag{2.8}
$$

We will apply a nonlinear Galerkin method for solving the variational problem (2.8).

REMARK 2.1. The trilinear form b defined in (2.7) is a different form from what we can expect in the Navier-Stokes equations. This form satisfies the zero dissipative property (2.10) and is usually taken for the Burgers equation (see, for example, [7]).

By the first Poincaré inequality and simple calculations, we have the following relations.

$$
\begin{aligned}
&a(\phi, \psi) = (A\phi, \psi), \\
&a(\psi, \psi) = |\psi|_V^2 \quad \text{for all } \phi \in \mathcal{D}(A),\ \psi \in V,
\end{aligned}
\tag{2.9}
$$

$$
b(\phi, \psi, \xi) + b(\phi, \xi, \psi) = 0 \quad \text{for all } \phi, \psi, \xi \in V.
\tag{2.10}
$$

Several nonlinear Galerkin methods have been reported in the literature [2,4,5,7 and references therein]. The choice of method is based on how to treat the low energy modes. We consider a classical method for our convergence analysis. But, our analyses can be easily extended to other methods.

Let $\{\phi_j\}$ be the orthonormal eigenvectors of A given by (2.3). For a finite dimensional approximation of problem (2.8) let H_m be the linear span of $\{\phi_j\}_{j=1}^m$, $\tilde{H}_m$ the linear span of $\{\phi_j\}_{j=m+1}^{2m}$, and let P_m be the orthogonal projection from H onto H_m. Let

$$
\begin{aligned}
u_m(t) &= \sum_{j=1}^{m} \alpha_{jm}(t)\phi_j, \quad \phi_j \in H_m, \\
z_m(t) &= \sum_{j=m+1}^{2m} \beta_{jm}(t)\phi_j, \quad \phi_j \in \tilde{H}_m.
\end{aligned}
\tag{2.11}
$$

Consider the following nonlinear Galerkin approximation problem.

Find a pair $\{u_m, z_m\}$ satisfies

(2.12)

$$< \frac{d}{dt}u_m, \phi > +\epsilon a(u_m,\phi) + b(u_m,u_m,\phi) + b(u_m,z_m,\phi) + b(z_m,u_m,\phi) = 0, \quad \phi \in H_m,$$

(2.13)
$$\epsilon a(z_m,\psi) + b(u_m,u_m,\psi) = 0, \quad \psi \in \tilde{H}_m,$$

together with the initial condition

(2.14)
$$u_m(0) = P_m u_0.$$

REMARK 2.2. (i) If $z_m = 0$, the system (2.12)-(2.14) becomes the classical Galerkin approximation on H_m. The existence and uniqueness of a solution to (2.12)-(2.14) defined on a maximal interval $[0, T_m)$ follows standard theory of ordinary differential equations. Moreover, from the next section, T_m can be taken as $+\infty$.

(ii) The nonlinear Galerkin approximation (2.12)-(2.14) is obtained by neglecting the terms $\frac{d}{dt}z_m(t)$ and $z_m z_m'$ in the classical Galerkin approach on $H_m \cup \tilde{H}_m$ from the low energy point of view.

(iii) Since (2.13) is a linear system of z_m it can be solved by an efficient numerical algorithm.

3. Convergence analysis

In this section we prove convergence properties of the pair $\{u_m, z_m\}$ obtained from the nonlinear Galerkin method (2.11)-(2.14). We use the same notations as in Section 2. First, we have the following lemma.

LEMMA 3.1.

