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SOBOLEV ORTHOGONAL POLYNOMIALS RELATIVE TO $\lambda p(c)q(c) + \langle \tau, p'(x)q'(x) \rangle$

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ABSTRACT. Consider a Sobolev inner product on the space of polynomials such as

$$\phi(p,q) = \lambda p(c)q(c) + \langle \tau, p'(x)q'(x) \rangle,$$

where τ is a moment functional and c and λ are real constants. We investigate properties of orthogonal polynomials relative to $\phi(\cdot,\cdot)$ and give necessary and sufficient conditions under which such Sobolev orthogonal polynomials satisfy a spectral type differential equation with polynomial coefficients.

1. Introduction

We consider a linear differential equation of order $N \geq 1$ of spectral type

$$L_N[y](x) = \sum_{i=1}^{N} \ell_i(x) y^{(i)}(x) = \lambda_n y(x), \tag{1.1}$$

where each $\ell_i(x) = \sum_{j=0}^{i} \ell_{ij} x^j$ is a polynomial of degree $\leq i$, independent of n, $\ell_N(x) \neq 0$, and λ_n is the eigenvalue parameter given by

$$\lambda_n = \ell_{11} n + \ell_{22} n(n-1) + \cdots + \ell_{NN} n(n-1)\cdots(n-N+1).$$

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Recently, many attempts have been made to find differential equations of the form (1.1) satisfied by polynomial sequences which are orthogonal relative to a symmetric bilinear form (so called a Sobolev inner product)

$$\phi(p,q) = \langle \sigma, p(x)q(x)\rangle + \langle \tau, p'(x)q'(x)\rangle, \tag{1.2}$$

where σ and τ are moment functionals.

Koekoek and Meijer [7] introduced the generalized Sobolev-Laguerre polynomials $\{L_n^{\alpha,M,N}(x)\}_{n=0}^{\infty}$, which are orthogonal relative to the Sobolev inner product

$$\phi(p,q) = \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} e^{-x}x^{\alpha}p(x)q(x)\,dx + Mp(0)q(0) + Np'(0)q'(0),$$

where $\alpha > -1$, $M \geq 0$ and $N \geq 0$. Later, J. Koekoek, R. Koekoek, and H. Bavinck [5] showed that when α is a non-negative integer, $\{L_n^{\alpha,M,N}(x)\}_{n=0}^{\infty}$ satisfy a differential equation of order $\leq 4\alpha + 10$ of the form (1.1)(see also [6]).

Jung, Kwon, Lee, and Littlejohn [4] found necessary and sufficient conditions for the differential equation (1.1) to have a sequence of polynomials orthogonal relative to $\phi(\cdot,\cdot)$ in (1.2) as solutions. When $\tau = 0$, this result is reduced to the result obtained by H. L. Krall [9] (see also [12]). In particular, Kwon and Littlejohn [11] classified all polynomial sequences, which are orthogonal relative to $\phi(\cdot,\cdot)$ in (1.2) and satisfy second order differential equations.

Motivated by several non-standard examples of Sobolev orthogonal polynomials found in [11], we will consider the following symmetric bilinear form

$$\phi(p,q) = \lambda p(c)q(c) + \langle \tau, p'(x)q'(x)\rangle, \tag{1.3}$$

where τ is a moment functional and c and λ are real constants. In [11], the case when τ is a classical moment functional is handled. An inner product similar to (1.3) was considered by Cohen [2].

We first give some properties of Sobolev orthogonal polynomials relative to $\phi(\cdot,\cdot)$ in (1.3) and find necessary and sufficient conditions for the differential equation (1.1) to have such Sobolev orthogonal polynomials as solutions. Finally, we give an example of a spectral type differential equation of order $N = 4$, which has such Sobolev orthogonal polynomials as solutions.

2. Sobolev orthogonal polynomials

In this work, all polynomials are assumed to be real polynomials in one variable. The space of all such polynomials is denoted by $\mathcal{P}$. We call any linear functional on $\mathcal{P}$ a moment functional. By a polynomial system(in short, PS)$\{P_n(x)\}_{n=0}^{\infty}$, we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$ for $n \geq 0$.

