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A WEAKLY DEPENDENCE CONCEPT IN MOVING AVERAGE MODELS

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ABSTRACT. We introduce a class of finite and infinite moving average (MA) sequences of multivariate random vectors with exponential marginals. The theory of dependence is used to show that in various cases the class of MA sequences consists of associated random variables. We utilize positive dependence properties to obtain some probability bounds for the multivariate processes.

1. Introduction

Univariate stationary moving average (MA) models, univariate stationary autoregressive moving-average ($ARMA$) models and discrete models are constructed by several researchers. Lawrance and Lewis [7], [12] presented stationary MA models with exponential distributions. Gaver and Lewis [5] presented univariate stationary $ARMA$-type models with gamma distributions. The models mentioned above have been used in various fields of applied probability and time series analysis; for example, they have been used to model and analyze univariate point processes with correlated service and correlated interarrival times (see Jacobs [7]). Details concerning univariate geometric MA processes and the corresponding point processes may be found in Langberg and Stoffer [12]. In this paper we present a class of MA sequences of multivariate random vectors with exponential marginals. Within the class of models, the sequences are classified according to their order of dependence on the past. We use the theory of dependence to show that in a variety of caseses the class of MA sequences are associated. We then apply the association to establish some probability bounds. In section 2 we define the multivariate exponential distribution which is the underlying

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distribution of our class, and present a variety of examples of such distributions. Further, in section 2 we define the concept of association and present a variety of multivariate exponential distributions that are associated. In section 3 we construct the class of MA sequences proving that they have exponential marginals and showing that if the underlying distribution is associated, so is the related MA sequence. In section 4 we consider how to relate the multivariate point processes to the multivariate exponential MA processes discussed in section 3. Also, in section 4 we extend a useful concept of Alzaid's weakly bivariate random variables to multivariate random vector and utilize positive dependence properties to obtain weakly positive orthant dependence ($WPOD$) for the multivariate processes.

2. Preliminaries

A primary stationary model in time series analysis is the $p \times 1$ moving average (MA) model given by

$$X(n) = \sum_{j=-\infty}^{\infty} A(j)\epsilon(n-j), \quad n = 0, \quad \pm 1, \pm 2, \cdots \tag{2.1}$$

where $A(j), j = 0, \pm 1, \pm 2, \cdots$, is a sequence of $p \times p$ parameter matrices such that $\sum_{j=-\infty}^{\infty} \|A(j)\| < \infty$, and $\epsilon(n), n = 0, \pm 1, \pm 2, \cdots$, is a sequence of uncorrelated $p \times 1$ random vectors with mean zero and common covariances matrix. It is well known that this model emerges from many physically realizable systems(see, for example, Hannan [6], p9). Now, we present definition of a multivariate exponential distribution.

DEFINITION 2.1. Let $(E_1, \cdots, E_n)$ be random variables assuming values in $(0, \infty)$. We say that $(E_1, \cdots, E_n)$ has multivariate exponential distribution (MVE) if E_i' s are exponential. Note that the $(n-1)$ dimensional marginals(hence $k-$dimensional marginal, $k = 1, 2, \cdots, n-1$) are MVE. In particular, the two dimensional marginals are bivariate exponential (BVE), so the one dimensional marginals are exponential.

EXAMPLE 2.2. (a) Let E be exponential. Then $(E, \cdots, E)$ is multivariate exponential. (b) Let $E_1, E_2, \cdots, E_k$ be independent exponentials. Then $(E_1, \cdots, E_k)$ has a multivariate exponential distribution. (c) Let $X_1, X_2, \cdots, X_{k+1}$ be independent exponential and put $E_1 = \{\min(X_1, X_{k+1})\}, E_2 = \{\min(X_2, X_{k+1})\}, \cdots, E_k = \{\min(X_k, X_{k+1})\}$. Then $(E_1, \cdots, E_k)$ has multivariate exponential distribution. (d) Let $(X_1, \cdots, X_k)$ be a random vector with continuous marginal distributions $F_1, \cdots, F_k$, respectively. Then the random vector $-\ln[1 - F_1(X_1)], \cdots, -\ln[1 - F_k(X_k)])$ is k-variate exponential. Finally we present a concept of positive dependence.

DEFINITION 2.3. Let $\underline{T} = (T_1, \cdots, T_n), n = 1, 2, \cdots$,be a multivariate random vector. We say that the random variables $T_1, \cdots, T_n$ are associated if for all pairs of measurable bounded functions $f, g : R^n \to R$ both non-decreasing in each argument $Cov(f(\underline{T}), g(\underline{T})) \geq 0$. The following lemma provides sufficient conditions for some of the multivariate distributions presented in Example 2.2 to be associated.

