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CENTRALLY SYMMETRIC ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

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ABSTRACT. We study centrally symmetric orthogonal polynomials satisfying an admissible partial differential equation of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda_n u,$$

where $A, B, \cdots, E$ are polynomials independent of n and λ_n is the eigenvalue parameter depending on n. We show that they are either the product of Hermite polynomials or the circle polynomials up to a complex linear change of variables. Also we give some properties of them.

1. Introduction

Classical orthogonal polynomials are the only orthogonal polynomials which satisfy a second order ordinary differential equation

$$\alpha(x)y'' + \beta(x)y' = (ax^2 + bx + c)y'' + (dx + e)y' = \lambda_n y, \tag{1.1}$$

where $\lambda_n = an(n-1) + dn$. They are Jacobi(including Gegenbauer), Laguerre, Hermite and Bessel polynomials. Among these polynomials, Gegenbauer and Hermite polynomials have the specific properties. For example, $\alpha(-x) = \alpha(x)$, $\beta(-x) = -\beta(x)$ and all the moments of odd order are zero.

As a natural generalization, we consider the problem of characterizing all centrally symmetric orthogonal polynomials(see Definition 2.3)

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satisfying a second order partial differential equation of the form

$$L[u] := Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda_n u, \tag{1.2}$$

where $A(x,y), \cdots, E(x,y)$ are polynomials and λ_n is the eigenvalue parameter. The differential equation (1.2) was first investigated by Krall and Sheffer [2]. They identified all weak orthogonal polynomials(See Definition 2.1) satifying the differential equation (1.1) but partially succeeded in showing that the polynomial solutions are orthogonal.

In this work, we characterize centrally symmetric orthogonal polynomials satisfying the differential equation (1.2) and show that they are either the product of Hermite polynomials or the circle polynomials. Also we give several properties of them.

2. Preliminaries

The set of all polynomials in two variables is denoted by $\mathcal{P}$. By a polynomial system(PS), we mean a sequence of polynomials $\{\phi_{mn}(x,y)\}_{m,n=0}^{\infty}$ such that $\deg \phi_{mn} = m+n$ for m and $n \geq 0$ and $\{\phi_{n-j,j}\}_{j=0}^{n}$ is linearly independent modulo polynomials of degree $\leq n-1$. For brevity, we denote $(\phi_{n0}, \phi_{n-1,1}, \cdots, \phi_{0n})^T$ by Φ_n and a PS $\{\phi_{mn}(x,y)\}_{m,n=0}^{\infty}$ by $\{\Phi_n\}_0^{\infty}$.

A PS $\{\mathbb{P}_n\}_0^{\infty}$ is called to be monic if

$$P_{m,n}(x,y) = x^m y^n + \text{lower degree terms.}$$

Any linear functional on the space of polynomials is called a moment functional. The action of a moment functional σ on a polynomial ϕ is denoted by $\langle \sigma, \phi \rangle$. Similarly, the action of σ on matrix Q with each entry Q_{ij} being a polynomial is the matrix defined by

$$\langle \sigma, Q \rangle = (\langle \sigma, Q_{ij} \rangle).$$

For any moment functional σ on $\mathcal{P}$, we define partial derivatives of σ by the formula

$$\langle \partial_x \sigma, \phi \rangle = -\langle \sigma, \partial_x \phi \rangle, \text{ and } \langle \partial_y \sigma, \phi \rangle = -\langle \sigma, \partial_y \phi \rangle,$$

and define the multiplication by a polynomial ψ on σ through the formula

$$\langle \psi \sigma, \phi \rangle = \langle \sigma, \psi \phi \rangle,$$

where ϕ in a polynomial.

DEFINITION 2.1. A PS $\{\Phi_n\}_0^\infty$ is a weak orthogonal polynomial system(WOPS) if there is a nonzero moment functional σ such that

$$\langle \sigma, \phi_{mn}\phi_{kl} \rangle = 0 \quad \text{if } m+n \neq k+l.$$

If $\langle \sigma, \phi_{mn}\phi_{kl} \rangle = K_{mn}\delta_{mk}\delta_{nl}$ where K_{mn} are nonzero(resp. positive) constants, we call $\{\Phi_n\}_0^\infty$ an orthogonal polynomial system(OPS)(resp. a positive-definite OPS). In this case, we say that $\{\Phi_n\}_0^\infty$ is a WOPS or an OPS(resp. positive-definte OPS) relative to σ.

DEFINITION 2.2. A moment functional σ is quasi-definite(resp. positive-definite) if there is an OPS(resp. positive-definite OPS) relative to σ.