For a moment functional σ and a polynomial $\phi(x)$, we define σ' and $\phi\sigma$ by

$$\langle \sigma', \psi(x) \rangle = -\langle \sigma, \psi'(x) \rangle \ (\psi \in \mathcal{P})$$

and

$$\langle \phi\sigma, \psi(x) \rangle = \langle \sigma, \phi(x)\psi(x) \rangle \ (\psi \in \mathcal{P}).$$

Then we have

$$(\phi\sigma)' = \phi'\sigma + \phi\sigma'.$$

We say that a moment functional σ is quasi-definite (respectively, positive-definite) if there is a PS $\{P_n(x)\}_{n=0}^{\infty}$ such that

$$\langle \sigma, P_m(x)P_n(x) \rangle = K_n\delta_{mn}, \quad m,\ n = 0, 1, 2, \cdots ,$$

where K_n are nonzero (respectively, positive) constants. Then we call $\{P_n(x)\}_{n=0}^{\infty}$ an orthogonal polynomial system (in short, OPS) relative to σ.

Similarly, we say that a symmetric bilinear form $\phi(\cdot, \cdot)$ of the form (1.2) is quasi-definite (respectively, positive-definite) if there is a PS $\{Q_n(x)\}_{n=0}^{\infty}$ such that

$$\phi(Q_m, Q_n) = K_n\delta_{mn}, \quad m,\ n = 0, 1, 2, \cdots ,$$

where K_n are nonzero (respectively, positive) constants. Then we call $\{Q_n(x)\}_{n=0}^{\infty}$ a Sobolev orthogonal polynomial system (in short, SOPS) relative to $\phi(\cdot, \cdot)$.

THEOREM 2.1. *Let $\phi(\cdot, \cdot)$ be a symmetric bilinear form as in (1.3). Then $\phi(\cdot, \cdot)$ is quasi-definite (respectively, positive-definite) if and only if $\lambda \neq 0$ and τ is quasi-definite (respectively, $\lambda > 0$ and τ is positive-definite).*

Furthermore if $\phi(\cdot,\cdot)$ is quasi-definite, then the monic SOPS $\{Q_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$ are given by

$$Q_0(x) = 1, \quad Q_n(x) = n\int_c^x P_{n-1}(t)\,dt \quad (n \geq 1), \tag{2.1}$$

where $\{P_n(x)\}_{n=0}^{\infty}$ is the monic OPS relative to τ.

In particular, we have

$$Q_n(c) = 0, \; n \geq 1; \tag{2.2}$$

$$P_n(x) = \frac{1}{n+1}Q'_{n+1}(x), n \geq 0; \tag{2.3}$$

$$\phi(Q_n, Q_n) = \begin{cases} \lambda & \text{if } n = 0 \\ n^2\langle\tau, P_{n-1}^2(x)\rangle & \text{if } n \geq 1. \end{cases} \tag{2.4}$$

PROOF. Assume that $\phi(\cdot,\cdot)$ is quasi-definite and let $\{Q_n(x)\}_{n=0}^{\infty}$ be the monic SOPS relative to $\phi(\cdot,\cdot)$. Then

$$\phi(Q_m, Q_n) = \lambda Q_m(c)Q_n(c) + \langle\tau, Q'_m(x)Q'_n(x)\rangle = K_n\delta_{mn} \quad (m,\; n \geq 0),$$

where K_n are nonzero constants. If we take $m = 0$, then $\phi(Q_0, Q_n) = \lambda Q_n(c)$ for all $n \geq 0$ and hence $\lambda \neq 0$ and $Q_n(c) = 0, n \geq 1$. Thus we obtain

$$\phi(Q_m, Q_n) = \langle\tau, Q'_m(x)Q'_n(x)\rangle = K_n\delta_{mn}, \; (m, n \geq 1).$$

Therefore, $\{Q'_n(x)\}_{n=1}^{\infty}$ is an OPS relative to τ so that τ is quasi-definite and $Q'_n(x) = nP_{n-1}(x)$, $n \geq 1$. In particular, we obtain relations (2.3) and (2.4).

Conversely, assume that τ is quasi-definite and $\lambda \neq 0$. Let $\{P_n(x)\}_{n=0}^{\infty}$ be the monic OPS relative to τ. If we define $\{Q_n(x)\}_{n=0}^{\infty}$ by (2.1), then it is easy to see that $\{Q_n(x)\}_{n=0}^{\infty}$ is the monic SOPS relative to $\phi(\cdot,\cdot)$ and so $\phi(\cdot,\cdot)$ is quasi-definite.

Now if τ is positive-definite and $\lambda > 0$, then $\phi(\cdot,\cdot)$ is quasi-definite and $\phi(Q_n, Q_n) > 0$, $n \geq 0$ by (2.4). Hence, $\phi(\cdot,\cdot)$ is positive-definite.