LEMMA 2.4. *Let $\underline{Q} = (Q_1, \cdots, Q_n)$ be a random vector with components assuming values in the set $\{1, 2, \cdots\}$ and let $\underline{R}(j) = (R_1(j), \cdots, R_n(j)), j = 1, 2, \cdots$, be an i.i.d sequence of nonnegative random vectors independent of $\underline{Q}$. If $Q_1, \cdots, Q_n$ are associated, and $R_1(1), \cdots, R_n(1)$ are associated, then $\sum_{j=1}^{Q_1} R_1(j), \cdots, \sum_{j=1}^{Q_n} R_n(j)$ are associated.*

PROOF. Let $f, g : R^n \to R$ be measurable bounded functions non-decreasing in each argument and let $X_1 = \sum_{j=1}^{Q_1} R_1(j), \cdots, X_n = \sum_{j=1}^{Q_n} R_n(j)$. First note that

$$\begin{aligned} &Cov\{f(X_1, \cdots, X_n), g(X_1, \cdots, X_n)\} \\ &= E\{Cov(f(X_1, \cdots, X_n), g(X_1, \cdots, X_n))|\underline{Q}\} \\ &+ Cov\{E\{f(X_1, \cdots, X_n)|\underline{Q}\}, E\{g(X_1, \cdots, X_n)|\underline{Q}\}\}. \end{aligned}$$

Now $E\{f(X_1, \cdots, X_n)|\underline{Q}\}$, and $E\{g(X_1, \cdots, X_n)|\underline{Q}\}$ are non-decreasing functions of $Q_1, \cdots, Q_n$. Since $Q_1, \cdots, Q_n$ are associated, we have

$$Cov\{E\{f(X_1, \cdots, X_n)|\underline{Q}\}, E\{g(X_1, \cdots, X_n)|\underline{Q}\}\} \geq 0.$$

Now, let $\underline{Q} = \max(Q_1, \cdots, Q_n)$. Since $f(X_1, \cdots, X_n)|\underline{Q}$ and $g(X_1, \cdots, X_n)|\underline{Q}$ are non-decreasing functions of $R_1(1), \cdots, R_1(\underline{Q}), \cdots, R_n(1), \cdots, R_n(\underline{Q})$, these random variables are associated (cf. Barlow and Proschan [3]). Thus

$$Cov\{f(X_1, \cdots, X_n)|\underline{Q},\ \ g(X_1, \cdots, X_n)|\underline{Q}\} \geq 0.$$

Consequently, $Cov\{f(X_1, \cdots, X_n),\ \ g(X_1, \cdots, X_n)\} \geq 0$ and $X_1, \cdots, X_n$ are associated. □

3. Construction of Moving Average Models

In this section we construct a class of finite and infinite M A sequences of k-variate random vectors, denote the class of sequences by $\{\underline{X}(n,m) = (X_1(n,m), \cdots, X_k(n,m), n = 0, \pm 1, \pm 2, \cdots\}, m = 1, 2, \cdots, \infty$ and show that each random vector $\underline{X}(n,m)$ has a k-variate exponential distribution with a vector mean that does not depend on n or m. Within the class of sequences the order of dependence on the past is indicated by parameter m. For each positive integer $m, \underline{X}(n,m)$ depends on the previous matrices $\{\underline{X}(n-1,m), \cdots, \underline{X}(n-m,m)\}$ while $\underline{X}(n,\infty)$ depend on all the preceding random vectors $\{\underline{X}(n-1,\infty), \underline{X}(n-2,\infty), \cdots\}$. After constructing the various models we present sufficient conditions for random variables $\{X_l(n_j,m)\}, l = 1, 2, \cdots, k; j = 1, 2, \cdots, r$ to be associated, where $r = 1, 2, \cdots$, and $n_1 < n_2 < \cdots < n_r \in \{0, \pm 1, \pm 2, \cdots\}$. Now we construct the exponential class of sequences. Some notation is needed.

