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k-NIL RADICAL IN BCI-ALGEBRAS II

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ABSTRACT. This paper is a continuation of [3]. We prove that if A is a quasi-associative (resp. an implicative) ideal of a BCI-algebra X then the k-nil radical of A is a quasi-associative (resp. an implicative) ideal of X. We also construct the quotient algebra $X/[A;k]$ of a BCI-algebra X by the k-nil radical $[A;k]$, and show that if A and B are closed ideals of BCI-algebras X and Y respectively, then

$$X/[A;k] \times Y/[B;k] \cong X \times Y/[A \times B;k].$$

By a *BCI-algebra* we mean an algebra $(X;*,0)$ of type (2, 0) satisfying the axioms:

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $x*y=y*x=0$ implies $x=y$,

for all x,y and z in X. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x*y=0$. A BCI-algebra X is said to be *p-semisimple* if $\{x \in X | 0 \leq x\} = \{0\}$.

In any BCI-algebra X, the following hold:

(1) $x*0=x$.
(2) $(x*y)*z=(x*z)*y$.
(3) $0*(0*(0*x))=0*x$.
(4) $0*(x*y)=(0*x)*(0*y)$.

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In what follows, X would mean a BCI-algebra unless otherwise specified.

A non-empty subset A of X is called a *subalgebra* of X if $x * y \in A$ whenever $x, y \in A$.

A non-empty subset A of X is called an *ideal* of X if $0 \in A$ and if $x * y, y \in A$ imply that $x \in A$. We note that if x is in an ideal A of X and $y \leq x$, then $y \in A$.

An ideal I of X is said to be *closed* if $0 * x \in I$ whenever $x \in I$. We note that every closed ideal of X is a subalgebra of X.

A mapping $f : X \to Y$ of BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

For any elements x, y in X, let us write $x * y^k$ for $(...((x * y) * y) * ...) * y$ where y occurs k times.

LEMMA 1 (Huang [2, Lemmas 1 and 2]). *For any x, y in X and any positive integer k, we have*

(i) $0 * (x * y)^k = (0 * x^k) * (0 * y^k)$.

(ii) $0 * (0 * x)^k = 0 * (0 * x^k)$.

DEFINITION 1 (Hong et al. [3]). Let A be a subset of X. For given positive integer k, the *k-nil radical* of A, denote by $[A; k]$, is the set of all elements of X satisfying $0 * x^k \in A$, i.e.,

$$[A; k] := \{x \in X : 0 * x^k \in A\}.$$

Note that $[A; k]$ may not contain A itself (see [3]).

LEMMA 2 (Hong et al. [3, Proposition 2]). *Let A be a subalgebra of X and k a positive integer. Then*

(i) *if $x \in [A; k]$, then $0 * x \in [A; k]$.*

(ii) *if $x * y \in [A; k]$, then $y * x \in [A; k]$.*

LEMMA 3 (Hong et al. [3, Theorem 1]). *If A is a subalgebra of X, then the k-nil radical of A is a subalgebra of X containing A for every positive integer k.*

LEMMA 4 (Hong et al. [3, Theorems 2 and 3]). *If A is a (closed) ideal of X, then the k-nil radical of A is a (closed) ideal of X for every positive integer k.*

DEFINITION 2 (Yue et al. [8]). A non-empty subset A of X is called a *quasi-associative ideal* of X if it satisfies

(i) $0 \in A$,

(ii) $x * (y * z) \in A$ and $y \in A$ imply $x * z \in A$,

for all $x, y, z \in X$.

THEOREM 1. *If A is a quasi-associative ideal of X, then so is the k-nil radical of A for every positive integer k.*

PROOF. Clearly $0 \in [A; k]$. Let $x, y, z \in X$ be such that $x * (y * z) \in [A; k]$ and $y \in [A; k]$. By using Lemma 1(i), we obtain

$$\begin{aligned} 0 * (x * (y * z))^k &= (0 * x^k) * (0 * (y * z)^k) \\ &= (0 * x^k) * ((0 * y^k) * (0 * z^k)) \in A \end{aligned}$$

and $0 * y^k \in A$. Since A is a quasi-associative ideal, it follows from Lemma 1(i) that

$$0 * (x * z)^k = (0 * x^k) * (0 * z^k) \in A \text{ or } x * z \in [A; k].$$

Hence $[A; k]$ is a quasi-associative ideal of X. □

DEFINITION 3 (Hoo [1]). An ideal A of X is said to be *implicative* if whenever $(x * y) * z \in A$ and $y * z \in A$ then $x * z \in A$.