Conversely, if $\phi(\cdot,\cdot)$ is positive-definite, then $\phi(Q_0, Q_0) = \lambda > 0$ and $\phi(Q_n, Q_n) = n^2\langle\tau, P_{n-1}^2(x)\rangle > 0, n \geq 1$ by (2.4). Hence τ is also positive-definite. □

Classical Hahn-Sonine Theorem([3,14]) says that only OPS's, whose derivatives are also OPS's, are classical OPS's, that is, Jacobi, Bessel, Laguerre, or Hermite polynomials. Theorem 2.1 provides examples of Sobolev OPS's whose derivatives are ordinary OPS's. In the following, we derive a kind of three term recurrence relation and Favard-type theorem.

From now on, we always assume that τ in (1.3) is quasi-definite and $\lambda \neq 0$ so that $\phi(\cdot,\cdot)$ is also quasi-definite. Then, the monic OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to τ satisfies the three term recurrence relation (see [1]):

$$(2.5) \qquad P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x),\ n \geq 0\ (P_{-1}(x) \equiv 0),$$

where c_0 is arbitrary and $c_n \neq 0$, $n \geq 1$ and the Christoffel-Darboux formula (see [1]) :

$$(2.6) \qquad \sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\langle\tau, P_k^2(x)\rangle} = \frac{1}{\langle\tau, P_n^2(x)\rangle} \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x - y}.$$

THEOREM 2.2. *The monic SOPS $\{Q_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$ satisfies the recurrence relation*

(2.7)
$$\int_c^x Q_{n+1}(t)\,dt = -\frac{n+1}{n+2}Q_{n+2}(x) + (x - b_n)Q_{n+1}(x) - \tilde{c}_n Q_n(x), \quad n \geq 0,$$

where $\tilde{c}_0 = 0$ and $\tilde{c}_n = \frac{n+1}{n}c_n (\neq 0)$, $n \geq 1$.

(Favard-type theorem) *Conversely, if $\{Q_n(x)\}_{n=0}^{\infty}$ is a PS such that $Q_0(x) = 1$, $Q_1(x) = x - c$, and $Q_n(x)$ for $n \geq 2$ are defined by the recurrence relation (2.7) with $\tilde{c}_n \neq 0$, $n \geq 1$, then for any $\lambda \neq 0$, $\{Q_n(x)\}_{n=0}^{\infty}$ is the monic SOPS relative to $\phi(\cdot,\cdot)$.*

PROOF. By (2.3), we may rewrite (2.5) as

$$(2.8)\quad \begin{cases} \frac{1}{2}Q_2'(x) = (x-b_0)Q_1'(x), \ n=0 \\ \frac{1}{n+2}Q_{n+2}'(x) = (x-b_n)\frac{1}{n+1}Q_{n+1}'(x) - c_n\frac{1}{n}Q_n'(x), \ n \geq 1. \end{cases}$$

For $n \geq 0$, by integrating both sides of (2.8) from c to x and using $Q_n(c) = 0$ for $n \geq 1$, we obtain (2.7).

Conversely, define $\{Q_n(x)\}_{n=0}^{\infty}$ by (2.7) and $Q_0(x) = 1$, $Q_1(x) = x - c$. Then, by induction, $\{Q_n(x)\}_{n=0}^{\infty}$ is a monic PS and $Q_n(c) = 0$, $n \geq 1$. By differentiating both sides of (2.7) with respect to x, we have

$$\frac{1}{n+2}Q_{n+2}'(x) = (x-b_n)\frac{1}{n+1}Q_{n+1}'(x) + \frac{n}{n+1}\tilde{c}_n\frac{1}{n}Q_n'(x), \ \ n \geq 1.$$

By Favard's theorem, there exists a quasi-definite moment functional τ such that $\{\frac{1}{n}Q_n'(x)\}_{n=1}^{\infty}$ is an OPS relative to τ. Now for any $\lambda \neq 0$, define a symmetric bilinear form $\phi(\cdot,\cdot)$ by (1.3). Then

$$\phi(Q_m, Q_n) = \begin{cases} 0, & m \neq n \\ \lambda \neq 0, & m = n = 0 \\ \langle \tau, (Q_n')^2 \rangle \neq 0, & m = n \geq 1. \end{cases}$$

