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ISOMORPHISMS OF CERTAIN $(4k-1)$-DIAGONAL ALGEBRAS $Alg\mathcal{L}_{2n}^{(4k-1)}$ AND $Alg\mathcal{L}_{2n+1}^{(4k-1)}$

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ABSTRACT. In this paper, we introduce $(4k-1)$-diagonal algebras $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ and investigate necessary and sufficient conditions that isomorphisms of $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ are spatially implemented.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, then $Alg\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and I. If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is a commutative if each pair of projections in $\mathcal{L}$ commutes. If $\mathcal{L}$ is a commutative subspace lattice, or CSL, then $Alg\mathcal{L}$ is called a CSL-algebra. An algebra $\mathcal{A}$ is $(4k-1)$-diagonal if there exists a countable partition $\{E_i\}$ of $\mathcal{H}$ so that every $A \in \mathcal{A}$ is block $(4k-1)$-diagonal with respect to the sequence $E_1, E_2, \cdots$. That is , we require

$$AE_i \subset \sum_{j=1}^{i+2k-1} \bigoplus E_j$$

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for $i = 1, 2, \cdots, 2k-1$ and

$$AE_i \subset \sum_{j=i-2k+1}^{i+2k-1} \bigoplus E_j$$

for all positive integer $i \geq 2k$ and all $A \in \mathcal{A}$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. An isomorphism $\varphi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is said to be spatially implemented if there is a bounded invertible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_1$. If $x_1, x_2, \cdots, x_m \in \mathcal{H}$, we denote by $[x_1, x_2, \cdots, x_m]$ the closed subspace spanned by the vectors $x_1, x_2, \cdots, x_m$. Let i and j be two nonzero natural numbers. Then E_{ij} is the matrix whose (i,j)-component is 1 and all other entries are zero. An $n \times n$ matrix D_n is said to be the backward identity matrix if the $(i, n-i+1)$-component is 1 for all i and all other entries are zero.

Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ and let k be a natural number such that $2k \leq n$. Then we denote by $\mathcal{L}_{2n}^{(4k-1)}$ the subspace lattice of orthogonal projections generated by the subspaces $[e_1]$, $[e_3], \cdots, [e_{2n-1}]$, $[e_2, e_1, e_3, \cdots, e_{2k+1}]$, $[e_4, e_1, e_3, \cdots, e_{2k+3}]$, $\cdots$, $[e_{2k}, e_1, e_3, \cdots, e_{4k-1}]$, $[e_{2k+2}, e_3, e_5, \cdots, e_{4k+1}], \cdots$, $[e_{2n-2k}, e_{2n-4k+1}, e_{2n-4k+3}, \cdots, e_{2n-1}], \cdots$, $[e_{2n-2}, e_{2n-2k-1}, e_{2n-2k+1}, \cdots, e_{2n-1}]$, $[e_{2n}, e_{2n-2k+1}, e_{2n-2k+3}, \cdots, e_{2n-1}]$.

Let $\mathcal{H}$ be a $(2n+1)$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n+1}\}$ and let k be a natural number such that $2k \leq n$. Then we denote by $\mathcal{L}_{2n+1}^{(4k-1)}$ the subspace lattice of orthogonal projections generated by the subspaces $[e_1], [e_3], \cdots, [e_{2n+1}]$, $[e_2, e_1, e_3, \cdots, e_{2k+1}]$, $[e_4, e_1, e_3, \cdots, e_{2k+3}]$, $\cdots$, $[e_{2k}, e_1, e_3, \cdots, e_{4k-1}]$, $[e_{2k+2}, e_3, e_5, \cdots, e_{4k+1}], \cdots$, $[e_{2n-2k+2}, e_{2n-4k+3}, e_{2n-4k+5}, \cdots, e_{2n+1}]$, $\cdots$, $[e_{2n-2}, e_{2n-2k-1}, e_{2n-2k+1}, \cdots, e_{2n+1}]$, $[e_{2n}, e_{2n-2k+1}, e_{2n-2k+3}, \cdots, e_{2n+1}]$. Then $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ are $(4k-1)$-diagonal algebras. Since the lattices $\mathcal{L}_{2n}^{(4k-1)}$ and $\mathcal{L}_{2n+1}^{(4k-1)}$ are CSL, they are reflexive and $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ are reflexive CSL-algebras. If $k = 1$, then the algebras $Alg\mathcal{L}_{2n}^{(3)}$ and $Alg\mathcal{L}_{2n+1}^{(3)}$ are tridiagonal, and isomorphisms of these algebras are spatially implemented [5]. If $k \geq 2$, then isomorphisms of $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ need not be spatially implemented.