NOTATION 3.1. Throughout, n range over the integer and m, j over the positive integer. Let $\underline{E}(n) = (E_1(n), E_2(n), \cdots, E_k(n))$ be i.i.d k-variate exponential random vector $(\lambda_1^{-1}, \cdots, \lambda_k^{-1}) : \lambda_1, \cdots, \lambda_k > 0$. Let $\beta_1(n,j), \cdots, \beta_k(n,j)$ be parameters taking values in [0,1] and let $\underline{B}(n,j) = diag\{\beta_1(n,j), \cdots, \beta_k(n,j)\}$. Further let $(I_1(n,j), \cdots, I_k(n,j))$ be independent k-variate random vectors independent of all the $\underline{E}(n)$ such that $I_1(n,j), \cdots, I_k(n,j)$ are Bernoulli with parameters $1-\beta_1(n,j), \cdots, 1-\beta_k(n,j)$, respectively. Let $V_q(n,j)$ be a h timesh random diagonal matrix defined by $V_q(n,j) = diag\{\prod_{i=q}^{j} I_1(n,i), \cdots, \prod_{i=q}^{j} I_k(n,i)\}, q \in \{1, 2, \cdots, j\}$, and for case of notation we put $V_1(n,j) = V(n,j)$.

Finally, let a sum(product) over an empty set of indices be equal to 0(1).

We now present the class of exponential sequences. For $m = 1, 2, \cdots$, and $n = 0 \pm 1, \pm 2, \cdots$, let

$$(3.1)\quad \underline{X}(n,m) = \sum_{r=0}^{m-1} \underline{V}(n,r)\underline{B}(n,r+1)\underline{E}(n-r) + \underline{V}(n,m)\underline{E}(n-m)$$

and

$$(3.2)\qquad \underline{X}(n,\infty) = \sum_{r=0}^{\infty} \underline{V}(n,r)\underline{B}(n,r+1)\underline{E}(n-r).$$

We show in Corollary 3.3 and Lemma 3.4 that for all $n, m, \underline{X}(n,m)$ and $\underline{X}(n,\infty)$ have k-variate exponential distributions.

LEMMA 3.2. *For $n = 0, \pm 1, \pm 2, \cdots$, and $m, q = 1, 2, \cdots$, let*

$$\underline{Y}_q(n,m) = \sum_{r=0}^{m-1} \underline{V}_q(n, r+q-1)\underline{B}(n, r+q)\underline{E}(n-r-q+1) + \underline{V}_q(n, m+q-1)\underline{E}(n-m-q+1).$$

Then for all n, m and $q, \underline{Y}_q(n,m)$ has a k-variate exponential distribution with mean vector $(\lambda_1^{-1}, \cdots, \lambda_k^{-1})$.

PROOF. We prove the result of the lemma by an induction argument on m. For $m = 1$,

$$\underline{Y}_q(n,1) = \underline{B}(n,q)\underline{E}(n-q+1) + \underline{V}_q(n.q)\underline{E}(n-q).$$

By computing the characteristic functions of the components of $\underline{Y}_q(n,1)$, we can verify that the results of the lemma hold for all n, q. Assume now that the results of the lemma hold for m, and all n, q. Noting that

$$\underline{Y}_q(n,m+1) = \underline{B}(n,q)\underline{E}(n-q+1) + \underline{V}_q(n,q)[\sum_{r=0}^{m-1} \underline{V}_q + 1(n, r+q)\underline{B}(n, r+q+1)\underline{E}(n-q+1) + \underline{V}_{q+1}(n, m+q)\underline{E}(n-m-q)].$$

We see that, by induction, the terms in the brackets are k-variate exponential with mean $(\lambda_1^{-1}, \cdots, \lambda_k^{-1})$. Since these terms are independent of $\underline{E}(n-q+1)$, it follows as in the case $m = 1$ and $m = 0$ that $\underline{Y}_q(n, m+1)$ has the appropriate distributions for all n and q. Note that $\underline{X}(n,m)$ given by (3.1) are equal to $\underline{Y}_1(n,m)$. Thus, we conclude from Lemma 3.2 that the following holds. □

COROLLARY 3.3. *For all n and $m, \underline{X}(n,m)$ has a k-variate exponential distribution with mean vector $(\lambda_1^{-}1, \cdots, \lambda_k^{-}1)$. Next, we show that if for all $i = 1, 2, \cdots, k$,*

$$\lim_{m\to\infty} \Pi_{j=1}^{m}[1 - \beta_i(n,j)] = 0. \tag{3.3}$$

LEMMA 3.4. *If condition (3.3) holds, then for all $n, \underline{X}(n,\infty)$ has a k-variate exponential distribution with mean vector $(\lambda_1^{-}1, \cdots, \lambda_k^{-}1)$.*

PROOF. Let m be a positive integer. By (3.3) $\underline{X}(n,m) \xrightarrow{P} \underline{X}(n,\infty)$ as mrarrow inf. Thus $\underline{X}(n,m) \xrightarrow{D} \underline{X}(n,\infty)$ as $m \to \infty$ and the result of the lemma follows from Corollary 3.3. Note that (3.3) to hold, it suffices that for alln and $i = 1, 2, \cdots, k, inf\{\beta_i(n,j), j = 1, 2, \cdots\} > 0$. □

Next, we investigate some of the dependence aspects of the class.

