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PROBABILITY INEQUALITIES FOR PRODUCT OF SYMMETRIZED POISSON PROCESSES AND THEIR APPLICATIONS

JOONG SUNG KWON

ABSTRACT. This paper considers the problem of deriving exponential probability inequalities for product symmetric Poisson processes. As an application they are used to show the existence of regular version of some product process derived from Lévy process.

1. Introduction

Let $(\mathbf{S}, \mathcal{S}, P)$ be a finite measure space. And let $(\Omega, \mathbf{P})$ denote an underlying probability space. Let $\mathcal{F}$ be a set of $\mathcal{S}$-measurable functions on $\mathbf{S}$. A process $Y = \{Y(f) | f \in \mathcal{F}\}$ on $\mathbf{S}$ is said to be a *Poisson process with parameter* $\bar{\lambda}$, where $\bar{\lambda} = P(\mathbf{S})$ if it has independent increments, in the sense that $Y(A_1)$, $Y(A_2)$, ... $Y(A_k)$ are independent whenever A_1, A_2, ... A_k are disjoint subsets of $\mathbf{S}$, and the marginal distributions are Poisson with parameters $P(A_k)$. We can represent such a Poisson process as follows : Let $\{U_i\}$ be a sequence of independent identically Q-distributed $\mathbf{S}$-valued random variables (the locations of points), where $Q = \frac{P}{P(\mathbf{S})}$ and $N = Y(\mathbf{S})$ (the number of points) be a Poisson random variable with parameter $\bar{\lambda} = P(\mathbf{S})$, independent of $\{U_i\}$. Then, for a function f on $\mathbf{S}$, define

$$Y(f) = \sum_{i=1}^{N} f(U_i). \tag{1.1}$$

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When we replace (1.1) by

$$\overline{Y}(f) = \sum_{i=1}^{N} \varepsilon_i f(U_i)$$

where $\{\varepsilon_i\}$ is a sequence of identically distributed symmetric Bernoulli random variables ($\mathbf{P}(\varepsilon_i = 1) = 1/2 = \mathbf{P}(\varepsilon_i = -1)$) which is independent of $\{U_i\}$ and N, we have a symmetrized Poisson process. Similarly replacing $\{\varepsilon_i\}$ by $\{X_i\}$, a sequence of identically distributed random variables, independent of $\{U_i\}$ and N, gives a compound Poisson process with marginal distribution

$$\hat{Y}(f) = \sum_{i=1}^{N} X_i f(U_i).$$

Let $(\mathbf{S}_i, \mathcal{S}_i, P_i)$, $i = 1, 2$ be finite measure spaces. Let $\{\varepsilon_i\}$ and $\{\varepsilon'_j\}$ be independent sequences of identically distributed symmetric Bernoulli random variables. And let $\{U_i\}$ and $\{V_j\}$ be sequences of identically Q_1- and Q_2- distributed $\mathbf{S}_1$- and $\mathbf{S}_2$- valued random variables respectively, where $Q_i = \frac{P_i}{P_i(\mathbf{S}_i)}$, $i = 1, 2$. Finally let $\{X_i\}$ and $\{X'_j\}$ be independent sequences of identically distributed random variables. Let Y_1 and Y_2 denote two independent Poisson processes on $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively. For a function f on $\mathbf{S}_1 \times \mathbf{S}_2$, we define

$$Y_1 \times Y_2(f) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} f(U_i, V_j), \tag{1.2}$$

$$\overline{Y}_1 \times \overline{Y}_2(f) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \varepsilon_i \varepsilon'_j f(U_i, V_j), \tag{1.3}$$

and

$$\hat{Y}_1 \times \hat{Y}_2(f) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} X_i X'_j f(U_i, V_j), \tag{1.4}$$

where $N_i = P_i(\mathbf{S}_i)$, $i = 1, 2$. These product processes naturally arise as building blocks of product of infinitely divisible(ID) processes when we use series representation for ID processes. However they are not studied enough to be understood. In this direction, some works for probability bounds have been done in [6] for (1.2) and in [7] for product Poisson processes in full generality. In section 2, in this paper, we will derive probability inequalities for (1.3) and (1.4).

In section 3, we will consider a special process which we call the product diagonal process. It arises in the study of product Lévy processes:

$$\sum_{n=0}^{\infty} \hat{Z}_n \times \hat{Z}'_n$$

where $\hat{Z}_n \times \hat{Z}'_n$ is of the type as in (1.4) with some constraints on $\{X_i\}$ and $\{X'_j\}$. Clearly $\hat{Z}_n \times \hat{Z}'_n$ is a random measure on $(\mathbf{S}_1 \times \mathbf{S}_2, \mathcal{S}_1 \times \mathcal{S}_2)$ but whether $\sum_{n=0}^{\infty} \hat{Z}_n \times \hat{Z}'_n(\cdot)$ is a random measure is quite far from clear. Our study is to find the biggest possible sub-family of σ-algebra defined on $\mathbf{S}_1 \times \mathbf{S}_2$ as possible which satisfies some measure like properties. In that section we will show the existence of regular version of product diagonal process. For this we use metric entropy to measure the size of the index families. In this context, a natural regularity on sample paths of processes is measure-like "outer continuity and inner limits" property that was used in [1] and [3].

