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BEST APPROXIMATION SETS IN LINEAR 2-NORMED SPACES

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Abstract. In this paper, we give some properties of the sets $D_z(x_o, G), P_{G,z}(x)$. We also provide the relation between $P_{G,z}(x)$ and Gâteaux derivatives.

1. Introduction

The concept of linear 2-normed spaces has been initially investigated by S. Gähler ([7]) and has been developed extensively by Y. J. Cho, C. Diminnie, R. Freese, S. Gähler, A. White and many other ([2], [7], [8], [11]).

Let X be a linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

(N_1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(N_2) $\|x, y\| = \|y, x\|$,

(N_3) $\|\alpha x, y\| = |\alpha|\|x, y\|$, where α is real,

(N_4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

$\|\cdot,\cdot\|$ is called a *2-norm* on X and $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space*. Some of the basic properties of 2-norms are that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for every $x, y \in X$ and every real number α.

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $V(x, y)$ be the subspace of X generated by x and y in X. For all $x, y \in X$ define

$$n(x, y|z)(t) = \frac{\|x + ty, z\| - \|x, z\|}{t}$$

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for any real t and $z \in X \setminus V(x, y)$.

Then the functional $n(x, y|z)(t)$ is non-decreasing of the real positive variable t for any fixed x, y in X and for arbitrary z in X. Moreover, the limit $\lim_{t\to 0^+} n(x, y|z)(t)$ exists ([1]).

Put $N_+(x, z)(y) = \lim_{t\to 0^+} n(x, y|z)(t)$, which is called the *right-hand Gâteaux derivative* of the 2-norm $\|\cdot, \cdot\|$ at (x, z) in the direction y. In [1], Y. J. Cho, S. S. Kim and A. White obtained the following properties of $N_+(\cdot, \cdot)(\cdot)$:

For every x, y in X and $z \in X \setminus V(x, y)$,

(1) $|N_+(x, z)(y)| \leq \|y, z\|$.

(2) $N_+(x, z)(y + y') \leq N_+(x, z)(y) + N_+(x, z)(y')$.

(3) $N_+(\alpha x, z)(\beta y) = \beta N_+(x, z)(y)$ for all reals $\alpha, \beta \geq 0$.

(4) $N_+(x, z)(0) = 0$ and $N_+(0, z)(y) = \|y, z\|$.

(5) $-N_+(x, z)(-y) = \lim_{t\to 0^-} (\|x + ty, z\| - \|x, z\|)/t \leq N_+(x, z)(y)$.

(6) $N_+(x, z)(\alpha x) = \alpha\|x, z\|$ for all real α.

Recently, some results on best approximation theory in linear 2-normed spaces have been obtained by S. Elumalai and R. Ravi ([4], [5]), I. Franić ([6]), S. S. Kim and Y. J. Cho ([9]), S. A. Mariadoss ([13]), R. Ravi ([19]). These papers are based on the reseach works in normed linear spaces made by G. Godini ([10]), T. D. Narang ([14], [15]). P. L. Papini ([16], [17]), P. L. Papini and I. Singer ([18]), I. Singer ([20]), and they have contributed to the study on the geometric structures of linear 2-normed spaces.

For a fixed $z \in X$, the function $p_z(x) = \|x, z\|, x \in X$, is a seminorm on X and the family $P = \{p_z : z \in X\}$ of seminorms generates a locally convex topology on X, which is called the *natural topology* induced by the 2-norm $\|\cdot, \cdot\|$.

In this paper, we give some properties of the sets $D_z(x_o, G)$, $P_{G,z}(x)$. We also provide the relation between $P_{G,z}(x)$ and Gâteaux derivatives.

2. $D_z(x_o, G)$ and $P_{G,z}(x)$

In this section, we define and study two subsets of X, denoted by $D_z(x_o, G)$ and $P_{G,z}(x)$.

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and G be an arbitrary non-empty subset of X and $x_o \in X$. Then for every $x \in X$ and for every $z \in X \setminus G$ which is independent of x and x_o, we have

$$d_z(x, G) \leq \|x - x_o, z\| + d_z(x_o, G), \tag{1}$$

where $d_z(x, G) = \inf_{g \in G} \|x - g, z\|$. For each $G \subset X$ and $x_o \in X$, we define

$$D_z(x_o, G) = \{x \in X | d_z(x, G) = \|x - x_o, z\| + d_z(x_o, G)\} \tag{2}$$

for any $z \in X \setminus G$ which is independent of x and x_o.

Note that $D_z(x_o, G) \neq \phi$ since $x_o \in D_z(x_o, G)$. If $x_o \in \operatorname{Int} G$, the interior of G, then $D_z(x_o, G) = \{x_o\}$.

