Comm. Korean Math. Soc. **12** (1997), No. 3, pp. 655–664

# ON THE EXISTENCE AND BEHAVIOR OF SOLUTIONS FOR PERTURBED NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The existence and behavior of a bounded solution for a perturbed nonlinear differential equation of the type

$$\text{(DE)} \qquad x'(t) + Ax(t) \ni G(x(t)), \quad t \in [0, \infty)$$

is considered. First, we consider the existence of a bounded solution with more simple assumptions using the concept of "the method of lines". Then we devote to study its behavior using recent results of almost non-expansive curve which is developed by Djafari Rouhani.

## 1. Introduction

Let $H$ be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We consider the existence and asymptotic behavior of a strong solution for the abstract perturbed nonlinear differential equation of the type

$$(\text{DE} : x_0) \qquad \begin{cases} x'(t) + Ax(t) \ni G(x(t)), & t \in [0, \infty), \\ x(0) = x_0 \end{cases}$$

in a Hilbert space where $A$ is a maximal monotone(possibly multi-valued) operator defined on $D(A) \subset H$ and $x_0 \in D(A)$.

Received April 10, 1997. Revised June 5, 1997.

1991 Mathematics Subject Classification: Primary 34K25; Secondary 34K30, 34L20, 47H15.

Key words and phrases: maximal monotone operator, method of lines, almost nonexpansive curve.

This paper was supported by NON-DIRECTED RESEARCH FUND, Korea Research Foundation, 1995.

Recently, Djafari Rouhani ([3]) has obtained the asymptotic behavior of quasi-autonomous system

$$x'(t) + Ax(t) \ni f(t), \quad x(0) = x_0 \tag{1}$$

in a Hilbert space using the concept of "almost non-expansive curve" which is defined later. In his paper, it is basically assumed that the generalized solution of (1) is assumed bounded (cf. Theorem 4.5 and 4.6 of [3]). On the other hand, Jung, Park and Kang ([5]) also considered the asymptotic behavior of perturbed nonlinear functional differential equation

$$x'(t) + Ax(t) + G(x)(t) \ni f(t), \quad x(0) = x_0 \tag{2}$$

in a Hilbert space using same concept. In [5], the existence of a generalized solution is relied on the result of Aizicovici ([1]). Also, many authors are studied the behavior of solutions for (2) using various conditions. The reader is refered the papers which are mentioned in [5].

The purpose of this paper is to show the existence of a bounded solution of (DE:$x_0$) directly using the concept of "the method of lines" which is used by Kartsatos ([6]) and to find certain conditions on the perturbation $G$ which are quite different from that of [5] to have same asymptotic behavior of solution for perturbed nonlinear differential equation (DE:$x_0$). "The method of lines" is used frequently to solve the existence problem of various type of differential equations. We refer to Tanaka ([8]), Kartsatos and Parrott ([7]), Ha, Shin and Jin ([4]).

## 2. Preliminaries

In this section, we give some definitions and recent results for almost non-expansive curve(ANEC). we refer the reader to related results of Djafari Rouhani([2], [3]). First, we denote by $PC_{loc}([0,\infty) : H)$ the space of all piecewise continuous functions $f : [0,T] \to H$ for every finite interval $[0,T]$, $T > 0$, and $\overline{B_k(0)}$ the closed ball of $H$ with center at zero and radius $k$ for fixed $k > 0$. We also denote by $A^o x$ the element of minimum norm in the closed convex set $Ax$ and $\to$ ($\rightharpoonup$) strong(weak) convergence in $H$.

DEFINITION 2.1. Let $u \in C([0,\infty) : H)$ and $\sigma_T = (1/T)\int_0^T u(t)dt$. The $(u(t))_{t\geq 0}$ is ANEC if $\forall r, s, h \geq 0$

$$\|u(r+h) - u(s+h)\|^2 \leq \|u(r) - u(s)\|^2 + \epsilon(r,s)$$

where $\lim_{r,s\to\infty} \epsilon(r,s) = 0$.

We note that a bounded curve $(u(t))_{t\geq 0}$ satisfying

$$\|u(r+h) - u(s+h)\| \leq \|u(r) - u(s)\| + \epsilon_1(r,s)$$

$\forall r, s, h \geq 0$ where $\lim_{r,s\to\infty} \epsilon_1(r,s) = 0$ is ANEC.

