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ON CHARACTERIZATIONS OF RIGHT(LEFT) SEMIREGULAR *po*–SEMIGROUPS

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Abstract. In this paper, we give the characterizations of right(left) semiregular *po*–semigroups using the notions of some type ideals.

1. Introduction

Recently S. K. Lee and Y. I. Kwon ([1]) introduced the right(left) semiregularity in a *po*–semigroup. In this paper, we give the characterizations of right(left) semiregular ordered semigroups.

Kehayopulu considered the ordered semigroups. A *po*–semigroup(: ordered semigroup) is an ordered set $(S, \leq)$ at the same time a semigroup such that

$$a \leq b \Longrightarrow ca \leq cb \text{ and } ac \leq bc$$

for all $c \in S$.

The following definitions are well known from references.

Definition 1. Let A be a non-empty subset of a *po*–semigroup S. A is called a *left*(resp. *right*) *ideal* of S if

1) $SA \subseteq A$(resp. $AS \subseteq A$).
2) $a \in A$ and $b \leq a$ for $b \in S \Longrightarrow b \in A$.

A is called an *ideal* of S if A is both a left and a right ideal of S.

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DEFINITION 2 ([2], [3]). Let B be a non-empty subset of a *po*−semigroup S. B is called a *bi-ideal* of S if

1) $BSB \subseteq B$.
2) $a \in B$ and $b \leq a$ for $b \in S \Longrightarrow b \in B$.

Every ideal and right(left) ideal is a bi-ideal.

DEFINITION 3 ([1]). An element a of a *po*−semigroup S is a *right* (resp. *left*) *semiregular element* if $a \leq axay$(resp. $a \leq x'ay'a$) for some $x, y, x', y' \in S$.

A subsemigroup T of S is *right*(resp. *left*) *semiregular* if all elements of T are right(left) semiregular (cf. also [2]).

NOTATION. For $H \subseteq S$, $(H] = \{t \in S : t \leq h$ for some $h \in H\}$.

We denote by $L(a)$(resp. $R(a), B(a)$) the left(resp. right, bi-) ideal and $I(a)$ the ideal of S generated by a.

One can easily prove that

$$L(a) = (a \cup Sa], \quad R(a) = (a \cup aS],$$

$$B(a) = (a \cup aSa] \text{ and } I(a) = (a \cup aS \cup Sa]$$

(cf. also [7]).

2. Main Results

LEMMA. *Let S be a po−semigroup. We have the following:*

1) *$A \subseteq (A]$ for any $A \subseteq S$.*
2) *If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.*
3) *If A is some types of ideal, then $A = (A]$.*
4) *$(A](B] \subseteq (AB]$ for all A and $B \subseteq S$.*
5) *For $A, B \subseteq S, (A \cap B] \neq (A] \cap (B]$, in general. In particular, if A and B are some types of ideals of S, then $(A \cap B] = (A] \cap (B]$.*

PROOF. For 1) - 4), we refer to [6, 7]. If A and B are some types of ideals of S, then

$$(A \cap B] \subseteq (A] \cap (B] = A \cap B \subseteq (A \cap B].$$

We prove the rest of condition 5), by the following example. □

EXAMPLE. We consider the ordered semigroup of the Example 2 in [8], defined by the following multiplication " $\cdot$ " and with the order

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(f,f),(a,b),(d,b),(d,c),(f,c)\}$$

·	a	b	c	d	f
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
f	d	d	c	d	c

For an easy way to check that this is an ordered semigroup, we refer to [7]. We consider the sets $A := \{b,c\}, B := \{b,f\}$. Then

$$(A] \cap (B] = (b,c] \cap (b,f] = \{a,b,c,d,f\} \cap \{a,b,d,f\} = \{a,b,d,f\},$$

but

$$(A \cap B] = (b] = \{a,b,d\}.$$

THEOREM 1. *A po–semigroup S is right semiregular if and only if for every bi-ideal B and every ideal I we have*

(1) $$B \cap I \subseteq (BI].$$

PROOF. If S is right semiregular, then for any $a \in B \cap I$ there exist $x, y \in S$ such that

$$a \leq axay \leq (axay)xay$$
$$\in (BSB)(SIS) \subseteq BI.$$

Thus $a \in (BI]$.

Conversely, suppose that

$$B \cap I \subseteq (BI]$$

for every bi-ideal B and every ideal I of S. Let $a \in S$. We consider the bi-ideal $B(a)$ of S generated by a and the ideal $I(a)$ of S generated by a. From Lemma and (1), we have

$$\begin{aligned} a \in B(a) \cap I(a) &\subseteq (B(a)I(a)] \\ &\subseteq ((a \cup aSa](a \cup aS \cup Sa \cup SaS]] \\ &\subseteq ((a^2 \cup a^2S \cup aSa \cup aSaS]. \end{aligned}$$

Then $a \leq t$ for some $t \in a^2 \cup a^2S \cup aSa \cup aSaS$.

If $t = a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aaaa$. Thus $a \in (aSaS]$.

If $t = a^2x \in a^2S$ for some $x \in S$, then $a \leq a^2x \leq a^2xax$. Thus $a \in (aSaS]$.

If $t = axa \in aSa$ for some $x \in S$, then $a \leq axa \leq axaxa$. Thus $a \in (aSaS]$.

If $t = axay \in aSaS$ for some $x, y \in S$, then $a \in (aSaS]$.

In any case, we have $a \in (aSaS]$. Hence S is right semiregular. $\square$

Using the method of the proof of Theorem 1, we can get the following theorem.

THEOREM 2. *A po–semigroup S is right semiregular if and only if for every right ideals R_1 and R_2, we have*

$$R_1 \cap R_2 \subseteq (R_1R_2]. \tag{2}$$

PROOF. Assume that S is right semiregular. Then for any $a \in R_1 \cap R_2$ there exist $x, y \in S$ such that

$$a \leq axay \in R_1SR_2S \subseteq R_1R_2.$$

Thus $R_1 \cap R_2 \subseteq (R_1R_2]$.

Conversely, suppose that

$$R_1 \cap R_2 \subseteq (R_1R_2]$$

for all right ideals R_1, R_2 of S. We consider the right ideal $R(a)$ of S generated by a. From Lemma and (2), we get

$$\begin{aligned} a \in R(a) \cap R(a) &\subseteq (R(a)R(a)] \\ &\subseteq ((a \cup aS](a \cup aS]] \\ &\subseteq (a^2 \cup a^2S \cup aSa \cup aSaS]. \end{aligned}$$

Hence, as the proof of Theorem 1, S is right semiregular. □

Using the method of the proof of Theorem 1 and Theorem 2, we can get the following theorem.

THEOREM 3. *Let S be a po−semigroup. The following are equivalent:*
1) *S is left semiregular.*
2) *For every ideal I and every bi-ideal B, $B \cap I \subseteq (IB]$.*
3) *For every left ideals L_1 and L_2, $L_1 \cap L_2 \subseteq (L_1L_2]$.*

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