In this paper we will investigate necessary and sufficient conditions that isomorphisms of $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ are spatially implemented.

2. Examples of Isomorphisms

EXAMPLE 2.1. Let $\mathcal{H}$ be a 10-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{10}\}$ and let $\mathcal{L}_{10}^{(7)}$ be the subspace lattice of orthogonal projections generated by the subspaces $[e_1]$, $[e_3]$, $[e_5]$, $[e_7]$, $[e_9]$, $[e_1, e_2, e_3, e_5]$, $[e_1, e_3, e_4, e_5, e_7]$, $[e_3, e_5, e_6, e_7, e_9]$, $[e_5, e_7, e_8, e_9]$ $[e_7, e_9, e_{10}]$. Then $Alg\mathcal{L}_{10}^{(7)}$ is a 7-diagonal algebra and $A \in Alg\mathcal{L}_{10}^{(7)}$ has the matrix form

$$\begin{pmatrix}
* & * & & * & & & & & & \\
& * & & & & & & & & \\
& * & * & * & & * & & & & \\
& & & * & & & & & & \\
& * & & * & * & * & & * & & \\
& & & & & * & & & & \\
& & & * & & * & * & * & & * \\
& & & & & & & * & & \\
& & & & & * & & * & * & * \\
& & & & & & & & & *
\end{pmatrix},$$

where all non-starred entries are zero. Let $\rho : Alg\mathcal{L}_{10}^{(7)} \to Alg\mathcal{L}_{10}^{(7)}$ be a linear map defined by $\rho(E_{ij}) = E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_{10}^{(7)}$ except E_{14} and $\rho(E_{14}) = 2E_{14}$. Then ρ is an isomorphism. It is easy to check that ρ is not spatially implemented.

EXAMPLE 2.2. Let γ_{ij} be nonzero complex numbers for all i, j ($i \neq j$) with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$(resp., $Alg\mathcal{L}_{2n+1}^{(4k-1)}$). Let $\rho : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ ($Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$) be a linear map defined by $\rho(E_{ii}) = E_{ii}$ for all i and $\rho(E_{ij}) = \gamma_{ij}E_{ij}$ for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$($Alg\mathcal{L}_{2n+1}^{(4k-1)}$). Then ρ is an isomorphism.

EXAMPLE 2.3. Let γ_{ij} be as in Example 2.2, and let α_{ij} be complex numbers for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$(resp., $Alg\mathcal{L}_{2n+1}^{(4k-1)}$).

Let $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}(Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)})$ be a linear map defined by

$\varphi(E_{2p-1,2p-1}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j} E_{2p-1,2j}$

$\varphi(E_{2q,2q}) = E_{2q,2q} - \sum_i \alpha_{2i-1,2q} E_{2i-1,2q}$

$\varphi(E_{2p-1,2q}) = \gamma_{2p-1,2q} E_{2p-1,2q}$

for all $E_{2p-1,2p-1}$, $E_{2q,2q}$ and $E_{2p-1,2q}$ in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Then φ is an isomorphism. In particular, if $\alpha_{ij} = 0$ for all i and j, then φ is precisely the ρ in Example 2.2. Let S be a matrix in $Alg\mathcal{L}_{2n}^{(4k-1)}(Alg\mathcal{L}_{2n+1}^{(4k-1)})$ whose (i,i)-component is 1 for all i and (i,j)-component is $-\alpha_{ij}$ for all $i,j(i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}(Alg\mathcal{L}_{2n+1}^{(4k-1)})$. Then $\varphi(A) = S\rho(A)S^{-1}$ for all A in $Alg\mathcal{L}_{2n}^{(4k-1)}(Alg\mathcal{L}_{2n+1}^{(4k-1)})$, where ρ is the isomorphism in Example 2.2.