DEFINITION 2.2. Given a bounded curve $(u(t))_{t\geq 0}$ in $H$, the asymptotic centre $c$ of $(u(t))_{t\geq 0}$ is defined as follows ; For every $y \in H$, let $\psi(y) = \limsup_{t\to\infty} \|u(t) - y\|^2$. Then $\psi$ is a continuous, strictly convex function on $H$ satisfying $\psi(y) \to \infty$ as $\|y\| \to \infty$. Thus $\psi$ achives its minimum on $H$ at a unique point $c$ which is called asymptotic centre of the curve $(u(t))_{t\geq 0}$.

We denote by $F(u(t))$ or for simplicity by $F$ the following subset (possibly empty ) of $H$ ;

$$F = \{q \in H \mid \lim_{t\to\infty} \|u(t) - q\| \text{ exists } \}.$$

LEMMA 2.3. *Let $(u(t))_{t\geq 0}$ be an ANEC in $H$. Then the followings are equivalent ;*

(1) $F \neq \emptyset$.
(2) $\liminf_{T\to\infty} \|\sigma_T\| < \infty$.
(3) *$\sigma_T$ converges weakly to the asymptotic centre of $(u(t))_{t\geq 0}$ as $T \to \infty$.*

LEMMA 2.4. *Let $(u(t))_{t\geq 0}$ be an ANEC in $H$ such that $\forall h \geq 0$, $u(t+h) - u(t) \rightharpoonup 0$ as $t \to \infty$. Then the followings are equivalent ;*

(1) $F \neq \emptyset$.
(2) $\liminf_{t\to\infty} \|u(t)\| < \infty$.
(3) *$u(t)$ converges weakly to $p \in H$ as $t \to \infty$.*

LEMMA 2.5. *Let $(u(t))_{t\geq 0}$ be any curve in $H$ such that $\lim_{t\to\infty}(u(t+h), u(t)) = \beta(h)$ exists uniformly in $h \geq 0$. Then $\sigma_T$ converges strongly to the asymptotic centre of $(u(t))_{t\geq 0}$ as $T \to \infty$.*

## 3. The existence of a bounded solution

In the following, we assume the following hypotheses hold ;
**(A.1)** The operator $G : PC_{loc}([0,\infty),\overline{B_k(0)}) \to H$ satisfies

$$\|G(x(t)) - G(y(t))\| \le \alpha(t)\|x(t) - y(t)\|$$

for $x, y \in PC_{loc}([0,\infty),\overline{B_k(0)})$ and $\alpha \in L^1([0,\infty),\mathcal{R}) \cap C([0,\infty),\mathcal{R})$.
**(A.2)** The operator $A : D(A) \subset H \to 2^H$ is a maximal monotone operator satisfying

$$(z - G(y(t)), x) \ge 0$$

for $y \in PC_{loc}([0,\infty),\overline{B_k(0)})$ and $z \in Ax$ where $x \in D(A)$ with $\|x\| > k$.

Let $T > 0$ be arbitrary but fixed. We shall first establish a theorem concerning the existence of step functions constructed by points satisfying certain equation which is so called approximate discrete scheme.

Lemma 3.1. *Let $x_0 \in D(A) \cap \overline{B_k(0)}$ and $\{t_j^n\}_{j=0}^n$ be a partition of the interval $[0,T]$, where $t_j^n = jh_n = jT/n$, $j = 0, 1, \cdots, n$. For sufficiently large $n$, there exists $\{w_j^n\}_{j=0}^n$ such that for $j = 1, 2, \cdots, n$*

$$\frac{w_j^n - w_{j-1}^n}{h_n} + Aw_j^n \ni G(\overline{w}_j^n(t_j^n)), \tag{3}$$

*where the step function $\overline{w}_j^n(t)$ is defined by*

$$\overline{w}_j^n(t) = \begin{cases} x_0, & t = 0, \\ w_i^n, & t \in (t_{i-1}^n, t_i^n], i = 1, 2, \cdots, j-1 \\ w_j^n, & t \in (t_{j-1}^n, T]. \end{cases}$$

*The function $x_n(t) = \overline{w}_n^n(t)$ is said to be an approximate solution of (DE:$x_0$).*
*Moreover, $\sup_{t\in[0,T]} \|\overline{w}_j^n(t)\| \le k$ for $j = 1, 2, \cdots, n$.*

Proof. Let $T_j : \overline{B_k(0)} \to H$ be defined by

$$T_j x = [I + h_n A]^{-1}(w_{j-1}^n + h_n G(\overline{x}(t_j^n))) \tag{4}$$

for $j = 1, 2, \cdots, n$, where

$$\overline{x}_j^n(t) = \begin{cases} x_0, & t = 0, \\ w_i^n, & t \in (t_{i-1}^n, t_i^n], i = 1, 2, \cdots, j-1 \\ x, & t \in (t_{j-1}^n, T]. \end{cases}$$