THEOREM 2. *If A is an implicative ideal of X, then the k-nil radical of A is also an implicative ideal of X for every positive integer k.*

PROOF. We note from Lemma 4 that $[A; k]$ is an ideal of X. Let $x, y, z \in X$ be such that $(x * y) * z \in [A; k]$ and $y * z \in [A; k]$. Then

$$\begin{aligned} 0 * ((x * y) * z)^k &= (0 * (x * y)^k) * (0 * z^k) \\ &= ((0 * x^k) * (0 * y^k)) * (0 * z^k) \in A \end{aligned}$$

and $0 * (y * z)^k = (0 * y^k) * (0 * z^k) \in A$. Since A is an implicative ideal, it follows that $0 * (x * z)^k = (0 * x^k) * (0 * z^k) \in A$ or equivalently $x * z \in [A; k]$. Hence $[A; k]$ is an implicative ideal of X. □

THEOREM 3. *Let $f : X \to Y$ be a homomorphism of BCI-algebras and let A be a subset of X. Then $f([A;k]) \subseteq [f(A);k]$ for every positive integer k.*

PROOF. Let $y \in f([A;k])$. Then there exists $x \in [A;k]$ such that $f(x) = y$. It follows that

$$0 * y^k = f(0) * (f(x))^k = f(0 * x^k) \in f(A)$$

so that $y \in [f(A);k]$, ending the proof. □

THEOREM 4. *Let $f : X \to Y$ be a homomorphism of BCI-algebras and let A be a subalgebra of Y. Then $f^{-1}([A;k])$ is a subalgebra of X containing $[f^{-1}(A);k]$ for every positive integer k.*

PROOF. Let $x, y \in f^{-1}([A;k])$. Then $f(x), f(y) \in [A;k]$. It follows from Lemma 3 that $f(x * y) = f(x) * f(y) \in [A;k]$ or equivalently $x * y \in f^{-1}([A;k])$, which shows $f^{-1}([A;k])$ is a subalgebra of X. To prove that $[f^{-1}(A);k] \subseteq f^{-1}([A;k])$, let $x \in [f^{-1}(A);k]$. Then $0 * x^k \in f^{-1}(A)$ which implies that $0 * (f(x))^k = f(0) * f(x^k) = f(0 * x^k) \in A$. Thus $f(x) \in [A;k]$ or equivalently $x \in f^{-1}([A;k])$. This completes the proof. □

Note that the inverse image of an ideal under a BCI-homomorphism is an ideal. Hence we have the following theorem

THEOREM 5. *Let $f : X \to Y$ be a homomorphism of BCI-algebras. If A is an ideal of Y, then $f^{-1}([A;k])$ is an ideal of X containing $[f^{-1}(A);k]$ for every positive integer k.*

Let X and Y be BCI-algebras. We define $*$ on $X \times Y$ by

$$(x, y) * (u, v) = (x * u, y * v) \text{ for every } (x, y), (u, v) \in X \times Y.$$

Then clearly $(X \times Y; *, (0,0))$ is a BCI-algebra.

Next we shall define the quotient algebra $X/[A;k]$ of X by $[A;k]$. Let A be a closed ideal of X and let k be a positive integer. We define a relation $\sim$ on X by $x \sim y$ if and only if $x * y \in [A;k]$ for every $x, y \in X$.

Then $\sim$ is an equivalence relation on X. In fact, by using Lemma 1(i), we have

$$0 * (x * x)^k = (0 * x^k) * (0 * x^k) = 0 \in A \text{ for every } x \in X,$$

which implies that $x * x \in [A; k]$ or equivalently $x \sim x$.