From Definition 2.1 and 2.2, we see that a PS $\{\Phi_n\}_0^\infty$ is an OPS(resp. a positive-definite OPS) relative to σ if and only if $\langle \sigma, \Phi_m\Phi_n^T \rangle = H_n\delta_{mn}$ and $H_n := \langle \sigma, \Phi_n\Phi_n^T \rangle$ is a nonsingular(resp. positive-definite) diagonal matrix. Also, it is easy to see that for any PS $\{\Psi_n\}_0^\infty$, $\langle \sigma, \Phi_n\Psi_n^T \rangle$ is nonsingular.

For any PS $\{\Phi_n\}_0^\infty$, there is a unique moment functional σ, which is called the canonical moment functional of $\{\Phi_n\}_0^\infty$, defined by the conditions

$$\langle \sigma, 1 \rangle = 1, \quad \langle \sigma, \phi_{mn} \rangle = 0, m+n \geq 1. \tag{2.1}$$

Note that if a PS $\{\Phi_n\}_0^\infty$ is a WOPS relative to σ, then σ is a constant multiple of the canonical moment functional of $\{\Phi_n\}_0^\infty$.

DEFINITION 2.3. [6] A moment functional σ is centrally symmetric if $\langle \sigma, x^m y^n \rangle = 0$ for $m+n =$ odd integer. We say that a PS $\{\Phi_n\}_0^\infty$ is a centrally symmetric OPS if there is a centrally symmetric quasi-definite moment functional σ such that $\{\Phi_n\}_0^\infty$ is an OPS relative to σ.

THEOREM 2.1. [1] *For a nonzero moment functional σ, the followings are equivalent.*

(i) *σ is quasi-definite.*
(ii) *There is a unique monic WOPS $\{\Phi_n\}_0^\infty$ relative to σ.*
(iii) *There is a monic WOPS $\{\Phi_n\}_0^\infty$ such that $H_n := \langle \sigma, \Phi_n\Phi_n^T \rangle, n \geq 0$ is nonsingular.*

The following algebraic characterization of orthogonality for polynomials in two variables is fundamental in the study of orthogonal polynomials in two variables.

THEOREM 2.2. [5, 6] *Let $\{\Phi_n\}_0^\infty$ be any PS. The followings are equivalent.*

(i) *$\{\Phi_n\}_0^\infty$ is a WOPS relative to a quasi-definite moment functional σ.*

(ii) *For $n \geq 0$ and $i = 1, 2$ there are matrices $\mathbf{A}_{ni}, \mathbf{B}_{ni}$ and $\mathbf{C}_{ni}$ such that*

  (a) $x_i\Phi_n = \mathbf{A}_{ni}\Phi_{n+1} + \mathbf{B}_{ni}\Phi_n + \mathbf{C}_{ni}\Phi_{n-1}$,

  (b) *rank* $\mathbf{C}_n = n+1$, *where* $\mathbf{C}_n = (\mathbf{C}_{n1}, \mathbf{C}_{n2})$.

Furthermore, σ is centrally symmetric if and only if $\mathbf{B}_{ni} = 0$ for all $n \geq 0$.

3. Main contents

We can see [2] that if the differential equation (1.2) has a PS $\{\mathbb{P}_n\}_0^\infty$ as solutions, then it must be of the form

$$\begin{aligned} L[u] &= Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y \\ &= (ax^2 + d_1x + e_1y + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy} \\ &\quad + (ay^2 + d_3x + e_3y + f_3)u_{yy} + (gx + h_1)u_x + (gy + h_2)u_y \\ &= \lambda_n u, \end{aligned} \tag{3.1}$$

where $\lambda_n = an(n-1) + gn$.

DEFINITION 3.1. The differential equation (3.1) is admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$ or equivalently $an + g \neq 0$ for $n \geq 0$.

We know that the differential equation (3.1) is admissible if and only if the differential equation (3.1) has a unique monic PS as solutions. Generally, the admissibility of (3.1) does not guarantee the existence of an OPS satisfying the differential equation (3.1).

PROPOSITION 3.1. [1] *If the differential equation (3.1) has a PS $\{\Phi_n\}_0^\infty$ as solutions, the canonical moment functional σ of $\{\Phi_n\}_0^\infty$ must satisfy*

$$L^*[\sigma] = 0, \tag{3.2}$$

$$(3.3)\qquad \begin{aligned}\langle \sigma, D\rangle = \langle \sigma, E\rangle = \langle \sigma, A + xD\rangle &= \langle \sigma, C + yE\rangle \\ = \langle \sigma, B + yD\rangle &= \langle \sigma, B + xE\rangle = 0.\end{aligned}$$

PROPOSITION 3.2. *If the differential equation* (3.1) *has a WOPS* $\{\Phi_n\}_0^\infty$ *as solutions, then the canonical moment functional* σ *of* $\{\Phi_n\}_0^\infty$ *satisfies*

$$(3.4)\qquad M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0,$$

$$(3.5)\qquad M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0.$$

PROOF. See [1] for the proof. □

THEOREM 3.3. *For any OPS* $\{\Phi_n\}_0^\infty$ *relative to* σ, *the following statements are equivalent.*

(i) $\{\Phi_n\}_0^\infty$ *satisfy the differential equation* (3.1).
(ii) σ *satisfies the moment equations* $M_1[\sigma] = M_2[\sigma] = 0$.