EXAMPLE 2.4. Let α_{ij} and γ_{ij} be as in Example 2.3.
Let $\varphi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ be a linear map defined by

$\varphi(E_{2p-1,2p-1}) = E_{2n-2p+3,2n-2p+3} + \sum_j \alpha_{2n-2p+3,2j} E_{2n-2p+3,2j}$

$\varphi(E_{2q,2q}) = E_{2n-2q+2,2n-2q+2} - \sum_i \alpha_{2i-1,2n-2q+2} E_{2i-1,2n-2q+2}$

$\varphi(E_{2p-1,2q}) = \gamma_{2n-2p+3,2n-2q+2} E_{2n-2p+3,2n-2q+2}$

for all $E_{2p-1,2p-1}$, $E_{2q,2q}$ and $E_{2p-1,2q}$ in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Then φ is an isomorphism. Let $\phi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ be a linear map defined by $\phi(A) = D_{2n+1}\varphi(A)D_{2n+1}$ for all A in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$, where D_{2n+1} is the $(2n+1) \times (2n+1)$ backward identity matrix. Then ϕ is an isomorphism and the (i,i)-component of $\phi(E_{ii})$ is 1 for all $i(1 \leq i \leq 2n+1)$.

3. Isomorphisms of $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $Alg\mathcal{L}_{2n+1}^{(4k-1)}$

Before we investigate the general isomorphisms $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ (resp., $Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$) we will consider special isomorphisms $\rho : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}(Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)})$ satisfying $\rho(E_{ii}) = E_{ii}$ for all i. Since $E_{ii}E_{ij}E_{jj} = E_{ij}$ for all i,j, we have $\rho(E_{ij}) = \rho(E_{ii}E_{ij}E_{jj}) = E_{ii}\rho(E_{ij})E_{jj}$. From this we have the following theorem.

THEOREM 3.1. *Let* $\rho : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ *(resp.,* $Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$*) be an isomorphism such that* $\rho(E_{ii}) = E_{ii}$ *for all* i*. Then* *there exist nonzero complex numbers γ_{ij} such that $\rho(E_{ij}) = \gamma_{ij}E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$ ($Alg\mathcal{L}_{2n+1}^{(4k-1)}$).*

THEOREM 3.2. *Let $\rho : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ (resp., $Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$) be an isomorphism such that $\rho(E_{ii}) = E_{ii}$ for all i and $\rho(E_{ij}) = \gamma_{ij}E_{ij}$, $\gamma_{ij} \neq 0$, for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$ ($Alg\mathcal{L}_{2n+1}^{(4k-1)}$). If $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_{2n}^{(4k-1)}$ ($Alg\mathcal{L}_{2n+1}^{(4k-1)}$), then $\rho(A) = RAR^{-1}$ for all A in $Alg\mathcal{L}_{2n}^{(4k-1)}$ ($Alg\mathcal{L}_{2n+1}^{(4k-1)}$), where R is a diagonal operator whose*
$(1,1)$-component is 1,
$(2,2)$-component is γ_{12}^{-1},
$(2p+1, 2p+1)$-component is $\prod_{m=1}^{p} \gamma_{2m+1,2m}(\prod_{m=1}^{p} \gamma_{2m-1,2m})^{-1}$,
$(2q+2, 2q+2)$-component is $\prod_{m=1}^{q} \gamma_{2m+1,2m}(\prod_{m=1}^{q+1} \gamma_{2m-1,2m})^{-1}$
for all $p = 1, 2, \cdots n-1$ or n and $q = 1, 2, \cdots, n-1$.

PROOF. Let $A = [a_{ij}]$ be in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Then $\rho(A) = [\gamma_{ij}a_{ij}]$ and $\gamma_{ii} = 1$ for all i. Comparing the components of $\rho(A)R$ with that of RA, we have $\rho(A)R = RA$. □

THEOREM 3.3. *Let ρ be as in Theorem 3.2. If ρ is spatially implemented by T, then T is diagonal and $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_{2n}^{(4k-1)}$($Alg\mathcal{L}_{2n+1}^{(4k-1)}$).*

PROOF. Let $T = [t_{ij}]$ be an invertible matrix. Since $\rho(E_{ii})T = TE_{ii}$ and $\rho(E_{ii}) = E_{ii}$ for all i, we have $E_{ii}T = TE_{ii}$. So $t_{ij} = 0$ for all $i, j (i \neq j)$ and $t_{ii} \neq 0$ for all i. Hence T is diagonal. Let $T = \sum_{i=1}^{2n} t_{ii}E_{ii}$ and let $\rho(E_{ij}) = \gamma_{ij}E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$. For any $p, q (p \neq q)$ with E_{pq} in $Alg\mathcal{L}_{2n}^{(4k-1)}$,

$$\rho(E_{pq})T = \gamma_{pq}E_{pq}(\sum_{i=1}^{2n} t_{ii}E_{ii}) = \gamma_{pq}t_{qq}E_{pq}$$

and

$$TE_{pq} = (\sum_{i=1}^{2n} t_{ii}E_{ii})E_{pq} = t_{pp}E_{pq}.$$

Since $\rho(E_{pq})T = TE_{pq}$, we have $\gamma_{pq}t_{qq} = t_{pp}$. Hence $\gamma_{pq} = t_{pp}t_{qq}^{-1}$. If E_{st} is in $Alg\mathcal{L}_{2n}^{(4k-1)}$, then $\gamma_{st} = t_{ss}t_{tt}^{-1}$. Hence $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_{2n}^{(4k-1)}$. □