(i) $|b(\phi,\psi,\xi)| \le |\phi|_V\, |\psi|_V\, |\xi|_V, \quad \phi,\psi,\xi \in V.$

(ii) $\pi\, |u_m|_H \le |u_m|_V \le (m\pi)\, |u_m|_H.$

(iii) $|b(u_m,u_m,\xi)| \le (m\pi)^2 |u_m|_H^2\, |\xi\, _H, \quad \xi \in V.$

(iv) $|b(u_m,u_m,\xi)| \le |u_m|_H^{\frac{1}{2}}\, |u_m|_V^{\frac{3}{2}}\, |\xi|_V, \quad \xi \in V.$

PROOF. (i) Let $\phi, \psi, \xi \in V$. Then

$$\begin{aligned}
|b(\phi,\psi,\xi)| &\le \frac{1}{3}\int_0^1 |((\phi\psi)' + \phi\psi')\xi|\, dx \\
&\le \frac{1}{3}\left(|\phi'|_{L^2}\, |\psi|_{L^4}\, |\xi|_{L^4} + 2|\phi|_{L^4}\, |\psi'|_{L^2}\, |\xi|_{L^4}\right) \\
&\le \frac{1}{3}\left(|\phi|_V\, |\psi|_H^{\frac{1}{2}}\, |\psi|_V^{\frac{1}{2}}\, |\xi|_H^{\frac{1}{2}}\, |\xi|_V^{\frac{1}{2}} + 2|\phi|_H^{\frac{1}{2}}\, |\phi|_V^{\frac{1}{2}}\, |\psi|_V\, |\xi|_H^{\frac{1}{2}}\, |\xi|_V^{\frac{1}{2}}\right) \\
&\le |\phi|_V\, |\psi|_V\, |\xi|_V, \quad \text{since } |\cdot|_H \le |\cdot|_V.
\end{aligned}$$

(ii) Let u_m be given by (2.11). Then we have

$$\begin{aligned}
\pi^2\, |u_m|_H^2 = \pi^2 \sum_{j=1}^{m} |\alpha_{jm}(t)|^2 &\le \sum_{j=1}^{m} (j\pi)^2 |\alpha_{jm}(t)|^2 = |u_m|_V^2 \\
&\le (m\pi)^2 \sum_{j=1}^{m} |\alpha_{jm}(t)|^2 = (m\pi)^2\, |u_m|_H^2.
\end{aligned}$$

(iii) From (ii), it is easy to see that $|Au_m|_H \le (m\pi)^2 |u_m|_H$. Therefore, for any $\xi \in V$, we have

$$\begin{aligned}
|b(u_m, u_m, \xi)| &\le |u_m|_{L^4}\, |u_m'|_{L^4}\, |\xi|_{L^2} \\
&\le |u_m|_H^{\frac{1}{2}}\, |u_m|_V^{\frac{1}{2}}\, |u_m|_V^{\frac{1}{2}}\, |Au_m|_H^{\frac{1}{2}}\, |\xi|_H \\
&\le (m\pi)^2\, |u_m|_H^2\, |\xi|_H.
\end{aligned}$$

(iv) Let $\xi \in V$. Then we have

$$\begin{aligned}
|b(u_m, u_m, \xi)| &\le |u_m|_{L^4}\, |u_m'|_{L^2}\, |\xi|_{L^4} \\
&\le |u_m|_H^{\frac{1}{2}}\, |u_m|_V^{\frac{1}{2}}\, |u_m|_V\, |\xi|_H^{\frac{1}{2}}\, |\xi|_V^{\frac{1}{2}} \\
&\le |u_m|_H^{\frac{1}{2}}\, |u_m|_V^{\frac{3}{2}}\, |\xi|_V.
\end{aligned}$$

□

LEMMA 3.2.

(i) $|u_m(t)|_H \le |u_m(0)|_H \, e^{-\sqrt{2\epsilon}\pi t}, \quad t \ge 0.$

(ii) $|u_m|_{L^2(0,\infty;V)} \le \frac{1}{\sqrt{2\epsilon}} \, |u_m(0)|_H.$

(iii) $|z_m|_{L^2(0,\infty;V)} \le \frac{1}{\sqrt{2\epsilon}} \, |u_m(0)|_H.$

(iv) $|z_m|_H \le \frac{1}{\sqrt{\epsilon}} \, |u_m|_H.$

(v) $(m+1)\,|z_m|_H \le \frac{1}{\epsilon\pi} \, |u_m|_H \, |u_m|_V.$

PROOF. Let $\phi = u_m$ in (2.12), $\psi = z_m$ in (2.13), and add the corresponding equalities. We then have

