Comm. Korean Math. Soc. **12** (1997), No. 3, pp. 769–778

STRONG LAW OF LARGE NUMBERS FOR LEVEL-WISE INDEPENDENT FUZZY RANDOM VARIABLES

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ABSTRACT. In this paper, we obtain a strong law of large numbers for sums of level-wise independent and level-wise identically distributed fuzzy random variables.

1. Introduction

Laws of large numbers for sums of independent random sets have been studied by Artstein and Hart [1], Artstein and Vitale [2], Puri and Ralescu [16], Taylor and Inoue [18], Uemura [19], etc. These results have been generalized to the case of fuzzy random variables by several people. A SLLN for sums of independent and identically distributed fuzzy random variables was obtained by Kruse [14], and a SLLN for sums of independent fuzzy random variables by Miyakoshi and Shimbo [15]. Also, Klement, Puri and Ralescu [12] proved some limit theorems which includes a SLLN , and Inoue [10] obtained a SLLN for sums of independent tight fuzzy random sets. Recently, Hong and Kim [9] generalized Marcinkiewicz law of large numbers to fuzzy random variables.

In this paper, we obtain a SLLN for sums of level-wise independent and level-wise identically distributed fuzzy random variables by using a metric which is stronger than one in works mentioned previously. The representation theorem of fuzzy numbers by Goetschel and Voxman [7] will be used.

Received April 30, 1997. Revised June 25, 1997.

1991 Mathematics Subject Classification: 60D05, 60F15.

Key words and phrases: Strong law of large numbers, Fuzzy random variables, Level-wise independent.

This research was supported by the Dongshin University research grants in 1996.

2. Preliminaries

In this section, we describe some basic concepts of fuzzy numbers. Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.
(2) $\tilde{u}$ is upper semicontinuous.
(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.
(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

For a fuzzy set $\tilde{u}$, we define

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \alpha = 0 \end{cases}$$

Then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the end points of the intervals $L_\alpha \tilde{u} = [u_\alpha^-, u_\alpha^+]$. We denote the family of all fuzzy numbers by $F(R)$.

THEOREM 2.1 ([7]). *For $\tilde{u} \in F(R)$, we denote $u^-(\alpha) = u_\alpha^-$ and $u^+(\alpha) = u_\alpha^+$. Then the followings hold;*

(1) *$u^-(\alpha)$ is a bounded increasing function on [0,1].*
(2) *$u^+(\alpha)$ is a bounded decreasing function on [0,1].*
(3) *$u^-(1) \leq u^+(1)$.*
(4) *$u^-(\alpha)$ and $u^+(\alpha)$ are left continuous on (0,1] and right continuous at 0.*
(5) *If $v^-(\alpha)$ and $v^+(\alpha)$ satisfy above (1)-(4), then there exists unique $\tilde{v} \in F(R)$ such that $v_\alpha^- = v^-(\alpha)$, $v_\alpha^+ = v^+(\alpha)$.*

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the parametrized representation $\{(u_\alpha^-, u_\alpha^+) | 0 \leq \alpha \leq 1\}$. Suppose now that $\tilde{u}, \tilde{v}$ are fuzzy numbers represented by $\{(u_\alpha^-, u_\alpha^+) | 0 \leq \alpha \leq 1\}$ and $\{(v_\alpha^-, v_\alpha^+) | 0 \leq \alpha \leq 1\}$, respectively. If we define

$$(\tilde{u} + \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda\tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}, & \lambda = 0 \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$, then

$$\tilde{u} + \tilde{v} = \{(u_\alpha^- + v_\alpha^-, v_\alpha^+ + v_\alpha^+) |\ 0 \leq \alpha \leq 1\},$$

$$\lambda\tilde{u} = \begin{cases} \{(\lambda u_\alpha^-, \lambda u_\alpha^+) |\ 0 \leq \alpha \leq 1\}, & \lambda \geq 0 \\ \{(\lambda u_\alpha^+, \lambda u_\alpha^-) |\ 0 \leq \alpha \leq 1\}, & \lambda < 0. \end{cases}$$