LEMMA 3.5. *Suppose that $E_1(1), \cdots, E_n(1)$ are associated and condition (3.3) holds then for all positive intgers m, k and all integers $n_1 < n_2 \cdots < n_r$, the random variables $\{X_i(n_j, m), i = 1, \cdots, k; j = 1, \cdots, r\}$ are associated.*

LEMMA 3.6. *If $E_1(1), \cdots, E_n(1)$ are associated and condition (3.3) holds then for all positive integers r and all integers $n_1 < n_2 < \cdots < n_r$, the random variables $\{X_i(n_j, \infty), i = 1, \cdots, k; j = 1, \cdots, r\}$ are associated.*

PROOF. By similar arguments to the ones given in the proof of Lemma 3.4 we conclude that the sequence $\{X_1(n_1,m), \cdots, X_k(n_1,m), \cdots, X_1(n_r,m), \cdots, X_k(n_r,m)\}$ converge in distribution as $m \to \infty$ to $\{X_1(n_1,\infty), \cdots, X_k(n_1,\infty), \cdots, X_1(n_r,\infty), \cdots, X_k(n_r,\infty)\}$.

By Lemma 3.5 the random variables $\{X_i(n_j, m), i = 1, \cdots, k; j = 1, \cdots, r\}$ are associated for all m. Consequently, the results of the lemma follows by Esary et al.[4], P.4. □

4. WPOD in Models

Throughout this section we fix $m, m = 1, 2, \cdots, \infty$, and hence suppress it from our notion, that is $\underline{X}(n, m)$ is represent by $\underline{X}(n)$. In the point process theory of the models, the behavior of the vector of sums $S_{\underline{x}}(r) = (S_{x_1}(r_1), \cdots, S_{x_k}(r_k))$ where $S_{x_i}(r_i) = \sum_{n=1}^{r_i} X_i(n), i = 1, 2, \cdots, k$, is of interest, $r_1, \cdots, r_k = 1, 2, \cdots$. For example, if $\underline{X}(n)$ is a vector of k-variate exponential interarrival times of a point process $\underline{M}_x(t) = (M_{x_1}(t_1), \cdots, M_{x_k}(t_k))$ which are the number of arrivals by $\times t_1, \cdots, t_k > 0$, then

$$\begin{aligned} &P\{\underline{M}_{x_1}(t_1) \le r_1, \cdots, \underline{M}_{x_k}(t_k) \le r_k\} \\ &= P\{S_{x_1}(r_1) > t_1, \cdots, S_{x_k}(r_k) > t_k\}. \end{aligned}$$

We now utilize positive dependence properties to obtain weakly positive orthant dependence for the sums $S_{\underline{x}}(r)$.
First, we define four concepts of weakly positive orthant dependence.

Definition 4.1. Let $q = 2, 3, \cdots$, and let $\underline{X} = (X_1, \cdots, X_q)$ be a random vector. We say that $\underline{X}$ is weakly positive upper orthant dependent of the first type(WPUOD1) if for all real numbers $t_1, t_2, \cdots, t_q$,

$$\begin{aligned} &\int_{x_1}^{\infty} \cdots \int_{x_q}^{\infty} P(X_i > t_i, i = 1, 2, \cdots, q) dt_1 \cdots dt_q \\ &\ge \int_{x_1}^{\infty} \cdots \int_{x_q}^{\infty} \prod_{i=1}^{q} P(X_i > t_i) dt_1 \cdots dt_q \end{aligned} \tag{4.1}$$

DEFINITION 4.2. Let $q = 2, 3, \cdots$, and let $\underline{X} = (X_1, \cdots, X_q)$ be a random vector. we say that $\underline{X}$ is weakly positive upper orthant dependent second type(WPUOD2) if for all real numbers $t_1, t_2, \cdots, t_q$,

$$
\begin{aligned}
&\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_q} P(X_i > t_i, t = 1, 2, \cdots, q) dt_1 \cdots dt_q \\
(4.2) \quad &\geq \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \prod_{i=1}^{q} P(X_i > t_i) dt_1 \cdots dt_q
\end{aligned}
$$

we say that the random vector $\underline{X} = (X_1, \cdots, X_q)$ is weakly positive upper orthant dependent (WPUOD) if it satisfies both (4.1) and (4.2)