$$< \frac{d}{dt} u_m, u_m > + \epsilon\, a(u_m, u_m) + \epsilon\, a(z_m, z_m) = 0,$$

since, from (2.10), $b(\phi,\psi,\xi) + b(\phi,\xi,\psi) = 0$ and hence $b(\phi,\psi,\psi) = 0$ for all $\phi,\psi,\xi \in V$. Hence, by (2.9),

$$\frac{1}{2}\frac{d}{dt}|u_m|_H^2 + \epsilon \left(|u_m|_V^2 + |z_m|_V^2\right) = 0. \tag{3.1}$$

Note that for any $\phi \in V$, $|\phi|_V \ge \pi\, |\phi|_H$. Thus, from (3.1), we have

$$\frac{1}{2}\frac{d}{dt}|u_m|_H^2 + \epsilon\pi^2 \left(|u_m|_H^2 + |z_m|_H^2\right) \le 0,$$
$$\frac{1}{2}\frac{d}{dt}|u_m|_H^2 + \epsilon\pi^2 \, |u_m|_H^2 \le 0.$$

Therefore, we have

$$|u_m(t)|_H^2 \le |u_m(0)|_H^2 \, e^{-2\epsilon\pi^2 t}, \quad t \ge 0. \tag{3.2}$$

(i) follows (3.2). To prove (ii) and (iii), integrate (3.1) from 0 to T, $T > 0$. Then we have, by (i),

$$\int_0^T |u_m|_V^2 \, dt + \int_0^T |z_m|_V^2 \, dt = \frac{1}{2\epsilon}\left(|u_m(0)|_H^2 - |u_m(T)|_H^2\right)$$
$$\le \frac{1}{2\epsilon}\, |u_m(0)|_H^2$$

By letting $T \to \infty$, (ii) and (iii) follow. For (iv), take $\psi = z_m$ in (2.13). Then

$$\epsilon\, |z_m|_V^2 = -b(u_m, u_m, z_m).$$

Since $z_m \in \tilde{H}_m$,

$$\epsilon\, ((m+1)\pi)^2\, |z_m|_H^2 \leq \epsilon\, |z_m|_V^2. \tag{3.3}$$

Hence,

$$\begin{aligned} \epsilon\, ((m+1)\pi)^2\, |z_m|_H^2 &\leq \epsilon\, |z_m|_V^2 = |-b(u_m, u_m, z_m)| \\ &\leq (m\pi)^2\, |u_m|_H^2\, |z_m|_H, \quad \text{by Lemma 3.1.(iii)}. \end{aligned}$$

Thus, (iv) follows the above inequality. On the other hand, from (3.3), $|z_m|_V \geq (m+1)\pi\, |z_m|_H$ and $|Au_m|_H \leq (m\pi)^2\, |u_m|_H$, we have

$$\begin{aligned} \epsilon((m+1)\pi)^2\, |z_m|_H^2 &\leq |b(u_m, u_m, z_m)| \leq |u_m|_H^{\frac{1}{2}}\, |u_m|_V\, |Au_m|_H^{\frac{1}{2}}\, |z_m|_H \\ &\leq (m\pi)\, |u_m|_H\, |u_m|_V\, |z_m|_H. \end{aligned}$$

Thus, (v) follows immediately the above inequality. □

From Lemma 3.2, we have the following theorem.

THEOREM 3.3. *Let $\{u_m, z_m\}$ be the solution for the system (2.11)-(2.14). Then both the sequences $\{u_m\}$ and $\{z_m\}$ are bounded in $L^\infty(0,\infty; H)$ and in $L^2(0,\infty;V)$ as $m \to \infty$. Moreover, the sequence $\{(m+1)z_m\}$ is also bounded in $L^2(0,\infty;H)$ as $m \to \infty$.*

For the boundedness of $\{\frac{d}{dt}u_m\}$, we have the following lemma.