PROOF. See [1] for the proof. □

Still, Theorem 3.3 holds for any WOPS $\{\Phi_n\}_0^\infty$ relative to a quasi-definite moment functional σ.

Krall and Sheffer made an important observation, which was used in classifying all WOPS satisfying the differential equation (3.1) ([2]).

LEMMA 3.4. *If the differential equation* (3.1) *has an OPS as solutions, then we have*

$$(3.6)\qquad f_2^2 - 4f_1f_3 \neq 0.$$

Now we are ready to state our main results.

THEOREM 3.5. *Let* $\{\Phi_n\}_0^\infty$ *be a WOPS relative to a centrally symmetric quasi-definite moment functional* σ. *If* $\{\Phi_n\}_0^\infty$ *satisfies the differential equation* (3.1),*then the differential equation* (3.1) *must be of the form*

$$(3.7)\qquad (ax^2+f_1)u_{xx}+(2axy+f_2)u_{xy}+(ay^2+f_3)u_{yy}+gxu_x+gyu_y = \lambda_n u.$$

PROOF. By Proposition 3.1, we have

$$\begin{aligned}
\langle \sigma, D\rangle &= h_1\sigma_{00} = 0, \quad \langle \sigma, E\rangle = h_2\sigma_{00} = 0\\
\langle \sigma, A + xD\rangle &= (a+g)\sigma_{20} + f_1\sigma_{00} = 0\\
\langle \sigma, C + yE\rangle &= (a+g)\sigma_{00} + f_3\sigma_{00} = 0\\
\langle \sigma, B + yD\rangle &= (a+g)\sigma_{11} + \frac{1}{2}f_2\sigma_{00} = 0
\end{aligned}$$

Then we have $h_1 = h_2 = 0$ and $\sigma_{20} = -\frac{f_1}{a+g}\sigma_{00}$, $\sigma_{11} = -\frac{f_2}{2(a+g)}\sigma_{00}$, $\sigma_{02} = -\frac{f_3}{a+g}\sigma_{00}$.

On the other hand, $M_1[\sigma] = M_2[\sigma] = 0$ implies that $\langle M_i[\sigma], x^m y^n\rangle = 0 (i = 1, 2)$ for $m + n = 2$. Thus we have the following equations for $d_i, e_i (i = 1, 2, 3)$

$$\begin{aligned}
&d_1\sigma_{20} + e_1\sigma_{11} = 0\\
&d_2\sigma_{20} + (2d_1 + e_2)\sigma_{11} + 2e_1\sigma_{02} = 0\\
&d_2\sigma_{11} + e_2\sigma_{02} = 0\\
&d_2\sigma_{20} + e_2\sigma_{11} = 0\\
&2d_3\sigma_{20} + (d_2 + 2e_3)\sigma_{11} + e_2\sigma_{02} = 0\\
&d_3\sigma_{11} + e_3\sigma_{02} = 0.
\end{aligned}$$

Since $f_2^2 - 4f_1f_3 \neq 0$ by Lemma 3.4, we can conclude that

$$d_i = e_i = 0 \quad (i = 1, 2, 3).$$

Thus the proof is complete. □

Krall and Sheffer showed that if $d_i = d_i = 0$ for $i = 1, 2, 3$, then we may take $f_1 = f_3 = 0$ in the differential equation (3.7) and considered the differential equation

$$(3.8) \qquad ax^2u_{xx} + (2axy + f_2)u_{xy} + ay^2u_{yy} + gxu_x + gyu_y = \lambda_n u$$

with $f_2 \neq 0$. They used a complex change of variables to transform (3.8) to show the following:

If $a = 0$, then (3.8) can be written as

$$u_{xx} + u_{yy} - xu_x - yu_y = -nu,$$

which has the product of Hermite polynomials as solutions.
If $a \neq 0$, then (3.8) can be transformed as

$$(3.9)\quad (x^2-1)u_{xx} + 2xyu_{xy} + (y^2-1)u_{yy} + gxu_x + gyu_y = n(n+g-1)u,$$

whose solutions are called the circle polynomials.