DEFINITION 4.3. Let $q = 2, 3, \cdots$, and let $\underline{X} = (X_1, \cdots, X_q)$ be a random vector. we say that $\underline{X}$ is weakly positive lower orthant dependent of the first type (WPLOD1) if for all real numbers $t_1, t_2, \cdots, t_q$,

$$
\begin{aligned}
&\int_{x_1}^{\infty} \cdots \int_{x_q}^{\infty} P(X_i \leq t_i, i = 1, 2, \cdots, q) dt_1 \cdots dt_q \\
(4.3) \quad &\geq \int_{x_1}^{\infty} \cdots \int_{x_q}^{\infty} \prod_{i=1}^{q} P(X_i \leq t_i) dt_1 \cdots dt_q
\end{aligned}
$$

DEFINITION 4.4. Let $q = 2, 3, \cdots$, and let $\underline{X} = (X_1, \cdots, X_q)$ be a random vector. We say that $\underline{X}$ is weakly positive lower orthant dependent of the second type (WPLOD2) if for all real numbers $t_1, t_2, \cdots, t_q$,

$$
\begin{aligned}
&\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_q} P(X_i \leq t_i, i = 1, 2, \cdots, q) dt_1 \cdots dt_q \\
(4.4) \quad &\geq \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_q} \prod_{i=1}^{q} p(X_i \leq t_i) dt_1 \cdots dt_q
\end{aligned}
$$

we say that the random vector $\underline{X} = (X_1, \cdots, X_q)$ is weakly positive lower orthant dependent (WPLOD) if it satisfies both (4.3) and (4.4). Moreover, we say that the random vector $\underline{X} = (X_1, \cdots, X_q)$ is weakly positive orthant dependent (WPOD) if it satisfies both WPUOD and WPLOD.

REMARK 4.5. (a) $\underline{X} = (X_1, X_2, \cdots, X_q)$ is POD and having finite covariance then $\underline{X} = (X_1, X_2, \cdots, X_q)$ is WPOD, (b) If $X_1, \cdots, X_q$ are associated then clearly $\underline{X}$ is WPUOD and WPLOD, (c) Let $f_1, \cdots, f_q : (-\infty, \infty) \to [0, \infty)$ be measurable nondecreasing (nonincreasing) and let $\underline{X} = (X_1, X_2, \cdots, X_q)$ be WPUOD1(WPUOD2) and WPLOD1(WPLOD2). Then

$$E\prod_{i=1}^{q} f_i(X_i) \geq \prod_{i=1}^{q} Ef_i(X_i). \tag{4.5}$$

For the sake completeness we present the following definition.

DEFINITION 4.6. Let X, Y be random variables. we say that X is stochastically less than or equal to Y and write $X \leq Y$ if for every real number $t, P(X > t) \leq P(Y > t)$.

REMARK 4.7. Let $f : (-\infty, \infty) \to [0, \infty)$ be a measurable nondecreasing function and let $X \leq Y$. Then $Ef(X) \leq Ef(Y)$ (see Lehmann [14])

Next, we discuss the inheritance of weakly positive orthant dependence properties.

THEOREM 4.8. *Suppose that for* $q = 1, 2, \cdots$, *the random variables* $\{X_i(n) : i = 1, 2, \cdots, k; n = 1, 2, \cdots, q\}$ *are associated. Then for* $r_1, \cdots, r_k = 1, 2, \cdots$, *and* $t_1, \cdots, t_k > 0$ *we have that* (i) $\{S_{x_i}(r_i) : i = 1, 2, \cdots, k\}$ *are associated and* (ii) $\{M_{x_i}(t_i) : t = 1, 2, \cdots k\}$ *are WPUOD1(WPUOD2) and WPLOD1(WPLOD2).*