2. Probability Inequalities

In this section we state and prove probability bounds for product of symmetric Poisson processes. Assume throughout that generic constants K, K_1, K_2 etc. can be different, in which they appear in the context. But it should not cause the problem.

Let $\{\varepsilon_i : i = 1, 2, \ldots\}$, $\{\varepsilon'_j : j = 1, 2, \ldots\}$, $\{U_i : i = 1, 2, \ldots\}$ and $\{V_j : j = 1, 2, \ldots\}$ be defined as in section 1. For any $\sigma(\mathcal{S}_1 \times \mathcal{S}_2)$-measurable function f with $|f| \le 1$ and for $m, n \ge 0$, write

$$S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_i \varepsilon'_j f(U_i, V_j).$$

and

$$\overline{S}_{mn} = (mn)^{-1/2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_i \varepsilon'_j f(U_i, V_j).$$

in which empty sums are taken to be zero. Define $\mathcal{F}_{mn}$ to be the σ-algebra generated by ε'_j, U_j, and V_j, $i \le m$ and $j \le n$. And define $\mathcal{F}^{(1)}_{mn}$ to be the σ-algebra generated by U_i and V_j, $i \le m$, and $j \le n$. Finally denote

$$\Lambda^2_{mn} := (mn)^{-2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\sum_{i=1}^{m} f(U_i, V_j) f(U_i, V_k) \right)^2 .$$

Under the developments, the first lemma is as follows.

LEMMA 2.1 ([7]). *Under the above notations, for $\eta > 0$ and for some constants K_1 and K_2, we have*

$$\mathbf{P}(\overline{S}_{mn} > \eta | \mathcal{F}^{(1)}_{mn}) \le \exp\left(-\frac{K_1}{{\Lambda_{mn}}^{1/2}} \eta \right) + \exp\left(-\frac{K_2}{\Lambda^{2/3}_{mn}} \eta^{4/3} \right).$$

LEMMA 2.2. *For $\eta > 0$, $0 \le f \le 1$ and $\delta = E(f(U_i, V_j))$, we have*

$$\mathbf{P}(\overline{S}_{mn} > \eta) \le \exp\left(-\frac{K_1}{\delta^{1/4}} \eta \right) + \exp\left(-\frac{K_2}{\delta^{1/3}} \eta^{4/3} \right) + \exp\left(-8 \min(m, n) \delta^{1/2} \right).$$

PROOF. By Hölder's inequality we have $\Lambda^2_{mn} \le T^2_{mn}$ with

$$T_{mn} := (mn)^{-1} \sum_{j=1}^{n} \sum_{i=1}^{m} f(U_i, V_j).$$

Hence by Theorem 2.6 [6],

$$\mathbf{P}(\Lambda_{mn} > \delta + \gamma) \le \exp\left\{ -\frac{\min(m, n)\gamma^2}{2(\sigma^2_f + \gamma/3)} \right\},$$

where $\sigma_f^2 = \mathrm{Var}(f(U_i, V_j))$. Take $\gamma = 8\delta^{1/2}$. Then

$$\frac{\min(m,n)\gamma^2}{2(\delta_f^2 + \gamma/3)} \geq \frac{64\min(m,n)\delta}{2(\delta^{1/2} + 8\delta^{1/2}/3)} \geq 8\min(m,n)\delta^{1/2}$$

and

$$\mathbf{P}(\Lambda_{mn} > 9\delta^{1/2}) \leq \exp\left(-8\min(m,n)\delta^{1/2}\right).$$

Thus

$$\mathbf{P}(\overline{S}_{mn} > \eta) \leq E\mathbf{P}\left([\overline{S}_{mn} > \eta] \cap [\Lambda_{mn} \leq 9\delta^{1/2}] | \mathcal{F}_{mn}^{(1)}\right) + \mathbf{P}\left(\Lambda_{mn} > 9\delta^{1/2}\right)$$
$$\leq \exp\left(-\frac{K_1}{3\delta^{1/4}}\eta\right) + \exp\left(-\frac{K_2}{3^{4/3}\delta^{1/3}}\eta^{4/3}\right) + \exp\left(-8\min(m,n)\delta^{1/2}\right).$$

□

Now we are ready to state and prove our main probability bound of the paper.

THEOREM 2.3. *Let $\overline{Y}_1$ and $\overline{Y}_2$ be independent symmetric Poisson Processes on $\mathbf{S}_1$ and $\mathbf{S}_2$ with parameters λ and μ respectively. Let f be a $\sigma(\mathcal{S}_1 \times \mathcal{S}_2)$−measurable function with $0 \leq f \leq 1$ such that $\delta = E(f(U_i, V_j)) \leq 1/64$. Then, for all $\eta > 0$ and for some constant K,*

$$\mathbf{P}(\overline{Y}_1 \times \overline{Y}_2(f) > \eta) \leq \exp\left(-K\frac{\eta}{(\lambda\mu)^{1/2}\delta^{1/4}}\right) + \exp\left(-K\frac{\eta^{4/3}}{(\lambda\mu)^{2/3}\delta^{1/3}}\right)$$
$$+ 4\exp\left(-4\min\{\lambda,\mu\}\delta^{1/2}\right).$$

