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AN ESTIMATE OF THE AREA OF CONSTANT BREADTH CURVES

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ABSTRACT. We obtain new upper bound of the area of a constant breadth curve. We estimate the area of a constant breadth curve in terms of its breadth and minimal radius and the arc length of its pedal curve.

1. Introduction

A constant breadth curve can be rotated inside the fixed square without losing contact with each of its sides. It is obvious that this property completely characterizes constant breadth curves. That is, all chords of a pedal curve of a constant breadth curve through the origin have the same length. This means that the origin of a constant breadth curve is an equichordal point of its pedal curve. Convex bodies of constant breadth are of general interest and studied by many mathematicians. Good references in this line are Chakerian and Groemer [1] and Gruber and Wills [3].

In this paper, we develop a new upper bound of the area of a curve of constant breadth involving the arc length $L(P)$ of its pedal curve as follows:

THEOREM. *If C is a curve of constant breadth ω and P is a pedal curve of C, then*

$$A(C) \leq \pi(\omega^2 - 2\tilde{r}^2) - \frac{L^2(P)}{4\pi}.$$

The equality holds if and only if C is a circle.

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2. Curves of constant breadth

Let C be a simple closed convex plane curve. Then the breadth $\omega(\theta)$ in the direction θ is the distance between two parallel lines of support to C which are perpendicular to given direction and which contain C between them. Then the breadth of C in a given direction θ is $h(\theta)+h(\theta+\pi)$, where $h(\theta)$ is the support function of C. If $\omega(\theta)$ is constant, C is called a curve of constant breadth.

Let C_1, C_2 be analytic closed convex curves on the plane whose support functions are $h_1(\theta)$ and $h_2(\theta)$, respectively. The closed convex curve C_{12} whose support function is $h_1(\theta)+h_2(\theta)$ is called the mixed curve of C_1 and C_2. Then the enclosing area $A(C_{12})$ of C_{12} has the form $A(C_{12}) = A(C_1)+A(C_2)+2A(C_1,C_2)$. The quantity $A(C_1,C_2)$ is called the mixed area of C_1 and C_2.

LEMMA 1. *Let C be a plane curve of constant breadth ω; and let $-C$ be a curve obtained by rotating C through deg 180. And denote $A(C,-C)$ the mixed area of C and $-C$. Let $A(C)$ be the Euclidean area of C. Then*

$$A(C)+A(C,-C)=\frac{\omega^2}{2}\pi. \tag{1}$$

PROOF. Note that $A(C)=\frac{1}{2}\int_0^{2\pi}\{h^2(\theta)-h'^2(\theta)\}d\theta$. Thus

$$\begin{aligned} 2A(C) &= \int_0^{\pi}\{h^2(\theta)+h^2(\theta+\pi)-(h'^2(\theta)+h'^2(\theta+\pi))\}d\theta \\ &= \omega^2\pi-2A(C,-C). \end{aligned}$$

So $A(C)+A(C,-C)=\frac{\omega^2}{2}\pi$. □

DEFINITION 1. Assume that C is a convex curve with the origin O in its interior. For any line l through O there is a supporting line of C which is perpendicular to l at a point Q. As l turns about O, the trace of the point Q is called a pedal curve of C.

Let C be a simple closed convex plane curve with the origin O in its interior and P be a pedal curve of C. Then the radial function $x(\theta)$ of P is equal to the support function $h(\theta)$ of C. P is not convex, in general. But it is star shaped with respect to the origin. Let $\tilde{r}$ and $\tilde{R}$ be the minimal and maximal radius of C, respectively, i. e., $\tilde{r}=\min\{r(\theta)|0\le\theta\le 2\pi\}$

and $\tilde{R} = \max\{r(\theta) | 0 \leq \theta \leq 2\pi\}$. Then one can check that

$$\tilde{r} \leq h(\theta) \leq \tilde{R}. \tag{2}$$

Now we want to prove our main theorem.

THEOREM 1. *Let C be a curve of constant breadth ω and P be a pedal curve of C. Then*

$$A(C) \leq \pi(\omega^2 - 2\tilde{r}^2) - \frac{L^2(P)}{4\pi}. \tag{3}$$

The equality holds if and only if C is a circle.