THEOREM 3.4. *Let* $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ *(resp.,* $Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$*) be an isomorphism. Then for each* i,

$$\varphi(E_{ii}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j}E_{2p-1,2j}$$

for some $p = 1, 2, \cdots, n (p = 1, 2, \cdots, n+1)$ *and some complex numbers* $\alpha_{2p-1,2j}$ *with* $E_{2p-1,2j}$ *in* $Alg\mathcal{L}_{2n}^{(4k-1)}$ $(Alg\mathcal{L}_{2n+1}^{(4k-1)})$ *or*

$$\varphi(E_{ii}) = E_{2q,2q} + \sum_l \alpha_{2l-1,2q}E_{2l-1,2q}$$

for some $q (1 \le q \le n)$ *and some complex numbers* $\alpha_{2l-1,2q}$ *with* $E_{2l-1,2q}$ *in* $Alg\mathcal{L}_{2n}^{(4k-1)}$ $(Alg\mathcal{L}_{2n+1}^{(4k-1)})$.

PROOF. Let $\varphi(E_{ii}) = [\alpha_{st}]$ be in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Then $[\alpha_{st}]^2 = \varphi(E_{ii})^2 = \varphi(E_{ii}^2) = [\alpha_{st}]$. Hence $\alpha_{ss} = 1$ or 0 for all s. Since $\alpha_{ss} = 0$ for all s implies $[\alpha_{st}] = [\alpha_{st}]^2 = 0$, we have $\alpha_{ss} = 1$ for some s. If $\alpha_{tt} \neq 0$ for some t such that $s \neq t$ and $1 \le t \le 2n$, then the (s, s)-component and the (t, t)-component of $\varphi(E_{ii})$ are 1. So there exists $j (j \neq i$ and $1 \le j \le 2n)$ such that one of the (s, s)-component or the (t, t)-component of $\varphi(E_{jj})$ is 1. Hence $0 = \varphi(E_{ii}E_{jj}) = \varphi(E_{ii})\varphi(E_{jj}) \neq 0$ which is a contradiction. Thus $\alpha_{ss} = 1$ for one and only one s. If $\alpha_{2p-1,2p-1} = 1$ for some p, then

$$\varphi(E_{ii}) = E_{2p-1,2p-1} + \sum_{l,j} \alpha_{2l-1,2j}E_{2l-1,2j}$$

for all l, j with $E_{2l-1,2j}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Since $\varphi(E_{ii}) = \varphi(E_{ii})^2$, we have

$$\varphi(E_{ii}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j}E_{2p-1,2j}$$

for some $\alpha_{2p-1,2j} \in \mathbf{C}$ with $E_{2p-1,2j}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. If $\alpha_{2q,2q} = 1$ for some q, then by similar calculation, we have

$$\varphi(E_{ii}) = E_{2q,2q} + \sum_l \alpha_{2l-1,2q} E_{2l-1,2q}$$

for some $\alpha_{2l-1,2q} \in \mathbf{C}$ with $E_{2l-1,2q}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. □

THEOREM 3.5. *Let* $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ *(resp.,* $Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$*) be an isomorphism.*

(1) *If the* (i,i)*-component of* $\varphi(E_{2p-1,2p-1})$ *is 1, then* i *is an odd number.*
(2) *If the* (j,j)*-component of* $\varphi(E_{2q,2q})$ *is 1, then* j *is an even number.*

PROOF. Let $\varphi(E_{2p-1,2p}) = [\lambda_{st}]$ be in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Suppose i is even, say $i = 2l$. Then

$$\varphi(E_{2p-1,2p-1}) = E_{2l,2l} + \sum_i \alpha_{2i-1,2l} E_{2i-1,2l}$$

for some $\alpha_{2i-1,2l}$ in $\mathbf{C}$ with $E_{2i-1,2l}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Hence

$$\begin{aligned}\varphi(E_{2p-1,2p}) &= \varphi(E_{2p-1,2p-1})\varphi(E_{2p-1,2p})\varphi(E_{2p,2p}) \\ &= \lambda_{2l,2l}(E_{2l,2l} + \sum_i \alpha_{2i-1,2l} E_{2i-1,2l})\varphi(E_{2p,2p})\end{aligned}$$