PROOF. By Lemma 2.2,

$$\mathbf{P}(\overline{Y_1} \times \overline{Y_2}(f) > \eta) = E[\mathbf{P}(\overline{S}_{N_1 N_2} > \frac{\eta}{(N_1 N_2)^{1/2}} | N_1, N_2)]$$
$$\leq E\left(\exp\left(-K\frac{\eta}{(N_1N_2)^{1/2}\delta^{1/4}}\right)\right) + E\left(\exp\left(-K\frac{\eta^{3/4}}{(N_1N_2)^{2/3}\delta^{1/3}}\right)\right)$$
$$+ E\left(\exp\left(-8\min\{N_1,N_2\}\delta^{1/2}\right)\right)$$
$$:= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.$$

If we take $M = e^2(\lambda\mu)^{1/2}$, we have

$$\begin{aligned}
\mathrm{I}_1 &:= E\left\{\exp\left(-K\frac{\eta}{(N_1N_2)^{1/2}\delta^{1/4}}\right) : (N_1N_2)^{1/2} < M\right\} \\
&\quad + E\left\{\exp\left(-K\frac{\eta}{(N_1N_2)^{1/2}\delta^{1/4}}\right) : (N_1N_2)^{1/2} \geq M\right\} \\
&\leq \exp(-K\frac{\eta}{M\delta^{1/4}}) + P(N_1N_2 \geq M^2) \\
&\leq \exp\left(-\frac{K}{e^2}\frac{\eta}{(\lambda\mu)^{1/2}\delta^{1/4}}\right) + \exp(-e^2\lambda) + \exp(-e^2\mu)
\end{aligned} \tag{2.1}$$

where Lemma 2.2 [3] is used in the last inequality. Similarly,

$$\mathrm{I}_2 \leq \exp\left(-\frac{K}{e^{3/8}}\frac{\eta^{3/4}}{(\lambda\mu)^{2/3}\delta^{1/3}}\right) + \exp(-e^2\lambda) + \exp(-e^2\mu). \tag{2.2}$$

Finally, we have

$$\begin{aligned}
\mathrm{I}_3 \leq E\left(\exp(-8N_1\delta^{1/2})\right) &= \exp\left(\lambda\left(\exp\left(-8\delta^{1/2}\right) - 1\right)\right) \\
&\leq \exp\left(-4\lambda\delta^{1/2}\right)
\end{aligned} \tag{2.3}$$

and

$$\mathrm{I}_3 \leq E\left(\exp\left(-8N_2\delta^{1/2}\right)\right) \leq \exp\left(-4\mu\delta^{1/2}\right). \tag{2.4}$$

Summing (2.1), (2.2), (2.3) and (2.4) up completes the proof. □

The above proof extends immediately to give bounds for the product of symmetric compound Poisson processes having uniformly bounded jumps. Without loss of generality assume that they are bounded by 1, namely $0 < X_i, X'_j \leq 1$ and write

$$\hat{Z}_1 \times \hat{Z}_2(f) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \varepsilon_i\varepsilon'_j X_i X'_j f(U_i, V_j).$$

THEOREM 2.4. *Under the above setups, let f be a $\sigma(\mathcal{S}_1 \times \mathcal{S}_2)-$measurable function with $0 \le f \le 1$ such that $\delta = E(f(U_i, V_j)) \le 1/64$. Then for $\eta > 0$ and for some constant $K > 0$,*

$$\mathbf{P}(\hat{Z}_1 \times \hat{Z}_2(f) > \eta) \le \exp\left(-K\frac{\eta}{(\lambda\mu)^{1/2}\delta^{1/4}}\right) + \exp\left(-K\frac{\eta^{3/4}}{(\lambda\mu)^{2/3}\delta^{1/3}}\right) + 4\exp\left(-4\min\{\lambda, \mu\}\delta^{1/2}\right).$$

3. Application

Let $\mathcal{B}(\mathbf{I}^d)$ be the Borel $\sigma-$algebra defined on $\mathbf{I}^d = [0,1]^d$ and let $\mathcal{A}$ be a sub-family of $\mathcal{B}(\mathbf{I}^d)$. Given a Borel set $A \subset \mathbf{I}^d$, let A^0 be the interior of A with respect to the relative topology on $\mathbf{I}^d$, and let $A^\delta = \{\mathbf{t} \in \mathbf{I}^d :$ the Euclidean distance from $\mathbf{t}$ to A is less than $\delta\}$ be the open $\delta-$neighborhood around A. Define the Hausdoff metric d_H by

$$d_H(A, B) = \inf\{\varepsilon : A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\}.$$

DEFINITION 3.1. A sub-family $\mathcal{A}$ of $\mathcal{B}(\mathbf{I}^d)$ is called totally bounded with inclusion with respect to d_H if

(1) $\mathcal{A}$ is closed with respect to d_H.
(2) for each $\delta > 0$, there is a finite subset $\mathcal{A}_\delta$ of $\mathcal{A}$ such that whenever $A \in \mathcal{A}$ there exists $B \in \mathcal{A}_\delta$ with $A \subset B^0 \subset B \subset A^\delta$.