Now, we define two metrics d, d^* on $F(R)$ by

(2.1) $$d(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v})$$

(2.2) $$d^*(\tilde{u}, \tilde{v}) = \int_0^1 d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) d\alpha$$

where d_H is the Hausdorff metric defined as

$$d_H(L_\alpha\tilde{u}, L_\alpha\tilde{v}) = \max(|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|).$$

Also, the norm $\|\tilde{u}\|$ of fuzzy number $\tilde{u}$ will be defined as

$$\|\tilde{u}\| = d(\tilde{u}, \tilde{0}) = \max(|u_0^-|, |u_0^+|).$$

3. Fuzzy random variables

Throughout this paper, $(\Omega, \mathcal{A}, P)$ denotes a complete probability space. If $\tilde{X} : \Omega \to F(R)$ is a fuzzy number valued function and B is a subset of R, then $\tilde{X}^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$\tilde{X}^{-1}(B)(\omega) = \sup_{x \in B} \tilde{X}(\omega)(x)$$

for every $\omega \in \Omega$. The function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable if for every closed subset B of R, the fuzzy set $\tilde{X}^{-1}(B)$ is measurable when considered as a function from Ω to $[0, 1]$. If we denote $\tilde{X}(\omega) = \{(X_\alpha^-(\omega), X_\alpha^+(\omega)|\ 0 \leq \alpha \leq 1\}$, then it is well-known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0, 1]$, X_α^- and X_α^+ are random variables in the usual sense (See Kim and Ghil [11]). Hence, if $\sigma(\tilde{X})$ is the smallest σ-field which makes $\tilde{X}$ a fuzzy random variable, then $\sigma(\tilde{X})$ is consistent with $\sigma(\{X_\alpha^-, X_\alpha^+|\ 0 \leq \alpha \leq 1\})$. This enables us to define the concept of independence for fuzzy random variables as in the case of classical random variables.

DEFINITION 3.1. Let $\tilde{X}, \tilde{Y}$ be two fuzzy random variables whose representations are $\{(X_\alpha^-, X_\alpha^+)|\ 0 \le \alpha \le 1\}$ and $\{(Y_\alpha^-, Y_\alpha^+)|\ 0 \le \alpha \le 1\}$, respectively.

(1) $\tilde{X}$ and $\tilde{Y}$ are called independent if the σ-fields $\sigma(\tilde{X})$ and $\sigma(\tilde{Y})$ are independent.
(2) $\tilde{X}$ and $\tilde{Y}$ are called level-wise independent if for each $\alpha \in [0,1]$ the σ-fields $\sigma(X_\alpha^-, X_\alpha^+)$ and $\sigma(Y_\alpha^-, Y_\alpha^+)$ are independent.
(3) $\tilde{X}$ and $\tilde{Y}$ are called level-wise identically distributed if for each $\alpha \in [0,1]$, (X_α^-, X_α^+) and (Y_α^-, Y_α^+) are identically distributed random vectors.

Note that the definitions (2) and (3) is firstly introduced in this paper.

DEFINITION 3.2. A fuzzy random variable $\tilde{X} = \{(X_\alpha^-, X_\alpha^+)|\, 0 \le \alpha \le 1\}$ is called integrable if for each $\alpha \in [0,1]$, X_α^- and X_α^+ are integrable, equivalently, $\int \|\tilde{X}\| dP < \infty$. In this case, the expectation of $\tilde{X}$ is defined by

$$E\tilde{X} = \int \tilde{X} dP = \{(\int X_\alpha^- dP, \int X_\alpha^+ dP)|\, 0 \le \alpha \le 1\}$$

4. Main Result

In this section, a *SLLN* with respect to the metric d defined as in (2.1) will be obtained. In earlier works, the metric d^* defined as in (2.2) have been used (see [9],[10],[12]). Note that d is stronger than d^*. First, we need a subspace $F_C(R)$ of $F(R)$. Let $F_C(R) = \{\tilde{u} \in F(R)|\ u_\alpha^-$ and u_α^+ are continuous when considered as functions of $\alpha\}$. Then it is known that $\tilde{u} \in F_C(R)$ if and only if for any $\beta \in (0,1)$, there exist at most two different x_1, x_2 such that $\tilde{u}(x_1) = \tilde{u}(x_2) = \beta$ (See [4] Theorem 5.1). Note that if $\tilde{X}$ is $F_C(R)$-valued, then $E\tilde{X} \in F_C(R)$.
Before we state the main result, we recall the following lemma which is well-known in the classical Analysis.