LEMMA 3.4. *The sequence $\{\frac{d}{dt}u_m\}$ is bounded in $L^2(0,\infty;V^*)$ as $m \to \infty$.*

PROOF. Note that for any $\xi \in V$, $| < Au_m, \xi > | \leq |u_m|_V\, |\xi|_V$. By Lemma 3.1.(i), for any $\xi \in V$,

$$\begin{aligned} |b(u_m, u_m, \xi)| &\leq |u_m|_V\, |u_m|_V\, |\xi|_V, \\ |b(u_m, z_m, \xi)| &\leq |u_m|_V\, |z_m|_V\, |\xi|_V, \\ |b(z_m, u_m, \xi)| &\leq |z_m|_V\, |u_m|_V\, |\xi|_V. \end{aligned}$$

Since $\{u_m\}$ and $\{z_m\}$ are bounded in $L^2(0,\infty;V)$ as $m \to \infty$ by Theorem 3.3, $< Au_m, \cdot >$, $b(u_m, u_m, \cdot)$, $b(u_m, z_m, \cdot)$ and $b(z_m, u_m, \cdot)$ are all bounded in $L^2(0,\infty;V^*)$ as $m \to \infty$, where V^* is the dual space of V with respect to the H-norm as before. Since the system (2.12)-(2.14) is equivalent to the following ordinary differential system

(3.4)
$$\frac{d}{dt}u_m + \epsilon A u_m + P_m\left(b(u_m,u_m,\cdot) + b(z_m,u_m,\cdot) + b(u_m,z_m,\cdot)\right) = 0,$$
$$z_m = -(\epsilon A)^{-1}(P_{2m} - P_m)(b(u_m,u_m,\cdot)),$$
$$u_m(0) = P_m u_0,$$

in V^*, the lemma follows. □

Finally, we have the following strong convergent results for the sequence $\{u_m\}$ obtained from our nonlinear Galerkin approxmation.

THEOREM 3.5. *For any given $u_0 \in H$, the solution u_m of (2.12)-(2.14) converges to the solution u of problem (2.5) in the following sense:*
$u_m \to u$ in $L^2(0,\infty;V)$ strongly, $u_m(t) \to u(t)$ in H strongly for all $t \geq 0$ as $m \to \infty$.

PROOF. By Theorem 3.3, the sequence $\{(m+1)z_m\}$ is bounded in $L^2(0,\infty;H)$ as $m \to \infty$. Thus, we have

(3.5) $$z_m \to 0 \quad \text{in } L^2(0,\infty;H) \text{ strongly as } m \to \infty.$$

Hence, again from Theorem 3.3,

(3.6) $$\begin{aligned} &z_m \to 0 \text{ in } L^2(0,\infty;V^*) \text{ weakly,} \\ &\text{and } L^\infty(0,\infty;H) \text{ weak-star as } m \to \infty. \end{aligned}$$

Also, from Theorem 3.3 and Lemma 3.4,

(3.7)
there exists an element u^* and a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ such that $u_{m_k} \to u^*$ in $L^2(0,\infty;V)$ weakly, and $L^\infty(0,\infty;H)$ weak-star as $m_k \to \infty$; $\frac{d}{dt}u_{m_k} \to \frac{d}{dt}u^*$ in $L^2(0,\infty;V^*)$ weakly as $m_k \to \infty$.

Thus, by a standard compactness theorem [8],

$$u_{m_k} \to u^* \quad \text{in } L^2(0,\infty;H) \text{ strongly as } m_k \to \infty. \tag{3.8}$$

We now let $\xi \in \tilde{H}_m$ be fixed and take $m_k \geq m$. Then

$$\begin{aligned} & b(u_{m_k}, u_{m_k}, \xi) - b(u^*, u^*, \xi) \\ =& b(u_{m_k} - u^*, u_{m_k}, \xi) + b(u^*, u_{m_k}, \xi) - b(u^*, u^*, \xi) \\ =& - b(u_{m_k} - u^*, \xi, u_{m_k}) - b(u^*, \xi, u_{m_k} - u^*) \to 0 \quad \text{as } m_k \to \infty . \end{aligned}$$