It is known that $\mathcal{A}$ is compact if $\mathcal{A}$ is totally bounded with inclusion. If $\mathcal{A}$ is totally bounded, a restriction on the size of $\mathcal{A}$ will be given through its log-entropy H, where $H(\delta)$ is defined to be the logarithm of the cardinality $(\delta, \mathcal{A}, d_H)$ of the smallest $\delta-$net $\mathcal{A}_\delta$. Define the exponent of metric entropy of $\mathcal{A}$ as follows:

$$r := \inf\{s > 0 | H(\delta) = O(\delta^{-s}) \quad \delta \to 0\}.$$

When $\mathcal{A}$ is the family of closed convex sets of $\mathbf{I}^d$, it is known that $r = (d-1)/2$(cf. [3]).

DEFINITION 3.2. A set function $\psi : \mathcal{A} \to \mathbf{R}$ is said to have outer continuity and inner limits at $A \in \mathcal{A}$ if

(1) $\psi(A_n) \to \psi(A)$ for any sequence $\{A_n\} \subset \mathcal{A}$ such that $A_n \supset A$ for all n and $\lim_{n\to\infty} d_H(A_n, A) = 0$.
(2) $\lim_{n\to\infty} \psi(A_n)$ exists for any sequence $\{A_n\} \subset \mathcal{A}$ such that $A_n \subset A^0$ for all n and $\lim_{n\to\infty} d_H(A_n, A^0) = 0$, where A^0 is the interior of A.

The space of all functions $\psi : \mathcal{A} \to \mathbf{R}$ that have outer continuity and inner limits at each $A \in \mathcal{A}$ is denoted by $\mathcal{D}(\mathcal{A})$.

DEFINITION 3.3. A process $Z = \{Z(B) : B \in \mathcal{B}(\mathbf{I}^d)\}$ is said to be a Lévy process indexed by $\mathcal{A}$ and having Lévy measure ν if it is an ID process with Lévy measure ν and if the sample paths are almost surely in $\mathcal{D}(\mathcal{A})$.

In [9], Pyke proposed one problem as follows : Let $\{\mu_n\}$ and $\{\nu_n\}$ be sequences of signed measures on measurable spaces $(R, \mathcal{R})$ and $(T, \mathcal{T})$ respectively. Write $(S, \mathcal{S})$ for the product measurable space $(R \times T, \mathcal{R} \times \mathcal{T})$. Under what conditions on $\{\mu_k, \nu_k\}$ and for what domains $\mathcal{A} \subset \mathcal{S}$ does the series $\rho = \sum_{k=1}^{\infty} \mu_k \times \nu_k$ converge with respect to $\|\cdot\|_{\mathcal{A}}$? where $\|\cdot\|_{\mathcal{A}}$ denotes the sup-norm defined by $\|\psi\|_{\mathcal{A}} = \sup_{A\in\mathcal{A}} |\psi(A)|$ for a set function $\psi : \mathcal{A} \to \mathbf{R}$.

In this section we will consider special random measures to give a partial answer to a variant of the problem.

Let Z_1 and Z_2 be independent symmetric stable processes with the same stable index $\alpha \in (1, 2)$ with Lévy(stable) measure $\nu(dx) = |x|^{-\alpha-1}$ dx, indexed by $\mathcal{A}_1$ ($\subset \mathcal{B}(\mathbf{I}^{d_1})$) and $\mathcal{A}_2$ ($\subset \mathcal{B}(\mathbf{I}^{d_2})$), respectively. Assume that the drift and Gaussian parts of Z_1 and Z_2 are zero. Then these Lévy processes have only jump discontinuities and have no fixed point of discontinuities. The basic structure of the non-Gaussian part of an Lévy processes is that of a convergent series of compound Poisson processes. Let $a_n = \beta^n$, $0 < \beta < 1$, $n \geq 1$ be a sequence of real numbers with $a_0 = +\infty$. We will stratify the heights of jumps of Z_i, $i = 1, 2$ by the sequence $\{a_n\}$ and we will refer to the sequence $\{a_n\}$ as a sequence of stratifications. Then, following the notations of [3], we will be able to write

$$Z_1 = \sum_{n=0}^{\infty} \hat{Z}_n, \quad \text{and} \quad Z_2 = \sum_{n=0}^{\infty} \hat{Z'}_n, \tag{3.1}$$

where the $\hat{Z}_n$'s and $\hat{Z'}_n$'s are independent symmetric compound Poisson processes with 'parameters' $\int_{a_n<|x|\leq a_{n-1}} \nu(dx)$, which equals the average number of jumps of $\hat{Z}_n$. Since Z_i is symmetric we can represent $\hat{Z}_n$ and $\hat{Z'}_n$ as follows;

$$\hat{Z}_n = \sum_{i=1}^{N_n} \varepsilon_{n_i}|X_{n_i}|\delta_{U_{n_i}}, \quad \text{and} \quad \hat{Z'}_n = \sum_{j=1}^{N'_n} \varepsilon'_{n_j}|X'_{n_j}|\delta_{V_{n_j}}, \tag{3.2}$$