Here we may assume $w_{j-1}^n \in \overline{B_k(0)}$. Since $[I + h_n A]^{-1}$ is non-expansive, $T_j$ becomes Lipschitz continuous. In fact, there is $K > 0$ such that $\alpha(t) \le K$ for all $t \in [0, \infty)$ since $\alpha \in C([0,\infty), \mathcal{R}) \cap L^1([0,\infty), \mathcal{R})$. Hence

$$\|T_j x - T_j y\| \le h_n \|G(\overline{x}(t_j^n)) - G(\overline{y}(t_j^n))\| \le h_n K \|x - y\|.$$

For sufficiently large $n$ such as $h_n K < 1$, $T_j$ is Lipschitz continuous with constant $KT/n < 1$. To show that $T_j : \overline{B_k(0)} \to \overline{B_k(0)}$, we put $u = T_j x$, $x \in \overline{B_k(0)}$ and assume that $\|u\| > k$. Then, since

$$z - G(\overline{x}(t_j^n)) + (u - w_{j-1}^n)/h_n = 0$$

for some $z \in Au$ by (4),

$$\begin{aligned} 0 &= (z - G(\overline{x}(t_j^n)), u) + (u - w_{j-1}^n)/h_n, u) \\ &\ge (u - w_{j-1}^n, u)/h_n \ge [\|u\|^2 - (w_{j-1}^n, u)]/h_n \\ &\ge [\|u\| - \|w_{j-1}^n\|]\|u\|/h_n \ge (\|u\| - k)\|u\|/h_n > 0. \end{aligned}$$

From this contradiction, we have that $u \in \overline{B_k(0)}$. Therefore, for each $j = 1, 2, \cdots, n$, $T_j$ has a unique fixed point in $\overline{B_k(0)}$. When the unique fixed point of $T_j$ on $\overline{B_k(0)}$ is denoted by $w_j^n$ for $j = 1, 2, \cdots, n$, we are able to define the step function $\overline{w}_j^n(t)$ on $[0, T]$ with $\sup_{t \in [0,T]} \|\overline{w}_j^n(t)\| \le k$. □

LEMMA 3.2. *Let $x_0 \in D(A) \cap \overline{B_k(0)}$. Then $\{\|w_j^n - w_{j-1}^n\|/h_n\}$ is uniformly bounded for sufficiently large $n$.*

PROOF. Assume that $n$ is sufficiently large so that $1 - h_nK > 0$, $h_n < 1$. From (3), we have

$$\begin{aligned}\|w_1^n - w_0^n\| &\le h_n\|G(\overline{w}_1^n(t_1^n))\| + h_n\|(I+h_nA)^{-1}w_0^n - w_0^n\| \\ &\le h_n\|G(\overline{w}(t_1^n)) - G(\overline{0}(t_1^n))\| + h_n\|G(\overline{0}(t_1^n))\| + h_n|A^0 w_0^n| \\ &\le h_n[\alpha(t_1^n)\|w_1^n\| + |A^o x_0|] \le h_n[Kk + |A^o x_0|] \equiv h_n C_1\end{aligned}$$

with $C_1 = Kk + \|G(\overline{0}(0))\| + |A^o x_0|$ where $\overline{0}$ is the zero function. For $j = 2, 3, \cdots, n$, we have similar result such as

$$\begin{aligned}\|w_j^n - w_{j-1}^n\| &\le \|w_{j-1}^n - w_{j-2}^n\| + h_n\|G(\overline{w}_j^n(t_j^n)) - G(\overline{w}_{j-1}^n(t_{j-1}^n))\| \\ &\le \|w_{j-1}^n - w_{j-2}^n\| + h_n\|G(\overline{w}_j^n(t_j^n)) - G(\overline{w}_{j-1}^n(t_j^n))\| \\ &\le \|w_{j-1}^n - w_{j-2}^n\| + h_n\alpha(t_j^n)\|w_j^n - w_{j-1}^n\|.\end{aligned}$$

It follows that

$$\max_{1\le i\le j}\|w_i^n - w_{i-1}^n\|/h_n \le K\max_{1\le i\le j}\|w_i^n - w_{i-1}^n\| + \max_{1\le i\le j-1}\|w_i^n - w_{i-1}^n\|/h_n.$$