Simiarly, $b(z_{m_k}, u_{m_k}, \xi) \to b(0, u^*, \xi) = 0$ and $b(u_{m_k}, z_{m_k}, \xi) \to b(u^*, 0, \xi) = 0$ as $m_k \to \infty$. Therefore, by letting $m_k \to \infty$ in (2.12) with $m = m_k$, u^* satisfies

$$< \frac{d}{dt}u^*, \xi > + \epsilon\, a(u^*, \xi) + b(u^*, u^*, \xi) = 0 \tag{3.9}$$

for all $\xi \in \tilde{H}_m$, and by continuity, for all $\xi \in V$. Furthermore,

$$u_{m_k}(0) \to u^*(0) \quad \text{weakly in } H. \tag{3.10}$$

Since $u_{m_k}(0) = P_{m_k}u_0$, $u^*(0) = u_0$. Thus, by (3.9), u^* is a solution of problem (2.8), i.e., $u^* = u$. Therefore, the whole sequence $\{u_m\}$ converges to u in the sense (3.7), i.e.,

$$\begin{aligned} & u_m \to u^* \text{ in } L^2(0,\infty;V) \text{ weakly, and } L^\infty(0,\infty;H) \text{ weak-star} \\ & \text{as } m \to \infty; \frac{d}{dt}u_m \to \frac{d}{dt}u^* \quad \text{in } L^2(0,\infty;V^*) \text{ weakly as } m \to \infty. \end{aligned} \tag{3.11}$$

We now prove the strong convergence of $u_m \to u$ in $L^2(0,\infty;V)$. From (3.9) and (3.11), we have

$$\frac{1}{2}\frac{d}{dt}|u|_H^2 + \epsilon\,|u|_V^2 = 0. \tag{3.12}$$

By integrating (3.1) and (3.12) from 0 to T, $T \geq 0$,

$$\begin{aligned} & \frac{1}{2}\left(|u_m(T)|_H^2 - |u_m(0)|_H^2\right) + \epsilon \int_0^T \left(|u_m|_V^2 + |z_m|_V^2\right)\, dt = 0, \\ & \frac{1}{2}\left(|u(T)|_H^2 - |u(0)|_H^2\right) + \epsilon \int_0^T |u|_V^2\, dt = 0. \end{aligned} \tag{3.13}$$

Thus, by (3.13),

$$\begin{aligned}
&\frac{1}{2}\,|u_m(T)-u(T)|_H^2+\epsilon\int_0^T\left(|u_m-u|_V^2+|z_m|_V^2\right)\,dt\\
=&\frac{1}{2}\left(|u_m(T)|_H^2-2<u_m(T),u(T)>_H+|u(T)|_H^2\right)\\
&\qquad+\epsilon\int_0^T\left(|u_m|_V^2-2<u_m,u>_V+|u|_V^2+|z_m|_V^2\right)\,dt\\
=&\frac{1}{2}\,|u_m(0)|_H^2-<u_m(T),u(T)>_H+\frac{1}{2}\,|u(T)|_H^2\\
&\qquad+\epsilon\int_0^T\left(|u|_V^2-2<u_m,u>_V\right)\,dt\\
\to\ \ &\frac{1}{2}\left(|u(0)|_H^2-|u(T)|_H^2\right)-\epsilon\int_0^T|u|_V^2\,dt=0\quad\text{as } m\to\infty\,.
\end{aligned}$$

Therefore, $u_m\to u$ in $L^2(0,T;V)$ strongly for all $T\geq 0$ and $u_m(t)\to u(t)$ in H strongly for all $t\geq 0$ as $m\to\infty$. On the other hand, by Lemma 3.2, Theorem 3.3 and (3.11), we can take $T=\infty$, i.e., $u_m\to u$ in $L^2(0,\infty;V)$ strongly as $m\to\infty$. □

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