where $\{\varepsilon_{n_i}\}$ and $\{\varepsilon'_{n_j}\}$ are families of independent identically distributed signed Bernoulli random variables, and $\{U_{n_i}\}$ and $\{V_{n_j}\}$ are families of independent identically distributed uniform random variables on $\mathbf{I}^{d_1}$ and $\mathbf{I}^{d_2}$ respectively, such that $\delta_{U_{n_i}}(A) = 1$ or 0 depending on $U_{n_i} \in A$ or not, similiary for $\delta_{V_{n_j}}$. Notice that $\{\varepsilon_{n_i}\}$, $\{\varepsilon'_{n_j}\}$ $\{U_{n_i}\}$, $\{V_{n_j}\}$ $\{X_{n_i}\}$ and $\{X'_{n_j}\}$ are all muturally independent families of suitable random variables. Clearly $\hat{Z}_n \times \hat{Z'}_n$, product of compound Poisson process, is a random measure. In this section we will study the problem of constructing a version of infinite series $\sum_{n=1}^{\infty} \hat{Z}_n \times \hat{Z}'_n$ of product compound Poisson processes on as large a sub-family $\mathcal{A}$ of the $\sigma-$field $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ as possible which is also in $\mathcal{D}(\mathcal{A})$. For this we list assumptions on $\mathcal{A} \subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ that will be used.

For any positive sequence $\{b_n\}$ with $b_n \to 0$ as $n \to \infty$, we have

(1) there exists a finite set $\mathcal{A}_{b_n} \subset \mathcal{A}$ such that for any $A \in \mathcal{A}$, there exist $A_{b_n}, A^+_{b_n} \in \mathcal{A}_{b_n}$ such that $A_{b_n} \subset A \subset A^+_{b_n}$ and $K_c b_n \leq |A^+_{b_n} \setminus A_{b_n}| \leq b_n$ and

(2) for any $A \in \mathcal{A}$, there exists a sequence of nets $A_n \in \mathcal{A}_{b_n}$ such that $|A \triangle A_n| \leq b_n$ and $K_c b_n \leq |A_{n-1} \triangle A_n| \leq b_n$, where K_c is some constant depending on the choice of $\{b_n\}$ and $|\cdot|$ is Lebesgue measure.

Note that (1) and (2) are more than 'totally bounded with inclusion' in the sense that: this assumption requires that approximating sets are not too close(this concept also was used in [2]).

THEOREM 3.4. *Let Z_1 and Z_2 be independent symmetric stable processes with the same stable index $\alpha \in (1,2)$ indexed by $\mathcal{A}_1$ $(\subset \mathcal{B}(\mathbf{I}^{d_1}))$ and $\mathcal{A}_2$ $(\subset \mathcal{B}(\mathbf{I}^{d_2}))$, respectively. Assume that Z_1 and Z_2 be represented as (3.1) with a geometric stratifying sequence $\{a_n\}$ and that $\mathcal{A}(\subset \sigma(\mathcal{A}_1 \times \mathcal{A}_2))$ with entropy index r satisfies (1) and (2) listed above. Finally assume that α and r satisfy $r < \frac{2-\alpha}{2(\alpha-1)}$. Then the infinite series $\sum_{n=1}^{\infty} \hat{Z}_n \times \hat{Z}_n'$ of product compound Poisson measures have the outer continuity and inner limits at every $A \in \mathcal{A}$. (That is, $\sum_{n=1}^{\infty} \hat{Z}_n \times \hat{Z}_n' \in \mathcal{D}(\mathcal{A})$).*

First of all, we need a lemma.

LEMMA. *For arbitrary events $\{E_n\}$, we have*

$$\sum_{n=1}^{\infty} \mathbf{P}^*(E_n) < \infty \Longrightarrow \mathbf{P}^*(E_n,\ i.o) = 0.$$

where $\mathbf{P}^*$ is the outer measure induced by $\mathbf{P}$.

PROOF. Since $\mathbf{P}^*(E_n) = \inf\{\mathbf{P}(B) : B \supset E_n, B \text{ is measurable}\}$ for all n, there exists B_n such that B_n is measurable and $\mathbf{P}^*(E_n) + 1/n^2 > \mathbf{P}(B_n)$. This implies $\sum_n \mathbf{P}(B_n) < \infty$. By the usual Borel-Cantelli lemma, this means that $\mathbf{P}(B_n,\ i.o.) = 0$. Since $B_n \supset E_n$, we have $\mathbf{P}^*(E_n,\ i.o.) = 0$. □

SKETCH OF THE PROOF OF THEOREM 3.4. Let $0 < \beta < 1$ be such that $\beta^j = a_j$ and $\delta_j = \beta^j$ with $j \in \mathbf{N}$. Let $T_n := \hat{Z}_n \times \hat{Z}_n'$ and $S_n := \sum_{j=1}^{n} T_j$. Let $\{\eta_n\}$ be a sequence of numbers such that

$$\sum_{n=1}^{\infty} \eta_n < \infty. \tag{3.3}$$

We will show that η_n may be selected so that $\sum_{n=1}^{\infty} \mathbf{P}^*(\|T_n\|_{\mathcal{A}} > 2\eta_n) < \infty$. It then follows that $\mathbf{P}^*(\|T_n\|_{\mathcal{A}} > 2\eta_n \text{ i.o.}) = 0$ by the Lemma stated above. Given $\epsilon > 0$, take N large enough so that $\sum_{n=N}^{\infty} 2\eta_n < \epsilon$, and then, depending on ω, choose $N_\omega \geq N$ so that if $n \geq N_\omega$, then $\|T_n(\omega)\|_{\mathcal{A}} \leq 2\eta_n$. That is, if $n, m \geq N_\omega$,

$$\|S_n(\omega) - S_m(\omega)\|_{\mathcal{A}} < \epsilon, \quad \text{or} \quad \|S_n - S_m\|_{\mathcal{A}} \to 0 \quad \text{a.s.},$$

as $m, n \to \infty$. Applying Lemma 2.1 (iii) of [3] will complete the proof of the theorem. □