Putting $P_n = 1 - Kh_n \in (0,1)$ by our assumption, we obtain that

$$\frac{P_n}{h_n}\max_{1\le i\le j}\|w_i^n - w_{i-1}^n\| \le \frac{1}{h_n}\max_{1\le i\le j-1}\|w_i^n - w_{i-1}^n\|$$

and

$$\begin{aligned}P_n\max_{1\le i\le j}\|w_i^n - w_{i-1}^n\|/h_n &\le \max_{1\le i\le j-2}\|w_i^n - w_{i-1}^n\|/(P_n h_n) \\ &\;\;\vdots \\ &\le \|w_1^n - w_0^n\|/({P_n}^{j-2}h_n) \\ &\le \|w_1^n - w_0^n\|/({P_n}^{j-1}h_n)\end{aligned}$$

which yields, for $j = 2, 3, \cdots, n$,

$$\max_{1\le i\le j}\|w_i^n - w_{i-1}^n\|/h_n \le C_1/{P_n}^n.$$

Since $1/{P_n}^n = 1/(1 - KT/n)^n$, there exists $C_2 > 0$ such that for sufficiently large $n$

$$\max_{1\le i\le n}\|w_i^n - w_{i-1}^n\|/h_n \le C_1\exp(KT) \equiv C_2, \quad j = 1, 2, \cdots, n.$$

From the above inequality, we have the desired result. □

REMARK 3.3. Let $x_n(t)$ be an $h_n$-approximate solution of (DE :$x_0$) for for sufficiently large $n$. We define the function

$$z_n(t) = w_{j-1}^n + (t - t_{j-1}^n)(w_j^n - w_{j-1}^n)/h_n, \quad t \in [t_{j-1}^n, t_j^n] \quad j = 1, \cdots, n.$$

It is easy to see that $z_n(t)$, $t \in [0, T]$, is Lipschitzian with constant $C_2$. Since for $t \in [0, T]$ there exists $j$ $(1 \le j \le n)$ such that $t \in [t_{j-1}^n, t_j^n]$ and

$$\|x_n(t) - z_n(t)\| = \|w_j^n - w_{j-1}^n - (t - t_{j-1}^n)(w_j^n - w_{j-1}^n)/h_n\| \le C_2 h_n,$$

we have for $t, s \in [0, T]$

$$\begin{aligned}\|x_n(t) - x_n(s)\| &\le \|x_n(t) - z_n(t)\| + \|z_n(t) - z_n(s)\| + \|z_n(s) - x_n(s)\| \\ &\le C_2|t - s| + 2C_2 h_n \le C_3(|t - s| + h_n).\end{aligned}$$

Consequently, there exists a constant $C_3$ such that

$$\|x_n(t) - x_n(s)\| \le C_3(|t - s| + h_n)$$

for $t, s \in [0, T]$.

From the previous lemmas and remark, we have shown the way of construction and boundedness for an approximate solution $x_n(t)$ of (DE:$x_0$) for sufficiently large $n$. Moreover, we have also shown that $x_n(t)$ is Lipschitz continuous. Now, we should show the existence of a continuous function on $[0, T]$ which is uniform limit of the approximate solutions as $n \to \infty$. When we compare our equation (DE:$x_0$) to the equation (FDE:$\phi$) in [5], we note that they are very similar except our equation has no delay term. Moreover, our perturbation $G$ does not depend on $t$ directly. Therefore, since the next processes are very close to those of Tanaka(Theorem 2.6, Theorem 3.3 of [5]), we omit proofs of the next sever al results.

THEOREM 3.4. *Let $x_0 \in D(A) \cap \overline{B_k(0)}$. The limit $x(t) = \lim_{n\to\infty} x_n(t)$ exists unformly for $t \in [0, T]$. Moreover, $x(t)$ is a unique strong solution of* (DE:$x_0$).

As the way to show the uniqueness of the previous theorem, the following inequality is crucial. It is also very important to assure that the solution is actually ANEC.

THEOREM 3.5. *Let $x_0, \hat{x}_0 \in D(A) \cap \overline{B_k(0)}$ and $x(t), \hat{x}(t)$ be the corresponding solutions on $[0,T]$ of* (DE:$x_0$) *and* (DE:$\hat{x}_0$), *respectively. Then, for $0 \le s \le t \le T$,*

$$\begin{aligned} \|x(t) - \hat{x}(t)\| &\le \|x(s) - \hat{x}(s)\| + \int_s^t \|G(x(\tau)) - G(\hat{x}(\tau))\| d\tau \\ &\le \|x(s) - \hat{x}(s)\| + \int_s^t \alpha(\tau)\|x(\tau) - \hat{x}(\tau)\| d\tau. \end{aligned}$$

## 4. Asymptotic behavior of solution

We devote to study the asymptotic behavior of solution of (DE:$x_0$) which is obtained in Theorem 3.4 in this section.