It is known that (3.9) has a positive-definite OPS as solutions if $g > 1$. Furthermore, (3.9) has an OPS as solutions if and only if $g \neq 1, 0, -1, \cdots$. (See [3] for details.)

We can summarize these results into the following:

THEOREM 3.6. *Let $\{\Phi_n\}_0^\infty$ be an OPS relative to a centrally symmetric quasi-definite moment functional σ. If $\{\Phi_n\}_0^\infty$ satisfies the differential equation* (3.1), *then they are either the product of Hermite polynomials or the circle polynomials with $g \neq 1, 0, -1, \cdots$.*

In the following, we investigate some properties of centrally symmetric OPS satisfying the differential equation (3.1) or (3.8).

THEOREM 3.7. *Let $\{\mathbb{P}_n\}_0^\infty$ be a monic WOPS relative to a centrally symmetric quasi-definite moment functional σ. If $\{\mathbb{P}_n\}_0^\infty$ satisfies the differential equation* (3.8), *then we have*

$$\mathbb{P}_n(-x, -y) = \mathbb{P}_n(x, y).$$

PROOF. Let $\mathbb{P}_n(x, y) = \sum_{j=0}^n A_j^n \mathbf{x}^j$ be the monic polynomial of degree n, where $A_{n+1}^n = 0, A_n^n = I_{n+1}$, A_j^n is an $(n+1) \times (j+1)$ matrix and $\mathbf{x}^j = (x^j, x^{j-1}y, \cdots, y^j)^T$. Then the coefficients $A_j^n (0 \leq j \leq n-1)$ satisfy the recursive equation

$$(\lambda_n - \lambda_j)A_j^n = A_{j+2}^n C_{j+2},$$

where $C_j = f_2 D_j^1 D_{j-1}^2$(See [3] for the definition of D_k^i.) Thus we have $A_{n-1}^n = A_{n-3}^n = \cdots = 0$ and

$$\mathbb{P}_n(-x,-y) = \sum_{j=0}^{n}(-1)^j A_j^n \mathbf{x}^j = (-1)^n A_n^n \mathbf{x}^n + (-1)^{n-2} A_{n-2}^n \mathbf{x}^{n-2} + \cdots$$
$$= (-1)^n \mathbb{P}_n(x,y).$$

This is the desired result. □

We say ([4]) that the differential operator $L[\cdot]$ in (3.1) belongs to the basic class if $A_y = C_x = 0$, that is,

$$A(x,y) = ax^2 + d_1 x + f_1, \quad C(x,y) = ay^2 + e_3 y + f_3.$$

If $L[\cdot]$ belongs to the basic class and $L[u] = \lambda u$, then $v = \partial_x^j \partial_y^k u (j,k \geq 0)$ satisfies

(3.10)
$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + (D + jA_x + 2kB_y)v_x + (E + 2jB_x + kC_y)v_y$$
$$= (\lambda - jD_x - kE_y - \frac{1}{2}j(j-1)A_{xx} - 2jkB_{xy} - \frac{1}{2}k(k-1)C_{yy})v.$$

Then we see that the differential equation (3.8) belongs to the basic class.

THEOREM 3.8. *Let $\{\mathbb{P}_n\}_0^\infty$ be the monic WOPS relative to a centrally symmetric quasi-definite moment functional σ. If $\{\mathbb{P}_n\}_0^\infty$ satisfies the differential equation (3.8) with $g \neq 1, 0, -1, \cdots$, then*

(i) *$P_{n0}(x,y) = P_{n0}(x)(n \geq 0)$ and $\{P_{n0}(x)\}_0^\infty$ is a classical OPS.*
(ii) *$P_{0n}(x,y) = P_{0n}(y)(n \geq 0)$ and $\{P_{0n}(y)\}_0^\infty$ is a classical OPS.*
(iii) *Set $P_{n-j,j}^{(x)} = \frac{1}{n-j+1}\partial_x P_{n-j+1,j}$ and $P_{n-j,j}^{(y)} = \frac{1}{j+1}\partial_y P_{n-j,j+1} (0 \leq j \leq n)$.*
Then $\{P_{n-j,j}^{(x)}\}_{n=0,j=0}^{\infty\ \ n}$ and $\{P_{n-j,j}^{(y)}\}_{n=0,j=0}^{\infty\ \ n}$ are monic WOPS relative to a centrally symmetric quasi-definite moment functional.

PROOF. If $\{\mathbb{P}_n\}_0^\infty$ is the product of Hermite polynomials, it is trivial. See [3] for the case $\{\mathbb{P}_n\}_0^\infty$ is the circle polynomials. □

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