DETAILS OF THE PROOF OF THEOREM 3.4. Note that $\hat{Z}_n$ and $\hat{Z}'_n$ are independent symmetric compound Poisson processes with parameter(modulo some constant(≤ 1) times)$\beta^{-\alpha n}$. Now consider nets $\mathcal{A}_{\delta_0}, \mathcal{A}_{\delta_1}, \mathcal{A}_{\delta_2}, \ldots, \mathcal{A}_{\delta_{k_n}}$ of $\mathcal{A}$ satisfying assumptions (1) and (2) where k_n will be chosen later to satisfy $\sum_{j=1}^{k_n} \eta_{n_j} = \eta_n$. Then for any $A \in \mathcal{A}$, set $\delta_n = \beta^n$.

$$T_n(A) = \sum_{j=1}^{k_n} \{T_n(A_j) - T_n(A_{j-1})\} + \{T_n(A) - T_n(A_{k_n})\},$$

where for $0 < K_c < 1$, $\quad A_0 = \emptyset, \quad A_j \in \mathcal{A}_{\delta_j}, \quad A^+_{k_n} \in \mathcal{A}_{\delta_{k_n}}$,

$$K_c\delta_j \leq |A_j \setminus A_{j-1}| \leq \delta_j \quad \text{and} \quad K_c\delta_j \leq |A_{j-1} \setminus A_j| \leq \delta_j,$$

$$A_{k_n} \subset A \subset A^+_{k_n}, \quad \text{and} \quad K_c\delta_{k_n} \leq |A^+_{k_n} \setminus A_{k_n}| \leq \delta_{k_n}.$$

(Here if $|A_j \setminus A_{j-1}| = 0$ or $|A_j \setminus A_{j-1}| = 0$, then without loss of generality we may assume that this condition holds).
Since

$$|T_n(A_j) - T_n(A_{j-1})| \leq |T_n(A_j \setminus A_{j-1})| + |T_n(A_{j-1} \setminus A_j)|,$$

if $K_c\delta_j \leq |A_j \setminus A_{j-1}| \leq \delta_j$, $K_c\delta_j \leq |A_{j-1} \setminus A_j| \leq \delta_j$, then by Theorem 2.4 we have

$$\mathbf{P}(|T_n(A_j) - T_n(A_{j-1})| > 2\eta_{n_j}) \leq 2p_{n_j},$$

where

$$p_{n_j} = 2\exp\left(-\frac{K\eta_{n_j}}{\beta^{(2-\alpha)n}\delta_j^{1/4}}\right) + 4\exp\left(-\frac{4K_c^{1/2}\delta_j^{1/2}}{\beta^{\alpha n}}\right).$$

And if A_{k_n}, $A^+_{k_n} \in \mathcal{A}_{\delta_{k_n}}$ with $K_c\delta_{k_n} \leq |A^+_{k_n} \setminus A_{k_n}| \leq \delta_{k_n}$, then

$$\mathbf{P}^*(|T_n(A) - T_n(A_{k_n})| > \eta_n \quad \text{for} \quad \text{some} \quad A \in \mathcal{A}, A_{k_n} \subseteq A \subseteq A^+_{k_n}) \leq q_n,$$

where

$$q_n = \max \mathbf{P}^* \left(\sup_{A_{k_n} \subset B \subset A_{k_n}^+} |T_n(B \setminus A_{k_n})| > \eta_n \right),$$

in which the maximum runs over all A_{k_n} and $A_{k_n}^+$ in $\mathcal{A}_{\delta_{k_n}}$ satisfying

$$K_c \delta_{k_n} \le |A_{k_n}^+ \setminus A_{k_n}| \le \delta_{k_n}.$$

Since $\beta^{n+1} < |X_{n_i}|,\ |X'_{n_j}| \le \beta^n$, we have

$$|\hat{Z}_1 \times \hat{Z}_2| \le \sum_{i=1}^{N_n} \sum_{j=1}^{N'_n} |X_{n_i}||X'_{n_j}|\delta_{(U_{n_i}, V_{n_j})} \le \sum_{i=1}^{N_n} \sum_{j=1}^{N'_n} \beta^{2n} \delta_{(U_{n_i}, V_{n_j})}.$$

Hence

$$q_n \le \mathbf{P}(Y_n \times Y'_n(A_{k_n}^+ \setminus A_{k_n}) > \frac{\eta_n}{\beta^{2n}}),$$

where Y_n and Y'_n denote Poisson processes on $\mathbf{I}^{d_1}$ and $\mathbf{I}^{d_2}$ with $Y_n(\mathbf{I}^{d_1}) = N_n$ and $Y'_n(\mathbf{I}^{d_2}) = N'_n$ respectively.

