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TOTALLY DISCONNECTED GROUPS, P-ADIC GROUPS AND THE HILBERT-SMITH CONJECTURE

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ABSTRACT. The following statement is known as the generalized Hilbert-Smith conjecture: If G is a compact group and acts effectively on a manifold, then G is a Lie group. In this paper we prove that the generalized Hilbert-Smith conjecture is equivalent to the following: A p-adic group cannot act effectively on a manifold. This fact is well known, but has never been published before.

1. Introduction

The following, known as the generalized Hilbert-Smith conjecture, which originated from Hilbert's 5-th problem, is the classic unresolved problem of topological transformation groups.

CONJECTURE : *If G is a compact group and acts effectively on a manifold, then G is a Lie group.*

It is known that this conjecture is equivalent to the each of the following:

(I) *A p-adic group cannot act effectively on a manifold.*

(II) *A compact 0-dimensional infinite group cannot act effectively on a manifold.*

Essentially the equivalence comes from the following statement which was made by F. Raymond in [10] without proof. The author wishes to thank F. Raymond for some helpful remarks. He also thanks the referee for valuable comments.

If G is a compact non-Lie group acting effectively on a manifold, then G contains a p-adic group for some prime p.

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We will give a detailed new proof of the above statement using basic concepts of transformation group theory and group theory. So that all the topologist can follow the proofs of this paper without any difficulty. We also refer the reader to [7], [10] which give several equivalent conditions to the Hilbert-Smith conjecture on a compact 3-manifold.

2. Preliminaries

In this section we will construct the p-adic transformation group using different methods and state well known propositions which we will use later.

Let p be a prime number and let D_p be the set of all formal series in powers of p:

$$g = a_0 + a_1 p + \dots + a_n p^n + \dots, \text{ each } a_n = 0, \dots, p-1.$$

If we add elements with infinite carry-over, then D_p forms an Abelian group and the topology is determined by the following choice of neighborhoods of the identity:

$$U_m = \{g \in D_p | a_i = 0 \text{ if } i < m\},\ m = 1, 2, \dots .$$

We call D_p the *p-adic group* and if D_p acts on a Hausdorff space M onto itself, as a group of homeomorphisms, we say that D_p is the *p-adic transformation group* acting on M.

A Cantor set can be defined as follows: Let

$$A_i = \{0, 1, \dots, p-1\} \text{ with discrete topology.}$$

If we give the product topology on $\prod_{i=1}^{\infty} A_i$, then $\prod_{i=1}^{\infty} A_i$ is homeomorphic to the Cantor set which is constructed by a geometric method [2, pg. 104] and it is totally disconnected and compact.

We define a map

$$\phi : \prod_{i=1}^{\infty} A_i \longrightarrow D_p, \text{ by } \phi(a_0, \dots, a_n, \dots) = (a_0 + \dots + a_n p^n + \dots).$$

Then the map ϕ is continuous and one to one since

$$\phi^{-1}(U_n) = \{< 0, 0, \dots, 0, a_n, a_{n+1}, \dots >\} \text{ for } U_n \subset D_p,$$

and the homeomorphism follows from the fact that D_p is a Hausdorff space and $\prod_{i=1}^{\infty} A_i$ is compact.

Another important construction of the p-adic group is the following: Let D_p be the p-adic group which we already constructed. Then

$$U_m = \{g \in D_p | a_i = 0 \text{ if } i < m\},\ m = 1, 2, \dots$$

form open subgroups and hence closed subgroups, since the cosets of U_m are open in D_p. We consider the sequence of quotient groups

$$D_p/U_0, D_p/U_1, \dots, D_p/U_n, \dots .$$

For $j > i$, let

$$h_{i,j} : D_p/U_j \longrightarrow D_p/U_i$$

be the homomorphisms defined by $gU_j \longrightarrow gU_i$. Then we have

$$D_p \simeq \varprojlim \{D_p/U_j\} \text{with bonding map } h_{i,j}.$$

We notice that D_p/U_i is a cyclic group of order p^i. Therefore we can also define the p-adic group as the inverse limit of cyclic groups of order p^i for $i = 1, 2, \dots$.