THEOREM 4.1. *Let* (A.1)-(A.2) *hold and $x_0 \in D(A) \cap \overline{B_k(0)}$. If $x$ is a strong solution of* (DE:$x_0$) *for every interval $[0,T]$, then the curve $(x(t))_{t \ge 0}$ is* ANEC *in $H$.*

PROOF. For $\forall r \ge s$, we put $\hat{x}(t) = x(t + (r-s))$. Then $\hat{x}(t)$ is a strong solution of (DE:$x(r-s)$). Now, applying the inequality (5) in the above theorem, we have

$$\begin{aligned} \|x(t) - x(t+(r-s))\| &= \|x(t) - \hat{x}(t)\| \\ &\le \|x(s) - \hat{x}(s)\| + \int_s^t \|G(x(\tau)) - G(\hat{x}(\tau))\| d\tau \\ &\le \|x(s) - \hat{x}(s)\| + \int_s^t \alpha(\tau)\|x(\tau) - \hat{x}(\tau)\| d\tau \\ &= \|x(s) - x(s+(r-s))\| + \int_s^t \alpha(\tau)\|x(\tau) - x(\tau+(r-s))\| d\tau. \end{aligned}$$

It follows that

$$\|x(t) - x(t+(r-s))\| \le \|x(s) - x(r)\| \exp\left(\int_s^t \alpha(\tau) d\tau\right).$$

Putting $t = s + h$ for some $h \geq 0$

$$\begin{aligned}
&\|x(s+h) - x(r+h)\| \\
&\quad \leq \|x(s) - x(r)\| \exp(\int_s^{s+h} \alpha(\tau)d\tau) \\
&\quad = \|x(s) - x(r)\| + \|x(s) - x(r)\|(\exp(\int_s^{s+h} \alpha(\tau)d\tau) - 1) \\
&\quad \leq \|x(s) - x(r)\| + \epsilon(r, s).
\end{aligned}$$

Since $\alpha \in L^1([0, \infty) : \mathcal{R})$ and $\|x(t)\| \leq k$, we have $\epsilon(r, s) \to 0$ as $r, s \to \infty$. Hence $(x(t))_{t \geq 0}$ is ANEC in $H$. □

By the above theorem, we assure that our solution of (DE:$x_0$) is an ANEC under the conditions. Hence, using the basic properties of ANEC, we have the following results which are same as those of [3].

THEOREM 4.2. *Let* (A.1)-(A.2) *hold and* $x_0 \in D(A) \cap \overline{B_k(0)}$. *If* $x$ *is a strong solution of* (DE:$x_0$) *for every interval* $[0, T]$, *then* $\sigma_T = (1/T) \int_0^T x(t)dt$ *converges weakly as* $T \to \infty$ *to the asymptotic centre of the curve*$(x(t))_{t \geq 0}$.

PROOF. It follows from Lemma 2.3 and Theorem 4.1. □

THEOREM 4.3. *Let* (A.1)-(A.2) *hold and* $x_0 \in D(A) \cap \overline{B_k(0)}$. *We assume that* $x$ *is a strong solution of* (DE:$x_0$) *for every interval* $[0, T]$ *and* $\forall h \geq 0$, $x(t + h) - x(t) \rightharpoonup 0$ *as* $t \to \infty$. *Then* $x(t)$ *converges weakly as* $t \to \infty$ *to the asymptotic centre of the curve* $(x(t))_{t \geq 0}$.

PROOF. It follows from Lemma 2.4 and Theorem 4.1. □

THEOREM 4.4. *Let* (A.1)-(A.2) *hold and* $x_0 \in D(A) \cap \overline{B_k(0)}$. *We assume that* $x$ *is a strong solution of* (DE:$x_0$) *for every interval* $[0, T]$ *and* $\lim_{t \to \infty}(x(t), x(t + h)) = \beta(h)$ *exists uniformly in* $h \geq 0$. *Then* $\sigma_T = (1/T) \int_0^T x(t)dt$ *converges strongly* $T \to \infty$ *to the asymptotic centre of the curve* $(x(t))_{t \geq 0}$.

PROOF. From Lemma 2.5, it is obvious. □

REMARK 4.5. The existence of a bounded strong solution of the nonlinear functional differential equation

$$\text{(FDE)} \qquad x'(t) + A(t)x(t) \ni G(t, x_t), \quad t \in [0, \infty)$$

is already considered even reflexive Banach space (cf. Theorem 6 of [6]). Also, we have results for an unbounded(not necessary bounded) solution of (1) in general Banach space by Djafari Rouhani ([2]). In the sequel study, we have some results on behavior of solution of the type

$$\text{(FDE)} \qquad x'(t) + Ax(t) \ni G(x_t), \quad t \in [0, \infty)$$

in general Banach space using the above results.

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