We denote

$$P_{G,z}(x) = \{g_o \in G : \|x - g_o, z\| = d_z(x, G)\} \tag{3}$$

and

$$P_{G,z}^{-1}(x_o) = \{x \in X : \|x - x_o, z\| = d_z(x, G)\}, \tag{4}$$

where $x_o \in G$.

We now provide some simple properties of $D_z(x_o, G)$.

LEMMA 2.1. *Let $g_n \in G, n = 1, 2, \cdots$, such that*

$$d_z(x_o, G) = \lim_{n \to \infty} \|x_o - g_n, z\|.$$

Then for each $x \in D_z(x_o, G)$ and for any $z \in X \setminus G$ which is independent of x and x_o,

$$d_z(x, G) = \lim_{n \to \infty} \|x - g_n, z\|$$

PROOF. For each $x \in D_z(x_o, G)$,

$$\begin{aligned} d_z(x,G) &= \|x - x_o, z\| + d_z(x_o, G) \\ &= \lim_{n\to\infty} (\|x - x_o, z\| + \|x_o - g_n, z\|) \\ &\geq \limsup_{n\to\infty} \|x - g_n, z\| \\ &\geq d_z(x, G). \end{aligned}$$

Therefore, we have $d_z(x,G) = \lim_{n\to\infty} \|x - g_n, z\|$. This completes the proof. □

LEMMA 2.2. (i) $\|y - x_o, z\| = \|y - x, z\| + \|x - x_o, z\|$.
(ii) $y - x + x_o \in D_z(x_o, G)$.

PROOF. (i) Let $x \in D_z(x_o, G)$ and $y \in D_z(x, G)$. Then, by (1) and (2), we have

$$\begin{aligned} &\|y - x_o, z\| \\ &\leq \|y - x, z\| + \|x - x_o, z\| \\ &= \Big(d_z(y,G) - d_z(x,G)\Big) + \Big(d_z(x,G) - d_z(x_o,G)\Big) \\ &= d_z(y,G) - d_z(x_o,G) \\ &\leq \|y - x_o, z\|. \end{aligned}$$

Therefore, it follows that $\|y - x_o, z\| = \|y - x, z\| + \|x - x_o, z\|$.

(ii) Take x and y as in the proof of (i). Then, from (1), we have

$$\begin{aligned} &d_z(y - x + x_o, G) \\ &\geq d_z(y,G) - \|y - (y - x + x_o), z\| \\ &= d_z(y,G) - \|x - x_o, z\| \\ &= \Big(\|y - x, z\| + d_z(x,G)\Big) - \|x - x_o, z\| \\ &= \|y - x, z\| + \Big(\|x - x_o, z\| + d_z(x_o,G)\Big) - \|x - x_o, z\| \\ &= \|(y - x + x_o) - x_o, z\| + d_z(x_o, G). \end{aligned}$$

Again, by (1), it follows that

$$d_z(y - x + x_o, G) = \|(y - x + x_o) - x_o, z\| + d_z(x_o, G),$$

which proves (ii). This completes the proof. □

LEMMA 2.3. *Let* $x \in D_z(x_o, G)$. *Then*
(i) $[x_o, x] = \{\lambda x_o + (1 - \lambda)x : 0 \leq \lambda \leq 1\} \subset D_z(x_o, G)$.
(ii) $D_z(x, G) \subset D_z(x_o, G)$.

PROOF. (i) Let $y = \lambda x_o + (1 - \lambda)x$ such that $0 \leq \lambda \leq 1$. Then we have

$$\begin{aligned} d_z(y, G) &\geq d_z(x, G) - \|x - y, z\| \\ &= \|x - x_o, z\| + d_z(x_o, G) - \|x - y, z\| \\ &= \|y - x_o, z\| + d_z(x_o, G). \end{aligned}$$

So, by (1), it follows that $d_z(y, G) = \|y - x_o, z\| + d_z(x_o, G)$ implies $y \in D_z(x_o, G)$.

(ii) Let $y \in D_z(x, G)$. Then, by Lemma 2.2 (i),

$$\begin{aligned} d_z(y, G) &= \|y - x, z\| + d_z(x, G) \\ &= \|y - x, z\| + \big(\|x - x_o, z\| + d_z(x_o, G)\big) \\ &= \|y - x_o, z\| + d_z(x_o, G). \end{aligned}$$

Therefore, we have $y \in D_z(x_o, G)$. This completes the proof. □

LEMMA 2.4. *Let* $x_o, y_o \in X$ *and* $\lambda \neq 0$. *Then*
(i) $D_z(x_o, G) + y_o = D_z(x_o + y_o, G + y_o)$.
(ii) $D_z(x_o, \lambda G) = \lambda D_z(x_o/\lambda, G)$.