Since $\varphi(E_{2p-1,2p}) \neq 0$, the $(2l,2l)$-component of $\varphi(E_{2p,2p})$ is 1. It is a contradiction because the $(2l,2l)$-component of $\varphi(E_{2p-1,2p-1})$ is 1. Hence i is an odd number. In the same way we can show that j is an even number. □

THEOREM 3.6. *Let* $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ *be an isomorphism. Then the* (i,i)*-component of* $\varphi(E_{ii})$ *is 1 for all* $i = 1, 2, \cdots, 2n$.

PROOF. Suppose the $(1,1)$-component of $\varphi(E_{2p-1,2p-1})$ is 1. Then for each $l = 1, 2, \cdots, k$, there exist q_l such that the $(2l,2l)$-component of

$\varphi(E_{2q_l,2q_l})$ is 1. Let $\varphi(E_{2p-1,2p-1}) = E_{11} + \sum_j \alpha_{1,2j}E_{1,2j}$, $\varphi(E_{2q_l,2q_l}) = E_{2l,2l} + \sum_i \beta_{2i-1,2l}E_{2i-1,2l}$ and $\varphi(E_{2p-1,2q_l}) = [\lambda_{st}]$. Then

$$\begin{aligned}\varphi(E_{2p-1,2q_l}) &= \varphi(E_{2p-1,2p-1})\varphi(E_{2p-1,2q_l})\varphi(E_{2q_l,2q_l}) \\ &= (\lambda_{11}\beta_{1,2l} + \lambda_{1,2l} + \alpha_{1,2l}\lambda_{2l,2l})E_{1,2l}\end{aligned}$$

Hence $E_{2p-1,2q_l} \in Alg\mathcal{L}_{2n}^{(4k-1)}$ and $\varphi(E_{2p-1,2q_l}) = \gamma_{1,2l}E_{1,2l}$ for some complex number $\gamma_{1,2l}$. If $p \neq 1$, we can choose $E_{2p-1,2q_m}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$ such that $E_{2p-1,2q_m} \notin \{E_{2p-1,2q_1}, E_{2p-1,2q_2}, \cdots, E_{2p-1,2q_k}\}$. Then

$$\varphi(E_{2p-1,2q_m}) = \varphi(E_{2p-1,2p-1})\varphi(E_{2p-1,2q_m})\varphi(E_{2q_m,2q_m}) = 0.$$

It is a contradiction. Hence $p = 1$.
Suppose the $(2,2)$-component of $\varphi(E_{2q,2q})$ is 1. Then $\varphi(E_{1,2q}) \neq 0$. Hence $q \leq k$. For each $l = 1, 2, \cdots, k+1$, there exist p_l such that the $(2l-1, 2l-1)$-component of $\varphi(E_{2p_l-1,2p_l-1})$ is 1. Let $\varphi(E_{2p_l-1,2p_l-1}) = E_{2l-1,2l-1} + \sum_j \alpha_{2l-1,2j}E_{2l-1,2j}$, $\varphi(E_{2q,2q}) = E_{22} + \sum_{i=1}^{k+1} \beta_{2i-1,2}E_{2i-1,2}$ and $\varphi(E_{2p_l-1,2q}) = [\lambda_{st}]$. Then

$$\varphi(E_{2p_l-1,2q}) = \varphi(E_{2p_l-1,2p_l-1})\varphi(E_{2p_l-1,2q})\varphi(E_{2q,2q}) = \gamma_{2l-1,2}E_{2l-1,2}$$

for some complex number $\gamma_{2l-1,2}$. Hence $E_{2p_l-1,2q} \in Alg\mathcal{L}_{2n}^{(4k-1)}$. If $q \neq 1$, we can choose $E_{2p_m-1,2q}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$ such that $E_{2p_m-1,2q} \notin \{E_{2p_1-1,2q}, E_{2p_2-1,2q}, \cdots, E_{2p_{k+1}-1,2q}\}$. Then

$$\varphi(E_{2p_m-1,2q}) = \varphi(E_{2p_m-1,2p_m-1})\varphi(E_{2p_m-1,2q})\varphi(E_{2q,2q}) = 0$$