The following proposition plays an important role in the study of the Hilbert-Smith conjecture. It is from the following proposition that a totally disconnected compact group can be approximated by finite groups; i.e., it is the inverse limit of finite groups.

PROPOSITION 2.1. [8, pg. 56] *If G is a totally disconnected compact group and a neighborhood U of e is given, then there is a compact normal open subgroup $H \subset U$ such that G/H is a finite group.*

The following proposition, known as the Structural Theorem for Locally Compact Groups, was proved by Yamabe [14].

PROPOSITION 2.2. [14. See also 8, pg. 175] *If G is a compact non-Lie group, then there exists a sequence of compact invariant subgroups $H_1 \supset H_2 \supset \dots$ such that $\lim H_i = e$, giving Lie factor groups, and G is the inverse limit of $\{G/H_i\}$.*

PROPOSITION 2.3. [8, pg. 237] *Every n-dimensional locally compact group G, in some neighborhood U of the identity, is the direct product of a compact totally disconnected group H and a local n-parameter Lie group R;*

$$U = H \times R.$$

We may suppose that R is ruled by one-parameter local subgroups. If K is a compact invariant subgroup of G such that G/K is a Lie group, then $\dim G/K \leq n$.

PROPOSITION 2.4. [15] *If G is a compact group acting effectively on a manifold and if every element of G is of finite order, then G is a finite group.*

3. Main Theorem

In this section we prove the following theorem which was stated by Raymond [10] without proof.

THEOREM 3.1. *2 If G is a compact non-Lie group acting effectively on a manifold, then G contains a p-adic group for some prime p.*

From the following lemma, we can characterize the inverse limit group.

LEMMA 3.1. [3, pg. 61] *The inverse limit A^* of the inverse system $A = \{A_i\,(i \in I);\, \pi_i^j\}$ has this property: if G is a group and if there are homomorphisms $\sigma_i : G \to A_i$ with commutative diagrams*

$$\begin{array}{ccc} G & & \\ {\scriptstyle \sigma_j}\downarrow & \searrow{\scriptstyle \sigma_i} & \\ A_j & \xrightarrow{\pi_i^j} & A_i \end{array} \qquad (i \leq j),$$

then there exists a unique homomorphism $\sigma : G \to A^*$

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & A^* \\ & {\scriptstyle \sigma_i}\searrow & \downarrow{\scriptstyle \pi_i} \quad (i \in I) \\ & & A_i \end{array}$$

for which all the diagrams are commutative [where π_i is the canonical homomorphism]. This property characterizes A^* and π_i up to isomorphism.

THEOREM 3.2. *Let H be a compact totally disconnected infinite group acting effectively on a manifold. Then H must contain a p-adic group, for some prime p.*

PROOF. Let H be a compact totally disconnected infinite group acting effectively on a manifold. Note that H must contain an element $g \in H$ with infinite order [1]. We consider a sequence of compact normal subgroups, $H_1 \supset H_2 \supset H_3 \supset \ldots$ with $\lim H_i = e$, as in Proposition 2.2 and $\lim_{\leftarrow} H/H_i \cong H$. Let g_i denote the coset gH_i in H/H_i, and let A_i be the finite subgroup of H/H_i which is the homomorphic image of the cyclic group S, generated by g. Then A_i is cyclic, since A_i is the homomorphic image of S. Consider $\{A_i\}$ and the onto homomorphism $\pi_i^j|_{A_j}$, where π_i^j is the natural projection from $H/H_j \to H/H_i$. Now we take the inverse limit of $\{A_i\}$ with bonding maps $\pi_i^j|_{A_j}$, denoted by A^*. Then, $A^* \cong \overline{S}$ [12] and by the following theorem, Theorem 3.3, $A^* \cong \prod_{p_m} Z_{p_m^{e_m}}$, where p_m is a prime number which appears in the product of prime powers of the order of A_i for some i. We note that $e_m = \infty$ means that $Z_{p_m^{e_m}}$ is a p_m-adic subgroup.