Thus $\{Q_n(x)\}_{n=0}^{\infty}$ is an SOPS relative to $\phi(\cdot,\cdot)$. □

THEOREM 2.3. *We have*

(i)

$$\sum_{k=0}^{n} \frac{Q_{k+1}'(x)Q_{k+1}'(y)}{\phi(Q_{k+1}, Q_{k+1})} = \frac{n+1}{n+2} \cdot \frac{1}{\phi(Q_{n+1}, Q_{n+1})} \cdot \frac{Q_{n+2}'(x)Q_{n+1}'(y) - Q_{n+2}'(y)Q_{n+1}'(x)}{x-y};$$

(ii)

$$\sum_{k=0}^{n} \frac{(Q_{k+1}'(x))^2}{\phi(Q_{k+1}, Q_{k+1})} = \frac{n+1}{n+2} \cdot \frac{1}{\phi(Q_{n+1}, Q_{n+1})}[Q_{n+2}''(x)Q_{n+1}'(x) - Q_{n+2}'(x)Q_{n+1}''(x)].$$

PROOF. (i) : By (2.3), (2.4) and (2.6), it is obvious.
(ii) : Letting $y \to x$ in (i), we obtain (ii). □

3. Differential equations

In this section, we shall give necessary and sufficient conditions for the differential equation (1.1) to have an SOPS of solutions, which are orthogonal relative to the symmetric bilinear form (1.3).

For a differential operator $L_N[\cdot]$ in (1.1) and a moment functional τ, we let

$$S_{k+1}(\tau) := \sum_{j=0}^{N-k} (-1)^{j+k} \binom{j+k}{k} (\ell_{j+k}\tau)^{(j)} - \ell_k \tau, \; 0 \le k \le N \tag{3.1}$$

and

$$\begin{aligned} R_{k+1}(\tau) := & \sum_{j=0}^{N-k+2} (-1)^{j+k} \binom{j+k-2}{k-2} (\ell_{j+k-2}\tau)^{(j)} - \ell_{k-2}\tau \\ & - 2(\ell_{k-1}\tau)' - (\ell_k \tau)'' \\ & - \sum_{j=0}^{N-k+1} (-1)^{j+k} \binom{j+k-1}{k-1} (\ell_{j+k-1}\tau')^{(j)} \\ & + \ell_{k-1}\tau' + (\ell_k \tau')' = 0, \quad 1 \le k \le N+2, \end{aligned} \tag{3.2}$$

where $\ell_k(x) = 0$ for $k \le 0$ or $k \ge N+1$ and $\binom{n}{k} = 0$ for $k < 0$. We also let $S_{N+2}(\tau) = S_{N+3}(\tau) = R_1(\tau) = 0$.

We now recall the following theorem, which is proved in [4].

THEOREM 3.1. *Assume that the symmetric bilinear form $\phi(\cdot,\cdot)$ as in (1.2) is quasi-definite and let $\{Q_n(x)\}_{n=0}^{\infty}$ be the monic SOPS relative to $\phi(\cdot,\cdot)$. Then, the followings are equivalent.*

(i) *$\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (1.1), that is,*

$$L_N[Q_n](x) = \lambda_n Q_n(x), \;\; n \ge 0.$$

(ii) *σ and τ satisfy $N+3$ functional equations*

$$R_{k+1}(\sigma,\tau) := R_{k+1}(\tau) - S_{k+1}(\sigma) = 0, \quad 0 \le k \le N+2. \tag{3.3}$$

(iii) *σ and τ satisfy $r+1$ functional equations*

$$R_{2k+2}(\sigma,\tau) = 0, \quad 0 \le k \le r := \left[\frac{N+1}{2}\right]. \tag{3.4}$$

Moreover, in this case, $N = 2r$ must be even.

From now on, we assume that $\phi(\cdot,\cdot)$ in (1.3) is quasi-definite and $\{Q_n(x)\}_{n=0}^{\infty}$ is the monic SOPS relative to $\phi(\cdot,\cdot)$. We also let $\{P_n(x)\}_{n=0}^{\infty}$ be the monic OPS relative to τ.

LEMMA 3.2. *If $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (1.1), then $\ell_i(c) = 0$, $1 \le i \le N$.*

PROOF. We recall that $Q_n(c) = 0$ for all $n \ge 1$. For $n = 1$, $L_N[Q_1](x) = \ell_1(x)Q_1'(x) = \lambda_1 Q_1(x)$ and so $\ell_1(c) = 0$. For $n = 2$, $L_N[Q_2](x) = \ell_2(x)Q_2''(x) + \ell_1(x)Q_2'(x) = \lambda_2 Q_2(x)$ and so $\ell_2(c) = 0$. Continuing the same process, we can easily see that $\ell_i(c) = 0$ for $1 \le i \le N$. □

THEOREM 3.3. *For the differential operator $L_N[\cdot]$ in (1.1), the followings are equivalent.*

(i) $L_N[Q_n](x) = \lambda_n Q_n(x)$, $n \ge 0$.