It is a contradiction. Hence $q = 1$.
In the same way we can show that the (i,i)-component of $\varphi(E_{ii})$ is 1 for all $i = 1, 2, \cdots, 2k-1$.
Suppose the $(2k, 2k)$-component of $\varphi(E_{2q,2q})$ is 1 and $q > k$. Then

$$\varphi(E_{1,2q}) = \varphi(E_{11})\varphi(E_{1,2q})\varphi(E_{2q,2q}) = \gamma_{1,2k}E_{1,2k}$$

for some complex number $\gamma_{1,2k}$. Since $E_{1,2k} \in Alg\mathcal{L}_{2n}^{(4k-1)}$ and $E_{1,2q} \notin Alg\mathcal{L}_{2n}^{(4k-1)}$, it is a contradiction. Hence $q = k$.

Suppose the $(2k+1, 2k+1)$-component of $\varphi(E_{2p-1,2p-1})$ is 1 and $p > k+1$. Then

$$\varphi(E_{2p-1,2}) = \varphi(E_{2p-1,2p-1})\varphi(E_{2p-1,2})\varphi(E_{22}) = \gamma_{2k+1,2}E_{2k+1,2}$$

for some complex number $\gamma_{2k+1,2}$. Since $E_{2k+1,2} \in Alg\mathcal{L}_{2n}^{(4k-1)}$ and $E_{2p-1,2} \notin Alg\mathcal{L}_{2n}^{(4k-1)}$, it is a contradiction. Hence $p = k+1$.
Similarly we can check that the (i,i)-component of $\varphi(E_{ii})$ is 1 for all $i = 2k+2, 2k+3, \cdots, 2n$. □

By an argument similar to that of Theorem 3.6 we can derive the following theorem.

THEOREM 3.7. *Let $\varphi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ be an isomorphism. Then the (i,i)-component of $\varphi(E_{ii})$ is 1 or the $(2n-i+2, 2n-i+2)$-component of $\varphi(E_{ii})$ is 1 for all $i = 1, 2, \cdots, 2n+1$.*

THEOREM 3.8. *Let φ be as in Theorem 3.5. If*
$\varphi(E_{2p-1,2p-1}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j}E_{2p-1,2j}$ *and*
$\varphi(E_{2q,2q}) = E_{2q,2q} + \sum_i \beta_{2i-1,2q}E_{2i-1,2q}$
for some complex numbers $\alpha_{2p-1,2j}$ and $\beta_{2i-1,2q}$, then there exist complex numbers γ_{ij} such that $\varphi(E_{ij}) = \gamma_{ij}E_{ij}$ and $\beta_{ij} = -\alpha_{ij}$.

PROOF. Let $\varphi(E_{2p-1,2q}) = [\lambda_{st}]$ be in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Then for each $E_{2p-1,2q}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$,

$$\begin{aligned}\varphi(E_{2p-1,2q}) &= \varphi(E_{2p-1,2p-1})\varphi(E_{2p-1,2q})\varphi(E_{2q,2q}) \\ &= (\lambda_{2p-1,2q} + \alpha_{2p-1,2q}\lambda_{2q,2q} + \lambda_{2p-1,2p-1}\beta_{2p-1,2q})E_{2p-1,2q}.\end{aligned}$$

Hence $\varphi(E_{2p-1,2q}) = \gamma_{2p-1,2q}E_{2p-1,2q}$ for some $\gamma_{2p-1,2q}$ in $\mathbf{C}$.
Let $A = E_{2p-1,2p-1} + E_{2p-1,2q} + E_{2q,2q}$. Then $A^2 = E_{2p-1,2p-1} + 2E_{2p-1,2q} + E_{2q,2q}$. Since $\varphi(A)^2 = \varphi(A^2)$, the $(2p-1, 2q)$-component $\varphi(A)^2$ is equal to the $(2p-1, 2q)$-component $\varphi(A^2)$. Hence $\alpha_{ij} = -\beta_{ij}$ for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$ or $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. □

From Theorems 3.6, 3.7 and 3.8, we can get the following theorems.

THEOREM 3.9. *Let $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ be an isomorphism. Then there exist complex numbers α_{ij} and $\gamma_{ij}(\gamma_{ij} \neq 0)$ for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$ such that*

$\varphi(E_{2p-1,2p-1}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j} E_{2p-1,2j}$
$\varphi(E_{2q,2q}) = E_{2q,2q} - \sum_i \alpha_{2i-1,2q} E_{2i-1,2q}$
$\varphi(E_{2p-1,2q}) = \gamma_{2p-1,2q} E_{2p-1,2q}$

for all $E_{2p-1,2p-1}$, $E_{2q,2q}$ and $E_{2p-1,2q}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$.

