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THE MEASURE OF THE UNIFORMLY HYPERBOLIC INVARIANT SET OF EXACT SEPARATRIX MAP

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ABSTRACT. In this work, using the exact separatrix map which provides an efficient way to describe dynamics near the separatrix, we study the stochastic layer near the separatrix of a one-degree-of-freedom Hamiltonian system with time periodic perturbation. Applying the twist map theory to the exact separatrix map, T. Ahn, G. I. Kim and S. Kim proved the existence of the uniformly hyperbolic invariant set(UHIS) near separatrix. Using the theorems of Bowen and Franks, we prove this UHIS has measure zero.

1. Introduction

In this paper, we are concerned with the area measure of the generator of the chaotic orbits in the stochastic layer near separatrices of a one-degree-of-freedom Hamiltonian system with time periodic perturbation.

In [1], using the exact separatrix map more efficient in describing the exact dynamics near separatrices for a general class of Hamiltonian systems than Chirikov separatrix map [2], T. Ahn, G. I. Kim and S. Kim proved the existence of uniformly hyperbolic invariant set(UHIS) in the stochastic layer near separatrices, which generates chaotic orbits and they computed the lower bound of the width of stochastic sea.

Since the theory of dynamics in the stochastic layers of Hamiltonian systems has many applications related to transport and diffusion [3, 4, 5], many authors [6, 7, 8, 9] have studied Hamiltonian transport and

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diffusion in the stochastic layer. These studies shows anomalous diffusion and specific transport with scaling law strongly related to the self-similar structure of islands and cantori in the stochastic layer which generically consists of islands with resonance, cantori and stochastic sea.

In this paper, we investigate the area measure of UHIS in order to estimate the role of stochastic sea in diffusion and transport. In section 2 of this paper, we explain the exact separatrix map and it's properties briefly. In section 3 we state two theorems for the existence of UHIS of the exact separatrix map and chaotic orbit generating property of UHIS and we prove that the area measure of UHIS of the separatrix map is zero by modifying this map to a C^2 diffeomorphism on torus. In section 4 we conclude our results.

2. Exact separatrix map

We consider one-degree-of-freedom Hamiltonian systems with time-periodic perturbations given by

$$\begin{aligned} \dot{x} &= JDH_0(X) + \epsilon JDH_1(x,\phi), \\ \dot{\phi} &= 1, \end{aligned} \qquad (x,\phi) \in \mathbb{R}^2 \times S^1, \tag{2.1}$$

where $H_0(x) + \epsilon H_1$ is a Hamiltonian function, $S^1 = \mathbb{R}/(T\mathbb{Z})$ the circle of length T, $T \in \mathbb{R}^+$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this section, we construct the separatrix map for this Hamiltonian system. First, we make the assumptions. (A.1) For $\varepsilon = 0$, the x component of (2.1) has a homoclinic orbit $x^0(t)$ to a hyperbolic fixed point x_0 whose trajectory is called the separatrix. (A.2) The interior of $x^0(t)$ is filled with a family of periodic orbits $x^h(t), h < 0$ of period $T(h)$ such that $H_0(x^h(t)) = h$ for all t and $T'(h) > 0$, and that

$$\lim_{h\to 0} x^h(t) = x^0(t) \quad \text{and} \quad \lim_{h\to 0} T(h) = \infty \quad \text{(See Figure 1)}$$

In the full phase space, x_0 corresponds to the periodic orbit $\gamma_0(t) = (x_0, t+\phi)$, which is a normally hyperbolic invariant manifold. By the invariant manifold theorem $\gamma_0(t)$ persists under the perturbation as $\gamma_\varepsilon(t)$, which has the stable and unstable manifolds (See Figure 2). In order to study the dynamics near the separatrix, let us consider a section $\Sigma \subset \mathbb{R}^2$ for the x component of (2.1) which transversally intersects the separatrix when $\epsilon = 0$. Then $\Sigma \times S^1$ is transversal to the flow of (2.1) for small enough ε. The transversality of $\Sigma \times S^1$ to the system (2.1) and an application of the implicit function theorem to our system yield that (ϕ, p) is a coordinate system of $\Sigma \times S^1$ for the cannonical variables of time ϕ and energy p [1]. Let $q(t; \phi, p) = (x(t), \phi(t))$ be a solution of (2.1) with the initial condition $(\phi, p) \in \Sigma \times S^1$. Then the return map(the exact separatrixmap) $S : \Sigma \times S^1 \to \Sigma \times S^1$ generated by the flow of (2.1) is given by

$$S(\phi, p) = \begin{pmatrix} \phi' \\ p' \end{pmatrix} = \begin{pmatrix} \phi + N(\phi, p) \mod T \\ p + \varepsilon M(\phi, p), \end{pmatrix} \tag{2.2}$$

where

$$N(\phi, p) = T(H_0(\phi, p)) + \varepsilon N_1(\phi, p),$$

$$M(\phi, p) = \int_0^T \frac{\partial H_1}{\partial \phi}(q(t; \phi, p))\, dt$$

and

$$N_1(\phi, p) = \int_0^T (\frac{\partial x_2}{\partial h} g_1 - \frac{\partial x_1}{\partial h} g_2)(q(t; \phi, p))\, dt$$

$$x = (x_1, x_2) \in \mathbb{R}^2$$

$$g_i = (-1)^i \frac{\partial H_1}{\partial x_i} \quad \text{for} \quad i = 1, 2 \quad (\text{ See [1]}).$$