(ii) *$\ell_i(c) = 0$, $1 \le i \le N$ and τ satisfies $N+2$ functional equations*

$$R_{k+1}(\tau) = 0, \quad 1 \le k \le N+2.$$

(iii) *$\ell_i(c) = 0$, $1 \le i \le N$ and τ satisfies $r+1$ functional equations*

$$R_{2k+2}(\tau) = 0, \quad 0 \le k \le r := \left[\frac{N+1}{2}\right].$$

Moreover, in this case, $N = 2r$ must be even.

PROOF. If $\ell_i(c) = 0, 1 \le i \le N$, then for $\sigma = \lambda\delta(x-c)$,

$$\ell_i(x)\sigma = \lambda\ell_i(x)\delta(x-c) = \lambda\ell_i(c)\delta(x-c) = 0,\ 1 \le i \le N$$

so that $S_{k+1}(\sigma) = 0, 0 \le k \le N$. Therefore, Theorem 3.3 comes immediately from Theorem 3.1, where $\sigma = \lambda\delta(x-c)$. □

PROPOSITION 3.4. *If $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (1.1), then $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation*

$$M_N[y](x) = \sum_{i=1}^{N} m_i(x)y^{(i)}(x) = \mu_n y(x), \tag{3.5}$$

where $\mu_n = \lambda_{n+1} - \ell_1'(x) = \lambda_{n+1} - \ell_{11}$ and

$$m_i(x) = \begin{cases} \ell_N, & i = N \\ \ell_i(x) + \ell_{i+1}'(x), & 1 \le i \le N-1. \end{cases} \tag{3.6}$$

Moreover, τ satisfies

$$\begin{aligned}\widetilde{S}_{k+1}(\tau) :=& \sum_{j=0}^{N-k}(-1)^{j+k}\binom{j+k}{k}(m_{j+k}\tau)^{(j)} - m_k\tau \\ =&0,\ 0 \le k \le N\ (m_0(x) \equiv 0)\end{aligned} \tag{3.7}$$

or equivalently

$$\begin{aligned}W_{k+1}(\tau) :=& \sum_{i=2k+1}^{N}(-1)^i\binom{i-k-1}{k}(m_i\tau)^{(i-2k-1)} \\ =&0, \quad 0 \le k \le r-1,\end{aligned} \tag{3.8}$$

where $r := [\frac{N+1}{2}]$.

PROOF. Assume that $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (1.1), i.e.,

$$L_N[Q_n](x) = \lambda_n Q_n(x),\ n \geq 0. \tag{3.9}$$

Differentiating both sides of (3.9) and then using $Q_n'(x) = nP_{n-1}(x)$, we obtain

$$\lambda_{n+1}P_n(x) = \sum_{i=0}^{N-1} \ell_{i+1}'(x)P_n^{(i)}(x) + \sum_{i=1}^{N} \ell_i(x)P_n^{(i)}(x),\ n \geq 0.$$

Thus (3.5) holds. It is then well known that τ satisfies (3.7) or equivalently (3.8) (see, for example, [12, Theorem 2.4] and [13, Theorem 5.3]). □

Now we shall obtain the relation between $R_{k+1}(\tau)$ and $\widetilde{S}_{k+1}(\tau)$.

PROPOSITION 3.5. *We have*

$$R_{k+1}(\tau) = \widetilde{S}_{k-1}(\tau) + \widetilde{S}_k'(\tau),\ 1 \leq k \leq N+1\ (\widetilde{S}_0(\tau) \equiv 0). \tag{3.10}$$