THEOREM 3.10. *Let $\varphi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)})$ be an isomorphism such that the (i, i)-component of $\varphi(E_{ii})$ is 1 for all i. Then there exist complex numbers α_{ij} and $\gamma_{ij}(\gamma_{ij} \neq 0)$ for all $i, j(i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ such that*

$\varphi(E_{2p-1,2p-1}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j} E_{2p-1,2j}$
$\varphi(E_{2q,2q}) = E_{2q,2q} - \sum_i \alpha_{2i-1,2q} E_{2i-1,2q}$
$\varphi(E_{2p-1,2q}) = \gamma_{2p-1,2q} E_{2p-1,2q}$

for all $E_{2p-1,2p-1}$, $E_{2q,2q}$ and $E_{2p-1,2q}$ in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$.

THEOREM 3.11. *Let $\varphi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ be an isomorphism such that the $(2n-i+2, 2n-i+2)$-component of $\varphi(E_{ii})$ is 1 for all i. Then there exist nonzero complex numbers γ_{ij} for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ such that $\varphi(E_{ij}) = \gamma_{2n-i+2,2n-j+2} E_{2n-i+2,2n-j+2}$.*

THEOREM 3.12. *Let $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ be an isomorphism and let $\varphi(E_{ij}) = \gamma_{ij} E_{ij}$, $\gamma_{ij} \neq 0$, for all E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$. If $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_{2n}^{(4k-1)}$, then there exists an invertible operator T in $Alg\mathcal{L}_{2n}^{(4k-1)}$ such that $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_{2n}^{(4k-1)}$.*

PROOF. Let $\varphi : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ be an isomorphism and let $\varphi(E_{ij}) = \gamma_{ij} E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$. Then there are complex numbers α_{ij} for all E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$ such that

$\varphi(E_{2p-1,2p-1}) = E_{2p-1,2p-1} + \sum_j \alpha_{2p-1,2j} E_{2p-1,2j}$
$\varphi(E_{2q,2q}) = E_{2q,2q} - \sum_i \alpha_{2i-1,2q} E_{2i-1,2q}$
$\varphi(E_{2p-1,2q}) = \gamma_{2p-1,2q} E_{2p-1,2q}$

for all $E_{2p-1,2p-1}$, $E_{2q,2q}$ and $E_{2p-1,2q}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$.

Let $\rho : Alg\mathcal{L}_{2n}^{(4k-1)} \to Alg\mathcal{L}_{2n}^{(4k-1)}$ be an isomorphism defined by $\rho(E_{ii}) = E_{ii}$ for all i and $\rho(E_{ij}) = \gamma_{ij}E_{ij}$ for all E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$. Then $\varphi(A) = S\rho(A)S^{-1}$ for all A in $Alg\mathcal{L}_{2n}^{(4k-1)}$, where S is a matrix in $Alg\mathcal{L}_{2n}^{(4k-1)}$ whose (i,i)-component is 1 for all $i = 1, 2, \cdots, 2n$ and (i,j)-component is $-\alpha_{ij}$ for all $i, j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$. From Theorem 3.2, there is a diagonal invertible operator R such that $\rho(A) = RAR^{-1}$ for all A in $Alg\mathcal{L}_{2n}^{(4k-1)}$. Put $T = SR$. Then $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_{2n}^{(4k-1)}$. □

THEOREM 3.13. *Let φ be as in Theorem 3.12. If φ is spatially implemented by T, then T is in $Alg\mathcal{L}_{2n}^{(4k-1)}$ and $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_{2n}^{(4k-1)}$.*

PROOF. Let $T = \sum_{i,j=1}^{2n} t_{ij}E_{ij}$ and $\varphi(E_{ij}) = \gamma_{ij}E_{ij}$ for all i and $j (i \neq j)$ with E_{ij} in $Alg\mathcal{L}_{2n}^{(4k-1)}$. For each $E_{2p-1,2q}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$,

$$\varphi(E_{2p-1,2q})T = \gamma_{2p-1,2q}E_{2p-1,2q}(\sum_{i,j=1}^{2n} t_{ij}E_{ij}) = \sum_{j=1}^{2n} \gamma_{2p-1,2q}t_{2q,j}E_{2p-1,j}$$

and

$$TE_{2p-1,2q} = (\sum_{i,j=1}^{2n} t_{ij}E_{ij})E_{2p-1,2q} = \sum_{i=1}^{2n} t_{i,2p-1}E_{i,2q}.$$

