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SOME PROPERTIES ON SPACES WITH NONCOMPACT GROUP ACTION

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ABSTRACT. The compact transformation group theory has been developed with lots of properties. Many properties which are satisfied on G-space for compact group G do not hold for noncompact case. To recover some theory on spaces with noncompact group action we give some restriction on G-spaces. Hence we introduce Cartan G-spaces and proper G-spaces for our goal and we prove some properties on these G-spaces with noncompact G.

0. Introduction

Let G be a locally compact Lie group if there is no special note. We consider a completely regular space X with a fixed action on G. If G is a compact Lie group then a lot of general theory of G-spaces has been developed. For the noncompact case we need to give some condition on G-space for which theory can be applied reasonably. For our purpose, first we define Cartan G-space. Then many of the statements which hold when G is compact are valid in this case. Also we are interested in some properties of orbit space X/G which has induced properties from X. If G is a compact Lie group, then the orbit space has more properties. For instance, if X is a G-space with compact Lie group G, then

(i) X/G is Hausdorff.

(ii) $\pi : X \to X/G$ is closed.

For the noncompact group G, any of the above properties do not hold in general. However the first case (i) holds if we give more restriction on Cartan G-space which satisfies the condition that given x, y in X there exist relatively thin neighborhoods U and V of x and y. This is called

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proper G-space. In case (ii) if the space has Cartan G-action, the orbit map is closed.

In this paper, we prove some properties on the Cartan G-spaces and more restrictive proper G-spaces and of course those properties are satisfied on G-spaces for compact G. In Section One, we give some preliminaries which include basic notations and definitions as background. In the second section, we extend some theory of G-spaces where G is a compact Lie group to theory of Cartan(or proper) G-spaces for G a locally compact Lie group. In Section Three, we study some porperty of ENR (Euclidean Neighborhood Retract) with the G-action for locally compact Lie group G.

1. Preliminaries

We denote G a locally compact Lie group with identity e which acts on completely regular space X. If X is a G-space then we write G_x for the isotropy group at x. We mean Gx for the orbit of x and if $S \subset X$ we write GS for the saturation of S, i.e. $GS = \{gs \mid g \in G, s \in S\}$. We let X/G denote the set of orbits Gx of G on X. Let $\pi : X \to X/G$ be a natural map by taking $\pi(x) = Gx$. Then X/G endowed with the quotient topology and X/G is called orbit space of X. The quotient map $\pi : X \to X/G$ is continuous open map. For the subgroup H of G, $X^H = \{x \in X \mid hx = x \ \forall \ h \in H\}$ is a fixed set under H. We define subsets of G

$$((U, V)) = \{g \in G \mid gU \cap V \neq \emptyset\}$$

where U and V are subsets of G-space X.

DEFINITION 1.1. If U and V are subsets of a G-space X then we say that U is thin relative to V if $((U, V))$ has compact closure in G. If U is thin relative to itself then we say that U is thin.

If U is thin relative to V then V is thin relative to U since $(gU \cap V) = g(U \cap g^{-1}V)$, hence U and V are relatively thin. If U and V are relatively thin then any translates g_1U and g_2V are also relatively thin and for $U' \subset U$, $V' \subset V$, U' and V' are relatively thin.

DEFINITION 1.2. A G-space X is Cartan G-space if every point of X has a thin neighborhood.

Definition 1.3. A subset S of a G-space X is a small subset of X if each point of X has a neighborhood which is thin relative to S. A G-space X is proper if each point of X has a small neighborhood.

If X is a G-space and A is an invariant subspace then a small subset of A is not necessarily a small subset of X. Small is relative notion unlike absolute concept of thin.

We state some important relation between Cartan G-space and proper G-space.

Proposition 1.4. [6] *A G-space X is proper if and only if X is a Cartan G-space and X/G is regular.*

Proof. See [6] (1.2.3, 1.2.8, and 1.2.5). □

Theorem 1.5. [6] *If X is a locally compact G-space then the following are equivalent.*

(a) *Given x, y in X there exist relatively thin neighborhoods U and V of x and y.*
(b) *X is a Cartan G-space and X/G is Hausdorff.*
(c) *X is a proper G-space.*
(d) *Every compact subset of X is small.*
(e) *Every compact subset of X is thin.*

2. Some Properties on G-space for Locally Compact Lie Group G

Now we prove some properties on Cartan G-space.

Proposition 2.1. *Let X be a Cartan G-space. Then the orbit map $\pi : X \to X/G$ is closed.*

Proof. Let A be closed in X. Then $\pi^{-1}(\pi(A)) = \{ga \mid g \in G, a \in A\} = G(A)$, saturation of A. Since π is a quotient map, it is enough to show $G(A)$ is closed in X. For any $a \in A$, there exists a net of points of A converging to a, say the net $\{a_\alpha\}$. Let y be adherent to $G(A)$ and let U be a thin neighborhood of y. We choose a net $\{g_\alpha a_\alpha\}$ in U converging to y. For fixed α_0, $(g_\alpha g_{\alpha_0}^{-1})(g_{\alpha_0} a_\alpha) = g_\alpha a_\alpha$ and hence $g_\alpha g_{\alpha_0}^{-1} \in ((U, U))$. By passing to a subnet, we can suppose that $g_\alpha g_{\alpha_0}^{-1}$ converges and hence that g_α converges, say to g. Then $y = \varinjlim g_\alpha a_\alpha = ga \in G(A)$. Therefore $G(A)$ is closed. □

PROPOSITION 2.2. *If X is a locally compact space then X/G is locally compact.*

PROOF. Let $\pi : X \to X/G$ be an orbit map. For $x \in U \subset X$, let $\overline{U}$ be a compact closure of U then $\pi(x) \in \pi(U) \subset \pi(\overline{U})$ where $\pi(\overline{U})$ is a compact closure containing $\pi(x)$. □