PROOF. For $1 \leq k \leq N+1$,

$$\begin{aligned}
&R_{k+1}(\tau) \\
&= \sum_{j=0}^{N-k+2} (-1)^{j+k}\binom{j+k-2}{k-2}(\ell_{j+k-2}\tau)^{(j)} - \ell_{k-2}\tau - 2(\ell_{k-1}\tau)' - (\ell_k\tau)'' \\
&\quad - \sum_{j=0}^{N-k+1} (-1)^{j+k}\binom{j+k-1}{k-1}((\ell_{j+k-1}\tau)' - \ell_{j+k-1}'\tau)^{(j)} + \ell_{k-1}\tau' + (\ell_k\tau')' \\
&= \sum_{j=0}^{N-k+2} (-1)^{j+k}\binom{j+k-2}{k-2}(\ell_{j+k-2}\tau)^{(j)} - \sum_{j=0}^{N-k+1} (-1)^{j+k}\binom{j+k-1}{k-1}(\ell_{j+k-1}\tau)^{(j+1)} \\
&\quad + \sum_{j=0}^{N-k+1} (-1)^{j+k}\binom{j+k-1}{k-1}(\ell_{j+k-1}'\tau)^{(j)} - (\ell_{k-2} + \ell_{k-1}')\tau - ((\ell_{k-1} + \ell_k')\tau)' \\
&= \sum_{j=0}^{N-k+2} (-1)^{j+k}\binom{j+k-2}{k-2}(\ell_{j+k-2}\tau)^{(j)} + \sum_{j=0}^{N-k+2} (-1)^{j+k}\binom{j+k-2}{k-1}(\ell_{j+k-2}\tau)^{(j)} \\
&\quad + \sum_{j=0}^{N-k+2} (-1)^{j+k}\left(\binom{j+k-2}{k-2} + \binom{j+k-2}{k-1}\right)(\ell_{j+k-1}'\tau)^{(j)} \\
&\quad - (\ell_{k-2} + \ell_{k-1}')\tau - ((\ell_{k-1} + \ell_k')\tau)'
\end{aligned}$$

$$= \sum_{j=0}^{N-k+2} (-1)^{j+k} \binom{j+k-2}{k-2} ((\ell_{j+k-2} + \ell'_{j+k-1})\tau)^{(j)} - (\ell_{k-2} + \ell'_{k-1})\tau$$

$$+ \sum_{j=0}^{N-k+2} (-1)^{j+k} \binom{j+k-2}{k-1} ((\ell_{j+k-2} + \ell'_{j+k-1})\tau)^{(j)} - ((\ell_{k-1} + \ell'_{k})\tau)'$$

$$= \sum_{j=0}^{N-k+2} (-1)^{j+k} \binom{j+k-2}{k-2} (m_{j+k-2}\tau)^{(j)} - m_{k-2}\tau$$

$$+ \left(\sum_{j=0}^{N-k+1} (-1)^{j+k-1} \binom{j+k-1}{k-1} (m_{j+k-1}\tau)^{(j)} - m_{k-1}\tau \right)'$$

$$= \widetilde{S}_{k-1}(\tau) + \widetilde{S}'_{k}(\tau).$$

□

Corollary 3.6. *We have* $R_{k+1}(\tau) = 0$, $k = 1, \cdots, N+1$ *if and only if* $\widetilde{S}_{k+1}(\tau) = 0$, $k = 0, 1, \cdots, N$.

Proof. By (3.10), it is clear that the conditions $\widetilde{S}_{k+1}(\tau) = 0$, $0 \le k \le N$ imply $R_{k+1}(\tau) = 0$, $1 \le k \le N+1$. Conversely, assume $R_{k+1}(\tau) = 0$, $1 \le k \le N+1$. For $k = 1$ in (3.10), we get $R_2(\tau) = \widetilde{S}'_1(\tau) = 0$ and so $\widetilde{S}_1(\tau) = 0$.

Assume that $\widetilde{S}_k(\tau) = 0$ up to $k = m$ for some $m \ge 1$. Then for $k = m+1$ in (3.10), we have $R_{m+2}(\tau) = \widetilde{S}_m(\tau) + \widetilde{S}'_{m+1}(\tau) = \widetilde{S}'_{m+1}(\tau) = 0$. Hence $\widetilde{S}_{m+1}(\tau) = 0$. □

Now we can obtain :

Theorem 3.7. *The followings are equivalent.*

(i) $\{Q_n(x)\}_{n=0}^{\infty}$ *satisfy the differential equation (1.1).*

(ii) $\ell_i(c) = 0$, $1 \le i \le N$ *and* τ *satisfies* $N+2$ *functional equations*

$$R_{k+1}(\tau) = 0, \quad 1 \le k \le N+2.$$

(iii) $\ell_i(c) = 0$, $1 \le i \le N$ *and* τ *satisfies* $r+1$ *functional equations*

$$R_{2k+2}(\tau) = 0, \quad 0 \le k \le r := [\frac{N+1}{2}].$$

(iv) $\{P_n(x)\}_{n=0}^{\infty}$ *satisfy the differential equation (3.5) and* $\sum_{j=0}^{N-k}(-1)^j m_{k+j}^{(j)}(c) = 0,\ 1 \leq k \leq N$.