Since $\varphi(E_{2p-1,2q})T = TE_{2p-1,2q}$, we have $\gamma_{2p-1,2q}t_{2q,2q} = t_{2p-1,2p-1}$. In the same way we can show that $\gamma_{2s-1,2t}t_{2t,2t} = t_{2s-1,2s-1}$, $\gamma_{2p-1,2t}t_{2t,2t} = t_{2p-1,2p-1}$ and $\gamma_{2s-1,2q}t_{2q,2q} = t_{2s-1,2s-1}$ for each $E_{2s-1,2t}, E_{2p-1,2t}$ and $E_{2s-1,2q}$ in $Alg\mathcal{L}_{2n}^{(4k-1)}$. Hence

$$\gamma_{2p-1,2q}\gamma_{2s-1,2t} = \gamma_{2p-1,2t}\gamma_{2s-1,2q}.$$

□

Let $\varphi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ be an isomorphism such that the (i,i)-component of $\varphi(E_{ii})$ is 1 for all i. Define the map $\phi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ by $\phi(A) = D_{2n+1}\varphi(A)D_{2n+1}$ for all A in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$, where

D_{2n+1} is the $(2n+1)\times(2n+1)$ backward identity matrix. Then ϕ is an isomorphism and the $(2n-i+2, 2n-i+2)$-component of $\phi(E_{ii})$ is 1 for all i. By an argument similar to that of Theorems 3.12 and 3.13, we can get the following theorem.

THEOREM 3.14. *Let* $\varphi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ *be an isomorphism and let* $\varphi(E_{ij}) = \gamma_{ij}E_{ij}$ *for all* E_{ij} *in* $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. *Then* $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ *for all* p, q, s *and* t *with* E_{pq}, E_{st}, E_{pt} *and* E_{sq} *in* $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ *if and only if there is an invertible operator* T *such that* $\varphi(A) = TAT^{-1}$ *for all* A *in* $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. *In this case* T *is in* $Alg\mathcal{L}_{2n+1}^{(4k-1)}$.

THEOREM 3.15. *Let* $\varphi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ *be an isomorphism and let* $\varphi(E_{ij}) = \gamma_{2n-i+2,2n-j+2}E_{2n-i+2,2n-j+2}$ *for all* E_{ij} *in* $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. *Then* $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ *for all* p, q, s *and* t *with* E_{pq}, E_{st}, E_{pt} *and* E_{sq} *in* $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ *if and only if there is an invertible operator* T *such that* $\varphi(A) = SAS^{-1}$ *for all* A *in* $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. *In this case* $S = D_{2n+1}T$, *where* $T \in Alg\mathcal{L}_{2n+1}^{(4k-1)}$ *and* D_{2n+1} *is the backward identity matrix.*

PROOF. Let $\phi : Alg\mathcal{L}_{2n+1}^{(4k-1)} \to Alg\mathcal{L}_{2n+1}^{(4k-1)}$ be a map defined by $\phi(A) = D_{2n+1}\varphi(A)D_{2n+1}$ for all A in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. Then ϕ is an isomorphism and the (i,i)-component of $\phi(E_{ii})$ is 1 for all $i = 1, 2, \cdots, n+1$. Now

$$\begin{aligned}\phi(E_{ij}) &= D_{2n+1}\varphi(E_{ij})D_{2n+1}\\ &= D_{2n+1}\gamma_{2n-i+2,2n-j+2}E_{2n-i+2,2n-j+2}D_{2n+1}\\ &= \gamma_{2n-i+2,2n-j+2}E_{ij}\end{aligned}$$

for all E_{ij} in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$. By Theorem 3.12 and Theorem 3.13, $\gamma_{pq}\gamma_{st} = \gamma_{pt}\gamma_{sq}$ for all p, q, s and t with E_{pq}, E_{st}, E_{pt} and E_{sq} in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$ if and only if ϕ is spatially implemented by T in $Alg\mathcal{L}_{2n+1}^{(4k-1)}$, that is

$$\varphi(A) = D_{2n+1}\phi(A)D_{2n+1} = D_{2n+1}TAT^{-1}D_{2n+1}.$$

□

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