Since the given perturbation is time periodic, $N(\phi, p)$ has the time periodic logarithmic singularity. Using Birkhoff theorem we can show that the singularity of $N(\phi, p)$ is represented as a graph given by $p = Y(\phi) - \hat{Y}(\phi)$ in the phase space in (ϕ, p). As the return maps used in studying transport and diffusion in Hamiltonian systems satisfy the domain dependent twist conditions and have logarithmic singularities [2, 10, 11, 12], the exact separatrixmap also satisfies those conditions, moreover infinite twist condition [1]. Let $(\phi', p') = S(\phi, p)$. By the twist map theory (see [13] for a review) there exists a generating function $G(\phi,\phi'), \phi,\phi' \in \mathbb{R}$ for the exact separatrixmap S such that

$$p = -D_1G(\phi,\phi') \quad \text{and} \quad p' = D_2G(\phi,\phi').$$

Here, D_1 and D_2 represent the derivatives with respect to the first and second arguments, respectively. Aubry and Abramovici introduced the concept of the anti-integrable-limit in the symplectic twist map which implies the singular limit where the generating function is expressed as the sum of one variable function. They proved that the dynamics in the anti-integrable limit is completely random and the system near the anti-integrable limit has a uniformly hyperbolic invariant set on which dynamics corresponds to random dynamics in the anti-integrable limit [14]. We can show that the exact separatrixmap has the anti-integrable limit with the anti-integrable states as the zeroes of $\hat{Y}(\phi) - Y(\phi) = 0$, where $Y(\phi)$ and $\hat{Y}(\phi)$ are graphs of differentiable functions given by the intersections of the stable and unstable manifolds of $\gamma_\epsilon(t)$ and $\Sigma \times S^1$ respectively [1].

3. Uniformly hyperbolic invariant set of the exact separatrix map

Throughout the rest of the paper, we write explicitly S and S^{-1} (2.2) in the following functional forms:

$$\text{(3.1)} \qquad S : \begin{cases} \phi' = \phi + N(\phi,p) \\ p' = p + \varepsilon M(\phi,p) \end{cases} \qquad S^{-1} : \begin{cases} \bar{\phi} = \phi - \hat{N}(\phi,p) \\ \bar{p} = p + \varepsilon \hat{M}(\phi,p), \end{cases}$$

where $N(\phi,p)$ and $\hat{N}(\phi,p)$ satisfy

$$N(Y(\phi) - \hat{Y}(\phi),\phi) = \infty, \quad \hat{N}(0,\phi) = \infty.$$

We assume the existence of transversal intersection of the stable and unstable manifolds of $\gamma_\epsilon(t)$, as is generically the case [15], that is, the function $Y(\phi) - \hat{Y}(\phi)$ has simple zeros. We denote the set of simple zeros of $\hat{Y}(\phi) - Y(\phi)$ by $\bar{\mathcal{A}}$. Let the set $\mathcal{A}$ be the lifted set of $\bar{\mathcal{A}}$ on the whole real line. Then we can get the anti-integrable limit of the separatrixmap. To prove that the separatrixmap has an invariant set near the anti-integrable limit, we need to define connected open sets near $(a_k, 0), a_k \in \mathcal{A}$ as follows (see Figure 3):

DEFINITION 3.1. *For sufficiently large K we choose a sequence $\{a_k\}_{-\infty}^{\infty}$ such that $a_k - a_{k-1} > K$ and $a_k \in \mathcal{A}$ for all $k \in N$. We define connected open sets near $(a_k, 0)$ for each k, $D(a_k; K)$, by*

$$D(a_k; K) = \{(\phi, p) : N(\phi, p) > K \quad \text{and} \quad \hat{N}(\phi, p) > K\}.$$

THEOREM 3.1. *For sufficiently large K and any sequence $\{a_k\}$ such that $a_k - a_{k-1} > K$ and $a_k \in \mathcal{A}$ for all k, there exists a unique orbit $\{(\phi_k, p_k)\}$ of S such that $(\phi_k, p_k) \in D(a_k; K)$ for all k.*

PROOF. See theorem 7 and its proof in [1].

Note that if we take the sequence $\{a_k\}$ arbitrarily random, the corresponding orbit $\{(\phi_k, p_k)\}$ should be also arbitrarily random. That is, theorem 3.1 implies that there is no rotational invariant circle(RIC) near separatrices; dynamics on $D(a_k; K)$ is chaotic.