(v) $\sum_{j=0}^{N-k}(-1)^j m_{k+j}^{(j)}(c) = 0,\ 1 \leq k \leq N$ *and* τ *satisfies* $\widetilde{S}_{k+1}(\tau) = 0,\ 0 \leq k \leq N$.

(vi) $\sum_{j=0}^{N-k}(-1)^j m_{k+j}^{(j)}(c) = 0,\ 1 \leq k \leq N$ *and* τ *satisfies* $W_{k+1}(\tau) = 0,\ 0 \leq k \leq r-1\ (r := [\frac{N+1}{2}])$.

Moreover, in this case, $N = 2r$ *must be even.*

PROOF. By Theorem 3.3 and Corollary 3.6, it suffices to show that the statements (i) and (iv) are equivalent.
(i) $\Rightarrow$ (iv) : Let $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (1.1). Then, by Proposition 3.4, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (3.5). If we solve the equations (3.6) for $\ell_i(x)$, then we obtain

$$\ell_i(x) = \sum_{j=0}^{N-i}(-1)^j m_{i+j}^{(j)}(x),\ 1 \leq i \leq N. \tag{3.11}$$

By Lemma 3.2, we then have $\sum_{j=0}^{N-k}(-1)^j m_{k+j}^{(j)}(c) = 0,\ 1 \leq k \leq N$.
(iv) $\Rightarrow$ (i) : Let $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (3.5) and $\sum_{j=0}^{N-k}(-1)^j m_{k+j}^{(j)}(c) = 0,\ 1 \leq k \leq N$. Define $\ell_i(x), 1 \leq i \leq N$, by (3.11). Then $\ell_i(c) = 0, 1 \leq i \leq N$, and (3.6) holds.

Since $Q'_n(x) = nP_{n-1}(x)$ (cf.(2.3)),

$$\mu_{n-1}Q'_n(x) = \mu_{n-1}nP_{n-1}(x) = n\sum_{i=1}^{N} m_i(x)P_{n-1}^{(i)}(x) = \sum_{i=1}^{N}(\ell_i(x) + \ell'_{i+1}(x))Q_n^{(i+1)}(x),$$

that is,

$$\mu_{n-1}Q'_n(x) = \sum_{i=1}^{N}(\ell_i(x) + \ell'_{i+1}(x))Q_n^{(i+1)}(x), \quad n \geq 1\ (\ell_{N+1}(x) \equiv 0).$$

Integrating both sides of the above equation from c to x, we obtain

$$\mu_{n-1}Q_n(x) = \sum_{i=1}^{N} \ell_i(x)Q_n^{(i)}(x) - \ell'_1(x)Q_n(x), \quad n \geq 1$$

since $Q_n(c) = 0, n \geq 1$. That is

$$\sum_{i=1}^{N} \ell_i(x) Q_n^{(i)}(x) = (\mu_{n-1} + \ell_{11}) Q_n(x) = \lambda_n Q_n(x), n \geq 1.$$

Therefore, $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy the differential equation (1.1). □

In particular, if $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy a second order differential equation

$$L_2[y](x) = \ell_2(x) y''(x) + \ell_1(x) y'(x) = \lambda_n y(x),$$

then $\{P_n(x)\}_{n=0}^{\infty}$ must be a classical OPS satisfying

$$\ell_2(x) y''(x) + [\ell_1(x) + \ell_2'(x)] y'(x) = (\lambda_{n+1} - \ell_1'(x)) y(x).$$

4. Example. For $N = 2$, Kwon and Littlejohn [11] found all SOPS's which are orthogonal relative to the Sobolev inner product (1.2) and satisfy second order differential equations of the form (1.1). They include all classical orthogonal polynomials as well as three more SOPS's, which are orthogonal relative to $\phi(\cdot,\cdot)$ in (1.3).