PROOF. Note that

$$\begin{aligned} L(P) &= \int_0^{2\pi} \sqrt{x^2(\theta) + x'^2(\theta)}d\theta = \int_0^{2\pi} \sqrt{h^2(\theta) + h'^2(\theta)}d\theta \\ &= \int_0^{\pi} \{\sqrt{h^2(\theta) + h'^2(\theta)} + \sqrt{h^2(\theta+\pi) + h'^2(\theta+\pi)}\}d\theta. \end{aligned}$$

Using the inequality $x^2 + y^2 \geq \frac{1}{2}(x+y)^2$, we get

$$L(P) \leq \sqrt{2}\int_0^{\pi} \sqrt{h^2(\theta) + h^2(\theta+\pi) + h'^2(\theta) + h'^2(\theta+\pi)}d\theta. \tag{4}$$

Since $h(\theta) + h(\theta+\pi) = \omega$, constant value, the value of $h'(\theta) + h'(\theta+\pi)$ is always equal to zero. Thus the right side of (4) is equal to

$$\sqrt{2}\int_0^{\pi} \sqrt{\omega^2 + 2\{h(\theta)h(\theta+\pi) - h'(\theta)h'(\theta+\pi)\} - 4h(\theta)h(\theta+\pi)}d\theta.$$

Now by using Cauchy-Schwartz inequality, $\int fg \leq \sqrt{\int f^2}\sqrt{\int g^2}$, we have

$$L(P) \leq \sqrt{2\pi\{\omega^2\pi + 2A(C,-C) - 4\int_0^{\pi} h(\theta)h(\theta+\pi)d\theta\}}. \tag{5}$$

By the inequality (2),

$$L(P) \leq \sqrt{2\pi\{\omega^2\pi + 2A(C,-C) - 4\pi\tilde{r}^2\}}. \tag{6}$$

Lemma 1 and the inequality (6) give the inequality (3).

Now we prove the second part of the theorem. First if C is a circle of radius r, then $P = C$. Thus the equality holds in (3).

Conversely, assume the equality holds in (3). Then the equality holds in (4). And the equality holds in (4) if and only if

$$h^2(\theta) + h'^2(\theta) = h^2(\theta + \pi) + h'^2(\theta + \pi). \tag{7}$$

Let X be the point at which the support line m_θ in the direction θ meets C and Y be the point at which $m_{\theta+\pi}$ meets C. Then the equation (7) means that the distance $d(O, X)$ is equal to $d(O, Y)$ because $|h'(\theta)|$ is the distance from X to the point H_1 at which the θ-directional line l_θ meets m_θ and $|h'(\theta + \pi)|$ is the distance from Y to the point H_2 at which $l_{\theta+\pi}$ meets $m_{\theta+\pi}$. Thus $h^2(\theta) + h'^2(\theta) = d^2(O, H_1) + d^2(H_1, X) = d^2(O, X)$ and $h^2(\theta + \pi) + h'^2(\theta + \pi) = d^2(O, H_2) + d^2(H_2, Y) = d^2(O, Y)$. Let ϕ be a map from $[0, 2\pi]$ to C such that $\phi(\theta)$ is the contact point of θ-directional support line of C. Since C is convex and of constant breadth C contains no segment. Thus ϕ is continuous. Let X_{t_i} and Y_{t_i} be the points at which $m_{\theta+t_i}$ and $m_{\theta+\pi+t_i}$ meets C, respectively, for $i = 0, 1$. If X_{t_2} is a point between X_{t_0} and X_{t_1}, then the tangent line to C at X_{t_2} is a $(\theta + t_2)$-directional support line of C for some $\theta + t_2 \in [\theta + t_0, \theta + t_1]$. Thus ϕ is an onto map. And if θ moves from $\theta + t_0$ to $\theta + t_1$, $0 \le t_0 \le \pi$, then since C is convex and of constant breadth X moves from X_{t_0} to X_{t_1} and Y moves from Y_{t_0} to Y_{t_1} and $d(O, X_{t_i}) = d(O, Y_{t_i})$, $i = 0, 1$, respectively. And for any point X_t between X_{t_0} and X_{t_1}, there is a point Y_t between Y_{t_0} and Y_{t_1} such that $d(O, X_t) = d(O, Y_t)$. This means that the arc $\widehat{Y_{t_0}Y_{t_1}}$ can be obtained from the arc $\widehat{X_{t_0}X_{t_1}}$ by rotation, so that the arc $\widehat{YX}$ can be obtained from the arc $\widehat{XY}$ by rotation. This means that C is centrally symmetric. Since C is a curve of constant breadth, C is a circle. □

COROLLARY 1. *Let α be the area of the region which is the inside of P and the outside of C. Then*

$$L^2(P) - 4\pi A(P) \le 4\pi\alpha.$$

PROOF. If C is a curve of constant breadth, then

$$\begin{aligned}\int_0^\pi h(\theta)h(\theta+\pi)d\theta &= \frac{1}{2}\int_0^\pi \{(h(\theta) + h(\theta+\pi))^2 - (h^2(\theta) + h^2(\theta+\pi))\}d\theta \\ &= \frac{\omega^2}{2}\pi - A(P).\end{aligned}$$

Thus from (5) we get $L(P) \le \sqrt{2\pi\{4A(P) + 2A(C, -C) - \omega^2\pi\}}$. By the equation (1), we get $L^2(P) - 4\pi A(P) \le 4\pi\alpha$. □

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