Now we suppose that the group H does not contain any p-adic subgroup. Then e_m is finite for each m.

We apply the above argument to the compact normal subgroup H_i for all i. Therefore, for every H_i, H_i contains a subgroup which is isomorphic to $\prod_{p_k} Z_{p_k^{e_k}}$ where e_k is finite for each k, since the group H does not contain any p-adic subgroup.

Consequently, given any $\epsilon > 0$ there exists an element h_ϵ such that the orbit of h_ϵ has diameter less than ϵ, and the order of h_ϵ is finite. This contradicts Newman's theorem [9] and completes the proof. □

REMARK 3.1 Newman's theorem [9] for periodic homeomorphisms on manifolds has also been proved by A. Dress [1] and P. A. Smith [11]. See also [Br, p.154]. Later H. T. Ku [6] extended Newman's theorem to actions of p-adic solenoids.

The following theorem is a consequence of Theorem 25.22 in [5, pg. 411]. In this paper we give a relatively simple new proof.

THEOREM 3.3. *Let $\{A_i\}$ be a sequence of finite cyclic groups such that for $j > i$, the order of A_j is a multiple of the order of A_i. Let $\pi_i^j : A_j \to A_i$ be an onto homomorphism. Then $\lim_{\leftarrow} A_i \cong \prod_{p_m} Z_{p_m^{e_m}}$, where p_m is a* *prime number which appears in the product of prime powers of the order of A_i for some i, and p_m's are all different prime numbers.*

PROOF. Let n_i be the order of A_i with $n_i = p_1^{a_{1i}} p_2^{a_{2i}} \dots p_k^{a_{ki}}$. Then

$$A_i \cong Z_{p_1^{a_{1i}}} \bigoplus \dots \bigoplus Z_{p_k^{a_{ki}}}$$

and for $j > i$, the order of A_j, $n_j = p_1^{a_{1j}} p_2^{a_{2j}} \dots p_k^{a_{kj}} \dots p_l^{a_{lj}}$ with $p_m^{a_{mi}} \mid p_m^{a_{mj}}$ for $m = 1, 2, \dots, k$. $\square$

Now we consider the inverse limit of each column corresponding to a prime p_m where p_m is a prime number which appears in the product of prime powers of the order of A_i for some i. [Notation: We call a sequence of subgroups of A_i's, where the sequence is isomorphic to $\{Z_{p_m^{e_{m1}}}, Z_{p_m^{e_{m2}}}, \dots, Z_{p_m^{e_{mi}}} \dots\}$, the column corresponding to a prime p_m. See the example in Figure 3.1]

$$A_1 \cong Z_2$$

$$\uparrow \quad \uparrow$$

$$A_2 \cong Z_4 + Z_3$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$A_3 \cong Z_{16} + Z_9 + Z_7 + Z_{11}$$

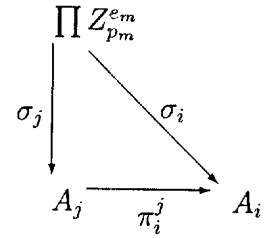
$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$A_4 \cong Z_{16} + Z_9 + Z_{49} + Z_{11} + Z_{19}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

FIGURE 3.1

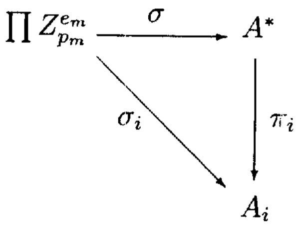
Then the inverse limit of each column corresponding to a prime p_m is isomorphic to $Z_{p_m{}^{e_m}}$, where $e_m = \infty$ means that $Z_{p_m{}^{e_m}}$ is a p_m-adic subgroup of the inverse limit $\lim_{\leftarrow} A_i$ with bonding map π_{mi}^j from the cyclic subgroup of A_j with order $p_m^{e_{mj}}$ to the cyclic subgroup of A_i with order $p_m^{e_{mi}}$ induced by π_i^j, which is a bonding map from A_j to A_i . We now show that the diagram commutes.