We now consider a fourth order differential equation :

$$M_4[y](x) = \sum_{i=1}^{4} m_i(x) y^{(i)}(x) = \mu_n y(x). \tag{4.1}$$

H. L. Krall [10] classified, up to a linear change of variable, all OPS's that satisfy the differential equation (4.1). They are four classical OPS's and three more classical-type OPS's (see also [8]):

Legendre-type polynomials $\{P_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfying

$$(1-x^2)^2 y^{(4)}(x) + 8x(x^2-1) y^{(3)}(x) + (4\alpha+12)(x^2-1) y''(x) + 8\alpha x y'(x) = \mu_n y(x) \quad (\alpha \neq \frac{-n(n-1)}{2}, n = 0, 1, \cdots)$$

Laguerre-type polynomials $\{R_n(x)\}_{n=0}^{\infty}$ satisfying

$$x^2 y^{(4)}(x) + 2x(2-x) y^{(3)}(x) + [x^2 - (2R+6)x] y''(x) + [(2R+2)x - 2R] y'(x) = \mu_n y(x) \quad (R \neq -1, -2, \cdots)$$

Jacobi-type polynomials $\{S_n^{\alpha,M}(x)\}_{n=0}^{\infty}$ satisfying

$$(x^2-x)^2y^{(4)}(x)+2x(x-1)[(\alpha+4)x-2]y^{(3)}(x)+x[(\alpha^2+9\alpha+14+2M)x \\ -2(3\alpha+6+M)]y''(x)+2[(\alpha+2)(\alpha+1+M)x-M]y'(x)=\lambda_n y(x) \\ (M>0,\alpha\neq -1,-2,\cdots).$$

Among these three differential equations, only the last one satisfies the condition (iv) in Theorem 3.7 with $c=1$ when $\alpha=0$.

Let

$$Q_0(x)=1 \text{ and } Q_n(x)=n\int_1^x S_{n-1}^{0,M}(t)\,dt,\ n\geq 1.$$

Then by Theorem 3.1 and Theorem 3.7, $\{Q_n(x)\}_{n=0}^{\infty}$ is the monic SOPS relative to the Sobolev inner product

$$\lambda p(1)q(1)+\frac{1}{M}p'(0)q'(0)+\int_0^1 p'(x)q'(x)\,dx\ (\lambda\neq 0, M>0).$$

Moreover, $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy

$$(x^2-x)^2y^{(4)}(x)+2x(x-1)(2x-1)y^{(3)}(x)+2[(1+M)x^2-Mx-1]y''(x)=\lambda_n y(x).$$

References

[1] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon Breach, N. Y., 1978.

[2] E. A. Cohen, *Theoretical properties of best polynomial approximation in* $W^{1,2}[-1,1]$, SIAM J. Math. Anal. **2** (1971), 187-192.

[3] W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Math. Z. **39** (1935), 634–638.

[4] I. H. Jung, K. H. Kwon, D. W. Lee, and L. L. Littlejohn, *Sobolev orthogonal polynomials and spectral differential equations*, Trans. Amer. Math. Soc. **347** (1995), 3629-3643.

[5] J. Koekoek, R. Koekoek, and H. Bavinck, *On differential equations for Sobolev-type Laguerre polynomials*, Delft Univ. of Tech., Report No. (1995), 95-79.

[6] R. Koekoek, *The search for differential equation for certain sets of orthogonal polynomials*, J. Comp. Appl. Math. **49** (1993), 111-119.

[7] R. Koekoek and H. G. Meijer, *A generalization of Laguerre polynomials.*, SIAM J. Math. Anal. **24** (1993), 768-782.

[8] A. M. Krall, *Orthogonal polynomials satisfying fourth order differential equations*, Proc. Roy. Soc. Edinburgh Sect. **A 87** (1982), 271-288.

[9] H. L. Krall, *Certain differential equations for Tchebycheff polynomials*, Duke Math. J. **4** (1938), 705-719.

[10] ———, *On orthogonal polynomials satisfying a certain fourth order differential equation*, The Penn. State College Studies No. 6, The Penn. State College, Pa. (1940).

[11] K. H. Kwon and L. L. Littlejohn, *Sobolev orthogonal polynomials and second order differential equations*, Rocky Mt. J. Math., to appear.

[12] K. H. Kwon, L. L. Littlejohn, and B. H. Yoo, *Characterizations of orthogonal polynomials satisfying differential equations*, SIAM J. Math. Anal. **25** (1994), 976-990.

[13] L. L. Littlejohn and D. Race, *Symmetric and symmetrizable ordinary differential expressions*, Proc. London Math. Soc. **3** (1990), 344-364.

[14] N. J. Sonine, *Über die angenäherte Berechnung der bestimmten Integrale und über die dabei vorkommenden ganzen Functionen*, Warsaw Univ. Izv. **18** (1887), 1–76; (In Russian.) Summary in Jbuch. Fortschritte Math. **19**, 282.

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