LEMMA 4.1. *Let (f_n) be a sequence of monotonic functions on $[0,1]$. If $f_n(x)$ converges pointwise to a continuous function $f(x)$ on $[0,1]$, then $f_n(x)$ converges to $f(x)$ uniformly.*

We now state the SLLN for sums of level-wise independent fuzzy random variables.

THEOREM 4.2. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise independent and level-wise identically distributed fuzzy random varibles with $E\|\tilde{X}_1\| < \infty$. If $E\tilde{X}_1 \in F_C(R)$, then*

$$d(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i, E\tilde{X}_1) \longrightarrow 0 \ \ a.s.$$

PROOF. Let $\tilde{X}_n = \{(X_{n\alpha}^-, X_{n\alpha}^+) | 0 \le \alpha \le 1\}$. Then for each $\alpha \in [0,1]$, $\{(X_{n\alpha}^-, X_{n\alpha}^+)\}$ is a sequence of independent and identically distributed random vectors with $E|X_{n\alpha}^-| < \infty$ and $E|X_{n\alpha}^+| < \infty$ in the classical sense. By Kolmogorov's strong law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^- \longrightarrow EX_{1\alpha}^- \ \ a.s.$$

and

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^+ \longrightarrow EX_{1\alpha}^+ \ \ a.s.$$

Now, let $\{r_k\}$ be a countable dense subset of $[0,1]$ with $r_0 = 0, r_1 = 1$. Then there exist $B_k \in \mathcal{A}$ with $P(B_k) = 0$ such that for each $\omega \notin B_k$

$$\frac{1}{n}\sum_{i=1}^{n} X_{ir_k}^-(\omega) \longrightarrow EX_{1r_k}^- \tag{4.1}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_{ir_k}^+(\omega) \longrightarrow EX_{1r_k}^+ \tag{4.2}$$

If we define $B = \cup_{k=0}^{\infty} B_k$, then $P(B) = 0$ and for each $\omega \notin B$, (4.1) and (4.2) hold for all r_k. Now, we will show that for each $\omega \notin B$

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^-(\omega) \longrightarrow EX_{1\alpha}^- \ \text{ uniformly in } \ \alpha \in [0,1].$$

By Lemma 4.1, it suffices to show that for each $\omega \notin B$, and each α,

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^{-}(\omega) \longrightarrow EX_{1\alpha}^{-}.$$

Let $\omega \notin B$ and $\epsilon > 0$ be fixed. Then by the continuity of $EX_{1\alpha}^{-}$ as a function of α, there exists $\delta > 0$ such that

$$|\alpha - \beta| < \delta \ \text{ implies } | \ EX_{1\alpha}^{-} - EX_{1\beta}^{-} \ | < \epsilon$$

If we take r_l, r_m so that $\alpha - \delta < r_l < \alpha < r_m < \alpha + \delta$, then

$$EX_{1r_m}^{-} - \epsilon < EX_{1\alpha}^{-} < EX_{1r_l}^{-} + \epsilon.$$

Hence, by the monotonicity of $X_{i\alpha}^{-}(\omega)$ with respect to α,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} X_{ir_l}^{-}(\omega) - EX_{1r_l}^{-} - \epsilon &< \frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^{-}(\omega) - EX_{1\alpha}^{-} \\ &< \frac{1}{n}\sum_{i=1}^{n} X_{ir_m}^{-}(\omega) - EX_{1r_m}^{-} + \epsilon \end{aligned}$$

which implies

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^{-}(\omega) \longrightarrow EX_{1\alpha}^{-}.$$