PROOF. Part of (i) follows from the fact that $\{S_{x_i}(r_i), i = 1, 2, \cdots, k\}$ are non-decreasing functions of associated random variables. To show (ii), let $f_i = \chi\{S_{x_i}(r_i) > t_i\}$ and $g_i = \chi\{S_{x_i}(r_i) \leq t_i\}, i = 1, 2, \cdots, k$, where χ is the indicator function. By Barlow and Proschan [3] p3, p30, f_i and $g_i, i = 1, 2, \cdots, k$ are associated since $S_{x_i}(r_i), i = 1, 2, \cdots, k$ are associated. Hence by (4.5)

$$\int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} P(M_{x_1}(t_1) > r_1, \cdots, M_{x_k}(t_k) > r_k)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} E(\prod_{i=1}^{k} f_i)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} E[\chi(S_{x_1}(r_1) > t_1)\cdots\chi(S_k(r_k) > t_k)]dr_1\cdots dr_k$$
$$\geq \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} E(\chi(S_{x_1}(r_1)) > t_1)\cdots E(\chi(S_{x_k}(r_k)) > t_k)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} \prod_{i=1}^{k} E(f_i)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} P(M_{x_1}(t_1) > r_1)\cdots P(M_{x_k}(t_k) > r_k)dr_1\cdots dr_k$$

and

$$\int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} P(M_{x_1}(t_1) \leq r_1, \cdots, M_{x_k}(t_k) \leq r_k)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} E(\prod_{i=1}^{k} g_i)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} E[\chi(S_{x_1}(r_1) > t_1)\cdots\chi(S_{x_k}(r_k) > t_k)]dr_1\cdots dr_k$$
$$\geq \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} E(\chi(S_{x_1}(r_1)) > t_1)\cdots E(\chi(S_{x_k}(r_k)) > t_k)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} \prod_{i=1}^{k} E(g_i)dr_1\cdots dr_k$$
$$= \int_{x_1}^{\infty}\cdots\int_{x_k}^{\infty} P(M_{x_1}(t_1) \leq r_1)\cdots P(M_{x_k}(t_k) \leq r_k)dr_1\cdots dr_k$$

The proof of WPUOD2(WPLOD2) is similar. □

Finally, for the stationary models we may obtain bounds for sums $S_{\underline{x}}(r)$ using gamma distribution. Now we construct on $S_{\underline{x}}(r)$.

THEOREM 4.9. *Assume that* $\beta_1(n_1,1),\cdots,\beta_k(n_k,1)$ *are equal to* $\beta_1,\cdots,\beta_k$*, respectively, for all* n*. Let* $Y_i(r_i,\theta_i), i=1,2,\cdots,k$ *be gamma random variables with parameter* (r_i,θ_i)*. Then* $S_{x_i}(r_i)\geq Y_i(r_i,\lambda_i\beta_i^{-1})$, $i=1,2,\cdots,k$*. If in addition the random variables* $\{X_i(n), i=1,2,\cdots,k, n=1,2,\cdots,q\}, q=1,2,\cdots$ *are associated, then for* $x_1,x_2,\cdots,x_k>0$,

$$\int_{s_1}^{\infty}\cdots\int_{s_k}^{\infty}P(S_{x_1}(r_1)>x_1,\cdots,S_{x_k}(r_k)>x_k)dx_1\cdots dx_k$$
$$\geq\int_{s_1}^{\infty}\cdots\int_{s_k}^{\infty}P(Y_1(r_1,\lambda_1\beta_1^{-1})>x_1)\cdots P(Y_k(r_k,\lambda_k\beta_k^{-1})>x_k)dx_1\cdots dx_k$$

(4.5) and

$$\int_{0}^{s_1}\cdots\int_{0}^{s_k}P(S_{x_1}(r_1)>x_1,\cdots,S_{x_k}(r_k)>x_k)dx_1\cdots dx_k$$
$$\geq\int_{0}^{s_1}\cdots\int_{0}^{s_k}P(Y_1(r_1,\lambda_1\beta_1^{-1})>x_1)\cdots P(Y_k(r_k,\lambda_k\beta_k^{-1})>x_k)dx_1\cdots dx_k$$

PROOF. From Equations (3.2) and (3.3) we see that $X_i(n)\geq\beta_iE_i(n)$, $i=1,2,\cdots k, n=1,2,\cdots$.
Hence

$$\begin{aligned}P(S_{x_i}(r_i)\geq x_i)&=P(\sum_{n=1}^{r_i}X_i(n)\geq x_i)\\&\geq P(\sum_{n=1}^{r_i}\beta_iE_i(n)\geq x_i)\\&=P(Y_i(r_i,\lambda_i\beta_i^{-1}\geq x_i), i=1,2,\cdots,k\end{aligned}$$

and first assertion is proved. Equation (4.5) now follows from the first assertion and the fact that since $S_{x_1}(r_1),\cdots,S_{x_k}(r_k)$ are associated from Theorem 4.9 they are also WPUOD.
The proof of WPLOD is similar. □

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