PROPOSITION 2.3. *If X is a proper G-space and N is a closed normal subgroup of G, then X^N is a proper G/N-space.*

PROOF. Since X is a proper G-space, X/G is regular by Proposition 1.4. For every $x \in X$, x has a thin neighborhood U such that $((U, U))$ is relatively compact in G. Recall G/N acts on X^N by $(gN)(x) = gNx = gx$. Then G/N action on X^N is equivalent to G-action on X^N and every subspace of a regular space is regular, and hence $X^N/(G/N)$ is regular. To show for every $x \in X^N$, x has a thin neighborhood U^* such that $((U^*, U^*))$ is relatively compact in G/N, we take $U^* = \{x \in U \mid nx = x$ for every $n \in N\} = U \cap X^N$ which is open in X^N. Moreover if p is the canonical map of G onto G/N it can be easily checked $p((U, U)) = ((U^*, U^*))$ since

$$\begin{aligned}((U^*, U^*)) &= \{gN \mid gNU^* \cap U^* \neq \emptyset\} \\ &= \{gN \mid gN(U \cap X^N) \cap (U \cap X^N) \neq \emptyset\}.\end{aligned}$$

□

DEFINITION 2.4. X is a Hilbert G-space if X is a real Hilbert space and each operation of G on X is an orthogonal linear transformation.

DEFINITION 2.5. If d is a metric for a G-space X then d is called invariant if $d(gx, gy) = d(x, y)$ for all $g \in G$ and $x, y \in X$, i.e. if each operation of G is an isometry.

THEOREM 2.6. [6] *If G is a Lie group and X is a separable, metrizable, proper G-space, then X admits an equivariant imbedding in a Hilbert G-space.*

COROLLARY 2.7. *Every separable, metrizable, proper G-space X admits an invariant metric.*

PROOF. This is an easy consequence of Theorem 2.6. □

LEMMA 2.8. *In a metric space X, X is separable if and only if X is a Lindelöf space.*

PROOF. Refer to point set topology. □

PROPOSITION 2.9. *Let X be a proper G-space. If X is separable metric then X/G is also separable metric.*

PROOF. By Corollary 2.7, let ρ be an invariant metric defined on X. We define

$$\bar{\rho}(\bar{x},\bar{y}) = inf\{\rho(x',y') \mid x' \in \bar{x},\ y' \in \bar{y} \text{ for } \bar{x},\ \bar{y} \in X/G\}.$$

Then since ρ is a metric, $\bar{\rho}(\bar{x},\bar{y}) \geq 0$ and $\bar{x} = \bar{y}$ if and only if $\bar{\rho}(\bar{x},\bar{y}) = 0$. Now

$$\begin{aligned}\bar{\rho}(\bar{x},\bar{y}) &= inf\{\rho(gx,g'y) \mid g,g' \in G\}\\ &\leq inf\{\rho(gx,g'y) + \rho(g'y,g''z) \mid g,g',g'' \in G\}\\ &= inf\{\rho(gx,g'y) \mid g,g' \in G\} + inf\{\rho(g'y,g''z) \mid g',g'' \in G\}\\ &= \bar{\rho}(\bar{x},\bar{y}) + \bar{\rho}(\bar{y},\bar{z}).\end{aligned}$$

Therefore $\bar{\rho}$ is a metric on X/G. $\bar{\rho}$ is induced from ρ and $\pi : X \to X/G$ is a continuous open map, hence $\bar{\rho}$ is consistent with the topology of X/G. Since X is a metric space, X is separable implies X is a Lindelöf space. The Lindelöf property is invariant under continuous surjections. Hence X/G is Lindelöf and also separable since X/G is a metric space. □

3. Application to G-ENR

Let G be a locally compact Lie group. We define a G-ENR(Euclidean Neighborhood Retract) to be a G-space X which is (G-homeomorphic to) a G-retract of some open G-subset in a G-module V. If we have no group G acting we simply talk about ENR's. We recall some local property of ENR. A space X is called locally contractible if every neighborhood V

of every point $x \in X$ contains a neighborhood W of x such that $W \subset V$ is null homotopic fixing x. We can see ENR is locally contractible [3]. A space is locally n-connected if every neighborhood V of every point x contains a neighborhood W such that any map $S^j \to W$, $j \leq n$ is null homotopic in V.

PROPOSITION 3.1. [3] *If $X \subset R^n$ is locally $(n-1)$ connected and locally compact then X is an ENR.*

REMARK. A separable metric space of dimension $\leq n$ can be embedded in R^{2n+1} [5]. Hence a space is an ENR if and only if it is locally compact, separable metric, finite dimensional and locally contractible.

PROPOSITION 3.2. *Let X be a proper G-ENR. Then the orbit space X/G is an ENR.*

PROOF. Since X is G-ENR, X is a retract of some open G subset in a G-module, i.e. $X \xrightarrow{i} U \xrightarrow{r} X$ and $r \circ i = id$. A retract of an ENR is an ENR. Hence we prove the proposition for X a differential G-manifold and then apply it to the manifold U. Let $\pi : X \to X/G$ be the quotient map. Then X/G is locally compact by Proposition 2.2 and separable metric by Proposition 2.9. By dimension theory [5] $dim X/G \leq dim X$. Hence X/G is finite dimensional. Let $x \in U \subset X/G$, U open. Then $\pi^{-1}(U)$ contains tubular neighborhood W of $\pi^{-1}(x)$. Hence $\pi(W)$ is contractible in X/G. Therefore X/G is locally contractible. By the above remark, we complete the proof. □

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