Similarly, it can be proved that for each $\omega \notin B$

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^{+}(\omega) \longrightarrow EX_{1\alpha}^{+} \ \text{ uniformly in } \ \alpha \in [0,1].$$

Therefore, for each $\omega \notin B$.

$$d\left(\frac{1}{n}\sum_{i=1}^{n} \tilde{X}_i(\omega), E\tilde{X}_1\right) \longrightarrow 0.$$

□

COROLLARY 4.3. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise independent and level-wise identically distributed $F_C(R)$-valued fuzzy random variables. There exists $\tilde{b} \in F_C(R)$ such that*

$$d\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i\ ,\ \tilde{b}\right) \longrightarrow 0 \quad a.s. \tag{4.3}$$

if and only if $E\|\tilde{X}_1\| < \infty$. Furthermore, if (4.3) holds, then $\tilde{b} = E\tilde{X}_1$.

PROOF. The sufficiency follows immediately from theorem 4.2. To prove the converse, if (4.3) holds, then for any $\alpha \in [0,1]$,

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^{-} \longrightarrow b_\alpha^{-} \quad a.s.$$

and

$$\frac{1}{n}\sum_{i=1}^{n} X_{i\alpha}^{+} \longrightarrow b_\alpha^{+} \quad a.s.$$

By the converse of Kolmogorov's strong law of large numbers,

$$E\left|X_{1\alpha}^{-}\right| < \infty, E\left|X_{1\alpha}^{+}\right| < \infty \quad \text{for each } \alpha \in [0,1]$$

which implies $E\|\tilde{X}_1\| < \infty$ and $\tilde{b} = E\tilde{X}_1$. □

EXAMPLE. Let $\tilde{u} \in F_c(R)$ be fixed and let $\{Y_n\}$ be i.i.d. with $E|Y_1| < \infty$ in the usual sense. Define $\tilde{X}_n(w)(x) = \tilde{u}(x - Y_n(w))$ i.e., $\tilde{X}_n(w)$ is the translation of $\tilde{u}$ by $Y_n(w)$ in x-axis. Then

$$X_{n,\alpha}^{-}(w) = u_\alpha^{-} + Y_n(w) \text{ and } X_{n,\alpha}^{+}(w) = u_\alpha^{+} + Y_n(w)$$

Hence the above theorem implies that

$$d\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{X}_i,\ E\tilde{X}_1\right) \longrightarrow 0 \quad a.s.$$

where $(E\tilde{X}_1)(x) = \tilde{u}(x - EY_1)$.

As a final result, we give a generalization of Chung's SLLN to the case of fuzzy random variables.

THEOREM 4.4. *Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables. If $\{\|\tilde{X}_n\|\}$ are independent random variables in classical sense and*

$$\sum_{n=1}^{\infty} \frac{1}{n} E\|\tilde{X}_n\| < \infty, \tag{4.4}$$

then

$$d\left(\frac{1}{n}\sum_{i=1}^{n} \tilde{X}_i, \frac{1}{n}\sum_{i=1}^{n} E\tilde{X}_i\right) \longrightarrow 0 \ a.s.$$

PROOF. First we note that

$$\begin{aligned} d(\frac{1}{n}\sum_{i=1}^{n} \tilde{X}_i, \frac{1}{n}\sum_{i=1}^{n} E\tilde{X}_i) &\leq \frac{1}{n}\sum_{i=1}^{n} d(\tilde{X}_i, E\tilde{X}_i) \\ &\leq \frac{1}{n}\sum_{i=1}^{n}(\|\tilde{X}_i\| + E\|\tilde{X}_i\|). \end{aligned}$$

Since $\{\|\tilde{X}_n\|\}$ is a sequence of independent random variables, (4.4) and Chung's law of large numbers yields

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(\|\tilde{X}_i\| - E\|\tilde{X}_i\|) = 0 \quad a.s. \tag{4.5}$$

Now, applying the Kronecker lemma to (4.4), we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E\|\tilde{X}_i\| = 0$$

which implies, together with (4.5),

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(\|\tilde{X}_i\| = 0 \quad a.s.$$

This gives the desired result. □

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