If $x \sim y$, then $x * y \in [A; k]$ and hence, from Lemma 2(ii), $y * x \in [A; k]$. Hence $y \sim x$.

Assume that $x \sim y$ and $y \sim z$. Then $x * y \in [A; k]$ and $y * z \in [A; k]$. Hence $(x * z) * (x * y) \leq y * z$ implies $x * z \in [A; k]$, since $[A; k]$ is an ideal. Therefore $x \sim z$. Consequently $\sim$ is an equivalence relation on X. Denote by C_x the equivalence class containing x, and by $X/[A; k]$ the set of all equivalence classes. We claim that $C_0 = [A; k]$. Let $x \in [A; k]$. Then

$$0 * (x * 0)^k = (0 * x^k) * (0 * 0^k) = (0 * x^k) * 0 = 0 * x^k \in A,$$

which implies that $x * 0 \in [A; k]$, i.e., $x \sim 0$. Hence $x \in C_0$. Conversely, let $x \in C_0$. Then $x \sim 0$ or $x * 0 \in [A; k]$. It follows that $0 * x^k = 0 * (x * 0)^k \in A$ so that $x \in [A; k]$. Hence $C_0 = [A; k]$.

Now we shall define a binary operation $*$ on $X/[A; k]$. For any $C_x, C_y \in X/[A; k]$, $C_x * C_y$ is defined as the class containing $x * y$. We can easily check that $(X/[A; k]; *, C_0)$ is a BCI-algebra which is called the *quotient algebra* of X by $[A; k]$.

LEMMA 5 (Jun et al. [4, Proposition 5]). *Let X and Y be BCI-algebras. For any $(x, y) \in X \times Y$, we have*

$$(0, 0) * (x, y)^k = (0 * x^k, 0 * y^k)$$

for every positive integer k.

THEOREM 6. *Let A and B be subsets of BCI-algebras X and Y, respectively and k a positive integer. Then*

(i) $[A; k] \times [B; k] = [A \times B; k]$.

(ii) *if A and B are closed ideals of X and Y respectively, then*

$$X/[A; k] \times Y/[B; k] \cong X \times Y/[A \times B; k].$$

PROOF. (i) We have that

$$\begin{aligned}[A \times B; k] &= \{(x, y) \in X \times Y | (0,0) * (x, y)^k \in A \times B\} \\ &= \{(x, y) \in X \times Y | (0 * x^k, 0 * y^k) \in A \times B\} \\ &= \{(x, y) \in X \times Y | 0 * x^k \in A, 0 * y^k \in B\} \\ &= \{x \in X | 0 * x^k \in A\} \times \{y \in Y | 0 * y^k \in B\} \\ &= [A; k] \times [B; k],\end{aligned}$$

proving (i).

(ii) We note that $[A; k] \times [B; k]$ is an ideal of $X \times Y$ whenever A and B are ideals of X and Y, respectively. Consider the natural homomorphisms

$$\pi_X : X \to X/[A; k], \quad x \mapsto C_x,$$

$$\pi_Y : Y \to Y/[B; k], \quad y \mapsto C_y.$$

Define a mapping $f : X \times Y \to X/[A; k] \times Y/[B; k]$ by $f(x, y) = (C_x, C_y)$ for every $(x, y) \in X \times Y$. Then clearly f is well-defined onto homomorphism. Moreover

$$\begin{aligned}Kerf &= \{(x, y) \in X \times Y | f(x, y) = ([A; k], [B; k])\} \\ &= \{(x, y) \in X \times Y | (C_x, C_y) = ([A; k], [B; k])\} \\ &= \{(x, y) \in X \times Y | C_x = [A; k], C_y = [B; k]\} \\ &= \{(x, y) \in X \times Y | 0 * x^k \in A, 0 * y^k \in B\} \\ &= \{x \in X | 0 * x^k \in A\} \times \{y \in Y | 0 * y^k \in B\} \\ &= [A; k] \times [B; k].\end{aligned}$$

By the first isomorphism theorem, we have

$$X \times Y/[A \times B; k] \cong X/[A; k] \times Y/[B; k].$$

This completes the proof. □

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