$(i \leq j)$,

Let $(a_1, a_2, \ldots) \in \prod_{p_m} Z_{p_m^{e_m}}$. Then $\sigma_i((a_1, a_2, \ldots) = (\pi_{1i}(a_1), \pi_{2i}(a_2), \ldots, \pi_{ki}(a_k))$ and $\sigma_j((a_1, a_2, \ldots) = (\pi_{1j}(a_1), \pi_{2j}(a_2), \ldots, \pi_{kj}(a_k), \ldots, \pi_{lj}(a_l))$. Then by the map $\pi_i^j : A_j \to A_i$, $\pi_i^j(\pi_{mj}(a_m)) = \pi_{mi}(a_m)$ for $m = 1, 2, \ldots, k$. So the diagram commutes.

Therefore, by Lemma 3.1, there exists a unique homomorphism $\sigma : \prod_{p_m} Z_{p_m^{e_m}} \to A^*$ such that the following diagram commutes, where $\sigma g = (\sigma_1(g), \sigma_2(g), \ldots)$.



$(i \in N)$

We show that σ is one to one. Let $a = (a_1, a_2, \ldots) \in Ker\, \sigma$. Then, by the commutativity of the diagram, we have $\sigma_i a = \pi_i \sigma a = 0$ for all i. Note that $\sigma_i a = (\pi_{1i}(a_1), \pi_{2i}(a_2), \ldots, \pi_{ki}(a_k)) = 0$. Therefore for all i and for all m, $\pi_{mi}(a_m) = 0$, and hence $a = 0$.

Now we show that σ is onto. Let $(b_1, b_2, \ldots) \in A^*$, where $b_i \in A_i$, $b_i = b_{1i} \bigoplus b_{2i} \bigoplus \cdots \bigoplus b_{mi}$. We define $b_{m*} = (b_{m1}, b_{m2}, b_{m3}, \ldots)$, where b_{mj} is the element of the column corresponding to p_m in A_j. Then b_{m*} is the element of the inverse limit induced by p_m column of $\{A_i\}$, which is isomorphic to $Z_{p_m{}^{e_m}}$. Then $(b_{1*}, b_{2*}, \ldots) \in \prod_{p_m} Z_{p_m^{e_m}}$. We claim that $\sigma(b_{1*}, b_{2*}, \ldots) = (b_1, b_2, \ldots)$. Note that $\sigma_i(b_{1*}, b_{2*}, \ldots) = (\pi_{1i}(b_{1*}), \pi_{2i}(b_{2*}), \ldots, \pi_{ki}(b_{k*})) = (b_{1i}, b_{2i}, \ldots, b_{ki}) = b_i$. Recall that $\sigma g = (\sigma_1(g), \sigma_2(g), \ldots)$. Therefore $\sigma(b_{1*}, b_{2*}, \ldots) = (\sigma_1(b_{1*}, b_{2*}, \ldots), \sigma_2(b_{1*}, b_{2*}, \ldots), \ldots) = (b_1, b_2, \ldots)$, so that σ is onto. This completes the proof. □

PROOF OF THEOREM 3.1. Let G be a compact non-Lie group. Then G contains a compact totally disconnected infinite subgroup H, by Proposition 2.3. Since G acts effectively, H also acts effectively. Thus H must contain p-adic group for some prime p, by Theorem 3.2. □

Now we show that the generalized Hilbert-Smith conjecture is equivalent to (I) and (II).

The generalized Hilbert-Smith conjecture is equivalent to (I): Let A_p be a p-adic group acting on a manifold M. Then, since A_p is a non-Lie group, A_p can not act effectively on M. Conversely, suppose there exists a compact group G acting effectively on M which is a non-Lie group. Then G contains a p-adic group for some prime p, by Theorem 3.1.

(I) is equivalent to (II): Suppose that there exists a compact 0-dimensional infinite group G acting effectively on M. Then, also by Theorem 3.1, G contains a p-adic group for some prime p. The converse is clear.

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