Consider $\mathbf{P}^*(\|T_n\|_{\mathcal{A}} > 2\eta_n)$ using above development

$$\begin{aligned}
\mathbf{P}^*(\|T_n\|_{\mathcal{A}} > 2\eta_n) &\le 2\sum_{j=1}^{k_n} \exp(2H(\delta_j)) \max \mathbf{P}(|T_n(A_j \setminus A_{j-1})| > \eta_{nj}) \\
&\quad + \exp(2H(\delta_{k_n})) \max \mathbf{P}^*(\sup_{A \in \mathcal{A}, A_{k_n} \subseteq A \subseteq A_{k_n}^+} |T_n(A) - T_n(A_{k_n})| > \eta_n) \\
&\le 2\sum_{j=1}^{k_n} \exp(2H(\delta_j)) p_{n_j} + 2\exp(2H(\delta_{k_n}))q_n := 2R_n + 2R_n^*,
\end{aligned}$$

where the first maximum runs over all $A_j \setminus A_{j-1}$, $A_j \in \mathcal{A}_{\delta_j}$ and $A_{j-1} \in \mathcal{A}_{\delta_{j-1}}$, and the second maximum is over all $A_{k_n}^+ \setminus A_{k_n}$, A_{k_n} , $A_{k_n}^+ \in \mathcal{A}_{\delta_{k_n}}$.

Remind that we need to show :

$$\sum_{n=1}^{\infty} R_n < \infty, \quad \sum_{n=1}^{\infty} R_n^* < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \eta_n < \infty.$$

□

Treatment of R_n-terms : $\sum_{n=1}^{\infty} R_n < \infty$.
Suppose that for $j = 1, 2, 3, \cdots, k_n$,

$$H(\delta_j) \leq \frac{K\eta_{n_j}}{4\beta^{(2-\alpha)n}\beta^{j/4}} \quad \text{and} \quad H(\delta_j) \leq \frac{K_c^{1/2}}{\beta^{(\alpha n - j/2)}}, \tag{3.4}$$

$$2\ln n + \ln k_n < \frac{K\eta_{n_j}}{2\beta^{(2-\alpha)n}\beta^{j/4}} \quad \text{and} \quad 2\ln n + \ln k_n < \frac{K_c^{1/2}}{\beta^{(\alpha n - j/2)}}. \tag{3.5}$$

(We will justify (3.4) and (3.5) in the latter part of the proof of theorem). Then

$$\begin{aligned}
&\exp(2H(\delta_j))p_{n_j} \\
&\leq 2\exp(2H(\delta_j) - \frac{K\eta_{n_j}}{\beta^{(2-\alpha)n}\beta^{j/4}}) + 4\exp(2H(\delta_j) - \frac{4K_c^{1/2}}{\beta^{(\alpha n - j/2)}}) \\
&\leq 2\exp(-\frac{K\eta_{n_j}}{2\beta^{(2-\alpha)n}\beta^{j/4}}) + 4\exp(-\frac{2K_c^{1/2}}{\beta^{(\alpha n - j/2)}}) \\
&\leq 6\exp(-2\ln n - \ln k_n).
\end{aligned}$$

Hence

$$\sum_{j=1}^{k_n} \exp(2H(\delta_j))p_{n_j} \leq 6n^{-2},$$

which implies $\sum_{n=1}^{\infty} R_n < \infty$.

Treatment of R_n^*-term : $\sum_{n=1}^{\infty} R_n^* < \infty$. For a bound of R_n^*, write $C = A_{k_n}^+ \setminus A_{k_n}$. Then

$$\begin{aligned}
q_n &= \mathbf{P}(\beta^n Y_n \times \beta^n Y_n'(A_{k_n}^+ \setminus A_{k_n}) > \eta_n) \\
&= \mathbf{P}(Y_n \times Y_n'(C) > \frac{\eta_n}{\beta^{2n}}, \ N_n N_n'|C| > V_n) + \mathbf{P}(Y_n \times Y_n'(C) \\
&\qquad > \frac{\eta_n}{\beta^{2n}}, \ N_n N_n'|C| \leq V_n) \qquad (3.6) \\
&\leq \mathbf{P}(N_n N_n'|C| > V_n) + \mathbf{P}\left(\frac{Y_n \times Y_n'(C)}{N_n N_n'|C|} > \frac{\eta_n}{\beta^{2n} V_n}\right)
\end{aligned}$$

where $V_n > 0$ is a constant which will be chosen later and $|C|$ denotes the Lebesgue measure of C. From the equality $2(\alpha-1) < \frac{\alpha}{r+1}$ and $2(\alpha-1) < \frac{2-\alpha}{r}$, take c such that

$$(3.7) \qquad 2(\alpha-1) < c < \frac{\alpha}{r+1} \wedge \frac{2-\alpha}{r}.$$

Furthermore, it is possible to choose $\kappa > 0$ to satisfy the following:

$$2(\alpha-1) < c-\kappa < c+\kappa < \alpha \quad \text{and} \quad \frac{2-\alpha}{2(\alpha-1)} > \frac{2-\alpha-\kappa}{c} > r.$$