PROOF. (i) Let $x \in D_z(x_o, G)$. Then we have

$$\begin{aligned} d_z(x + y_o, G + y_o) &= d_z(x, G) \\ &= \|x - x_o, z\| + d_z(x_o, G) \\ &= \|x + y_o - (x_o + y_o), z\| + d_z(x_o + y_o, G + y_o). \end{aligned}$$

Therefore, $x + y_o \in D_z(x_o + y_o, G + y_o)$.

Conversely, let $y \in D_z(x_o + y_o, G + y_o)$. Then we have

$$\begin{aligned} d_z(y - y_o, G) &= d_z(y, G + y_o) \\ &= \|y - y_o - x_o, z\| + d_z(x_o + y_o, G + y_o) \\ &= \|(y - y_o) - x_o, z\| + d_z(x_o, G). \end{aligned}$$

Therefore, it follows that $y - y_o \in D_z(x_o, G)$ and so

$$D_z(x_o, G) + y_o = D_z(x_o + y_o, G + y_o).$$

(ii) Let $x \in D_z(x_o, \lambda G)$. Then we have

$$\begin{aligned} d_z(x/\lambda, G) &= \frac{1}{|\lambda|} d_z(x, \lambda G) \\ &= \frac{1}{|\lambda|}\big(\|x - x_o, z\| + d_z(x_o, \lambda G)\big) \\ &= \Big\|\frac{x}{\lambda} - \frac{x_o}{\lambda}, z\Big\| + d_z(x_o/\lambda, G). \end{aligned}$$

Therefore, $x/\lambda \in D_z(x_o/\lambda, G)$.

Conversely, let $x \in D_z(x_o/\lambda, G)$. Then we have

$$\begin{aligned} d_z(\lambda x, \lambda G) &= |\lambda| d_z(x, G) \\ &= |\lambda|\Big(\Big\|x - \frac{x_o}{\lambda}, z\Big\| + d_z(x_o/\lambda, G)\Big) \\ &= \|\lambda x - x_o, z\| + d_z(x_o, \lambda G). \end{aligned}$$

Therefore, $\lambda x \in D_z(x_o, \lambda G)$.

In particular, if G is a subspace of X, then

$$D_z(\lambda x_o + g_o, G) = \lambda D_z(x_o, G) + g_o$$

for every $g_o \in G$ and $\lambda \neq 0$. This completes the proof.

LEMMA 2.5. *Let $G \subset G_1$ and $x_o \in X$, where G_1 is a subset of X such that*

$$d_z(x_o, G) = d_z(x_o, G_1). \tag{5}$$

Then $D_z(x_o, G_1) \subset D_z(x_o, G)$.

PROOF. Let $x \in D_z(x_o, G_1)$. Then, by (5), we have

$$\begin{aligned} d_z(x, G) &\geq d_z(x, G_1) \\ &= \|x - x_o, z\| + d_z(x_o, G_1) \\ &= \|x - x_o, z\| + d_z(x_o, G). \end{aligned}$$

By (1), it follows that $d_z(x, G) = \|x - x_o, z\| + d_z(x_o, G)$. Therefore, by (2), $x \in D_z(x_o, G)$. This completes the proof. □

THEOREM 2.6. (i) *$P_{G,z}(x_o) \subset P_{G,z}(x)$ for every $x \in D_z(x_o, G)$.*
(ii) *$D_z(x_o, G) = P^{-1}_{G,z}(x_o)$ for every $x_o \in \overline{G}$.*

PROOF. (i) Let $x \in D_z(x_o, G)$ and $g_o \in P_{G,z}(x_o)$. Then, by Lemma 2.2 (i),

$$\begin{aligned} d_z(x, G) &= \|x - x_o, z\| + d_z(x_o, G) \\ &= \|x - x_o, z\| + \|x_o - g_o, z\| \\ &= \|x - g_o, z\|, \end{aligned}$$

which proves that $g_o \in P_{G,z}(x)$.

(ii) Let $x_o \in \overline{G}$ and let $x \in D_z(x_o, G)$. Then we have

$$d_z(x, G) = \|x - x_o, z\| + d_z(x_o, G) = \|x - x_o, z\|,$$

where $x_o \in \overline{G}$. This shows that $x_o \in P_{G,z}(x)$, i.e., $x \in P^{-1}_{G,z}(x_o)$ and so $D_z(x_o, G) \subset P^{-1}_{G,z}(x_o)$.

Conversely, let $x \in P^{-1}_{G,z}(x_o)$. Then $x_o \in P_{G,z}(x)$. Since $x_o \in \overline{G}, d_z(x_o, G) = 0$. Hence, we have $d_z(x, G) = \|x - x_o, z\| + d_z(x_o, G)$ implies $x \in D_z(x_o, G)$. Therefore, it follows that $D_z(x_o, G) = P^{-1}_{G,z}(x_o)$ for $x_o \in \overline{G}$. This completes the proof. □

For a subset G of a linear 2-normed space X and for each $b \geq 0$, the b-extension of G denoted by G_b and defined by

$$G_b = \{x \in X : d_z(x, G) \leq b, \quad z \in X\}.$$

If $b = 0$, then $d_z(x, G) = \inf_{g \in G} \|x - g, z\| = 0$. This shows that $x \in \overline{G}$ and for each $x \in \overline{G}$, the best approximation of x is itself. Hence, $G_b = \overline{G}$.