Now we choose $k_n = [cn]$, $V_n = \frac{\eta_n}{10\beta^{2n}}$, $\eta_n = \beta^{\kappa n}$ and $\eta_{n_j} = \frac{\eta_n}{k_n}$ for $j = 1, 2, \dots, k_n$, where $[x]$ denotes the greatest integer which is less than or equal to x. Then trivially this choice of η_n satisfies (3.3) and $\sum_{j=1}^{k_n} \eta_{n_j} = \eta_n$. Further, for sufficiently large n,

$$\frac{\eta_n}{10\beta^{2n}\beta^{k_n}} > \frac{e^4K^2}{\beta^{2\alpha n}}.$$

So that, by Lemma 2.2 [3],

$$(3.8) \qquad \mathbf{P}(N_nN_n'|C| > \frac{\eta_n}{10\beta^{2n}}) \le 2\exp(-\frac{e^2K}{\beta^{\alpha n}}),$$

and since, without loss of generality, for large n it is possible to assume $|C| \le 1/4$, by Theorem 2.5 [6]

$$(3.9) \qquad \mathbf{P}\left(\frac{Y_n \times Y_n'(C)}{N_nN_n'|C|} > 10\right) \le 4\exp(-\frac{2KK_c}{\beta^{(\alpha-c)n}}).$$

Substitutions (3.8) and (3.9) into (3.6) gives $q_n \le 6\exp(-\frac{2KK_c}{\beta^{(\alpha-c)n}})$. Therefore,

$$R_n^* \le 6\exp(2H(\delta_{k_n}) - \frac{2KK_c}{\beta^{(\alpha-c)n}}).$$

Suppose that

$$(3.10) \qquad H(\delta_{k_n}) < \frac{KK_c}{2\beta^{(\alpha-c)n}} \quad \text{and} \quad 2\ln n < \frac{KK_c}{\beta^{(\alpha-c)n}}.$$

(We will justify (3.10) in the latter part of the proof of theorem). Then $R_n^* \leq 6n^{-2}$, which implies $\sum_{n=1}^{\infty} R_n^* < \infty$.

Finally to finish the proof of Theorem 3.4, what remains is that our choice of c and κ satisfies (3.4), (3.5) and (3.10). Now since we eventually concern about the summabilities of $\sum_{n=1}^{\infty} R_n$ and $\sum_{n=1}^{\infty} R_n^*$, it suffices to show that (3.4), (3.5) and (3.10) are true when n is sufficiently large.

For the first inquality of (3.4), notice that

$$H(\delta_j) \leq \frac{K\eta_{n_j}}{4\beta^{(2-\alpha)n}\beta^{j/4}} \Leftrightarrow (\frac{1}{\beta})^{rj} \leq \frac{K}{4}\frac{\beta^{\kappa n}}{[cn]}(\frac{1}{\beta})^{(2-\alpha)n+\frac{j}{4}}$$
$$\Leftrightarrow \frac{K}{4}(\frac{1}{\beta})^{((2-\alpha)-\kappa)n}(\frac{1}{\beta})^{(\frac{1}{4}-r)j} \geq [cn].$$

Since $(\frac{1}{\beta})^{\frac{1}{4}j}(\frac{1}{\beta})^{-rj} \geq \beta^{rk_n}$ for $j = 1, 2, ..., k_n$, it suffices to show

$$\frac{K}{4}(\frac{1}{\beta})^{((2-\alpha)-\kappa)n}(\frac{1}{\beta})^{-rk_n} \geq [cn]. \tag{3.11}$$

But it follows from the choice of κ that $\frac{2-\alpha-\kappa}{c} > r$ and $(2-\alpha)n - \kappa n - r[cn] \to \infty$. Therefore the left side of (3.11) grows geometrically and the right side does polynomially. Hence we see that the first inequality of (3.4) holds as n is large enough. The validity of the second inequality of (3.4) can be shown in a similar way from the fact $\alpha - c - rc > 0$.

For the validity of the inequalities of (3.5), notice that

$$2\ln n + \ln[cn] < \frac{K\eta_{n_j}}{2\beta^{(2-\alpha)n}\beta^{j/4}} \Leftrightarrow [cn](2\ln n + \ln[cn]) < (\frac{1}{\beta})^{(2-\alpha)n}(\frac{1}{\beta})^{-\kappa n}(\frac{1}{\beta})^{\frac{j}{4}}.$$

It suffices to show that

$$[cn](2\ln n + \ln[cn]) < (\frac{1}{\beta})^{(2-\alpha)n}(\frac{1}{\beta})^{-\kappa n}(\frac{1}{\beta})^{\frac{1}{4}}.$$

But our choice of κ satisfies $\frac{2-\alpha-\kappa}{c} > r > 0$. From this, we see that the left side grows polynomially and the right side geometrically. The second inequality can be shown to hold from $\alpha - c > 0$.

Lastly, notice that, since $\frac{\alpha-c}{c} > r$

$$H(\delta_{k_n}) < \frac{KK_c}{2\beta^{(\alpha-c)n}} \Leftrightarrow \frac{2}{KK_c} \leq (\frac{1}{\beta})^{(\alpha-c)n}(\frac{1}{\beta})^{-rk_n}$$

and

$$(\alpha-c)n - rk_n = (\alpha-c)n - r[cn] \geq (\alpha-c-rc)n - r \to \infty \quad \text{as} \quad n \to \infty,$$

from which (3. 10) follows.

REMARK 3.5. Theorem 3.4 gives some partial answers for the question raised in the beginning of the section, for conditions on $\mathcal{A}$ in terms of entropy when $\{\mu_n\}$ and $\{\nu_n\}$ are sequences of some special random signed measures.

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Department of Mathematics
Sun Moon University
Chungnam 336-840, Korea