THEOREM 2.7. *Let $G \subset X, x_o \in X$ and $b \leq d_z(x_o, G)$. Then*

$$D_z(x_o, G) = D_z(x_o, G_b).$$

PROOF. From [3], for each $x \in X$ with $d_z(x, G) \geq b$, we have

$$d_z(x, G) = d_z(x, G_b) + b. \tag{6}$$

Let $y \in D_z(x_o, G)$. Then

$$d_z(y, G) = \|y - x_o, z\| + d_z(x_o, G) \geq b,$$

since $d_z(x_o, G) \geq b$. So, by (6), we have

$$d_z(y, G) = d_z(y, G_b) + b, \qquad d_z(x_o, G) = d_z(x_o, G_b) + b.$$

Hence, we have

$$\begin{aligned} d_z(y, G_b) &= d_z(y, G) - b \\ &= \|y - x_o, z\| + d_z(x_o, G) - b \\ &= \|y - x_o, z\| + d_z(x_o, G_b). \end{aligned}$$

Therefore, it follows that $y \in D_z(x_o, G_b)$ and so $D_z(x_o, G) \subset D_z(x_o, G_b)$.

Conversely, let $x \in D_z(x_o, G_b)$ such that $x \neq x_o$. Then $x \notin G_b$ and so $d_z(x, G) > b$. By (6) and by $x \in D_z(x_o, G_b)$, we have

$$d_z(x, G_b) = \|x - x_o, z\| + d_z(x_o, G_b) = \|x - x_o, z\| + d_z(x_o, G) - b.$$

Thus, it follows that

$$\begin{aligned} d_z(x, G) &= d_z(x, G_b) + b \\ &= \|x - x_o, z\| + d_z(x_o, G_b) + b \\ &= \|x - x_o, z\| + d_z(x_o, G). \end{aligned}$$

which implies that $x \in D_z(x_o, G)$ and so $D_z(x_o, G_b) \subset D_z(x_o, G)$. This completes the proof. $\square$

THEOREM 2.8 ([19]). *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space. Let G be a linear subspace of X, $x \in X \setminus \overline{G}$ and $z \in X \setminus [x, G]$, where $[x, G]$ is the space generated by x and G. Then for $g_o \in G, g_o \in P_{G,z}(x)$ if and only if $g_o \in P_{G,z}(\alpha x + (1-\alpha)g_o)$ for all $\alpha \in R$.*

By using Theorem 2.8, we have following:

THEOREM 2.9. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and let G be a linear subspace of X, $x \in X \setminus \overline{G}$ and $z \in X \setminus [x, G]$. Then the following statements are equivalent:*

(i) *For $g_o \in \overline{G}, g_o \in P_{G,z}(x)$.*
(ii) *$N_+(x - g_o, z)(g_o - g) \geq 0$ for all $g \in G$.*

PROOF. (i) implies (ii): Assume that (i) holds. Then by theorem 2.8, we have $g_o \in P_{G,z}(\alpha x + (1-\alpha)g_o)$ for $\alpha \geq 1$. Thus, for all $g \in G$,

$$\|\alpha x + (1-\alpha)g_o - g_o, z\| \leq \|\alpha x + (1-\alpha)g_o - g, z\|,$$

which implies that

$$\|x - g_o, z\| \leq \|x - g_o + \frac{1}{\alpha}(g_o - g), z\|.$$

Taking $t = 1/\alpha$, we have $\|x - g_o + t(g_o - g), z\| \geq \|x - g_o, z\|$. Therefore, we have

$$\begin{aligned} N_+(x - g_o, z)(g_o - g) &= \lim_{t \to 0^+} \frac{\|x - g_o + t(g_o - g), z\| - \|x - g_o, z\|}{t} \\ &\geq 0. \end{aligned}$$

(ii) implies (i): Assume that (ii) holds. Then, since

$$\frac{\|x - g_o + t(g_o - g), z\| - \|x - g_o, z\|}{t}$$

is non-decreasing function of the real positive variable t, for any fixed $x \in X \setminus \overline{G}, z \in X \setminus [x, G]$ and $g \in G$, we have

$$\|x - g_o + t(g_o - g), z\| \geq \|x - g_o, z\|$$

for $t > 0$. So, for $t = 1$, it follows that $\|x - g, z\| \geq \|x - g_o, z\|$ for every $g \in G$. Therefore, we have $g_o \in P_{G,z}(x)$. This completes the proof. □

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