DEFINITION 3.2. *The projection of the orbits in theorem 3.1 to $R \times S^1$ forms an invariant set. We denote this invariant set by Λ_K.*

Let $\Lambda_{K,K'}$ be a subset of Λ_K obtained by the set of projections of the orbits corresponding to the sequences $\{a_k\}$ in theorem 3.1 with $K < |a_k - a_{k-1}| \leq K'$. Then $\Lambda_{K,K'}$ is a compact invariant set. We refer readers to Ref. [16] for the definition of the uniform hyperbolicity. The uniform hyperbolicity of $\Lambda_{K,K'}$ follows from a general result proved by MacKay, Aubry and Baesens [16].

THEOREM 3.2. *$\Lambda_{K,K'}$ are uniformly hyperbolic for sufficiently large K and K'.*

PROOF. See theorem 10 and its proof in [1].

Note that uniform hyperbolicity implies the strong persistence under small perturbation. So, by theorem 3.2, $\Lambda_{K,K'}$ is structurally stable. No we concern the role of the UHIS in the stochastic layer in transport and diffusion. To do this , first of all, we estimate the Lebesque measure of the UHIS in the stochastic layer. Concerning the role of the chaotic trajectories of the standard map, Aubry and Abramovici suggested the following conjecture: The Lebesque measure of the full set of uniformly hyperbolic trajectories is zero [14]. This conjecture was proved by Aubry, MacKay and Baesens [16]. In analogy with the proof of this conjecture, we prove that the UHIS $\Lambda_{K,K'}$ of the exact separatrixmap has zero measure.

THEOREM 3.3. *The set $\Lambda_{K,K'}$ has measure zero for the sufficiently large K, K' with $K < K'$.*

PROOF. To prove this, we modify the exact separatrixmap S to C^2 diffeomorphism of period L in p on cylinder.

We define a new connected open sets $D(a_k;K;K')$ including $\Lambda_{K,K'}$ in $D(a_k;K)$ by

$$D(a_k;K;K') = \{(\phi,p) : K < N(\phi,p) < K' \quad \text{and} \quad K < \hat{N}(\phi,p) < K'\}.$$

By the smooth Uryson's lemma, there exists a C^2 diffeomorphism $M^*(N^*)$ on cylinder respectively such that

(i) $M^*(N^*)|_{D(a_k;K;K')} = M(N)|_{D(a_k;K;K')}$.

(ii)$M^*(N^*)(\phi,p+L) = M^*(N^*)(\phi,p)$ for some sufficiently large L.

With these diffeomorphisms, we modify the exact separatrixmap S (3.1) to S^* given by

$$S^* : \begin{cases} \phi' = \phi + N^*(\phi,p) \\ p' = p + \varepsilon M^*(\phi,p) \end{cases}$$

The modified map S^* can be considered as acting on the torus T^2. Since the set $\Lambda_{K,K'}$ is also the UHIS of S^*, by theorem 3.1, the set $\Lambda_{K,K'}$ is closed, and by theorem 3.2 it forms a uniformly hyperbolic set. By a theorem of Bowen referred to in ref [17], a uniformly hyperbolic set for a C^2 diffeomorphism of a compact manifold, either has measure zero or is the whole manifold [18, 19, 20]. So, if the measure of $\Lambda_{K,K'}$ is not zero, the modified map of the separatrixmap would be Anosov. By a theorem of Franks [21], Anosov system implies a hyperbolic action on the first homology $H_1(T^2, R)$. That is, if $f : T^2 \to T^2$ is any Anosov diffeomorphism, then f_* induced by f on the first homology group with real coefficients has no root of unity among its eigenvalues [21, 22, 23].

Whereas the action of the modified map S^* on homology is $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Hence $\Lambda_{K,K'}$ has measure zero.

4. Conclusion

We have proved that the UHIS near the separatrices for a general one degree of freedom Hamiltonian system under time periodic perturbation has measure zero. This fact implies that the role of the UHIS near separatrices in transport and diffusion is negligible. That is, the transport and diffusion in the stochastic layer is determined by the role of cantori and islands with resonance.

So,the diffusion generated by the transport through turnstiles on cantori and islands chains must determine the global diffusion in the stochastic layer. By the recent results of many authors [7, 9, 24], this means that the diffusion in Hamiltonian chaos should be anomalous.

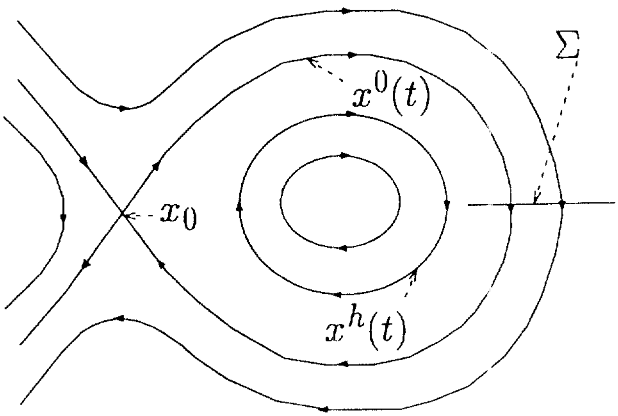

FIGURE 1: The unperturbed phase space of the x component of (2.1) with the homoclinic orbit $x^0(t)$ to a hyperbolic equilibrium x_0 and the transversal section Σ are shown.

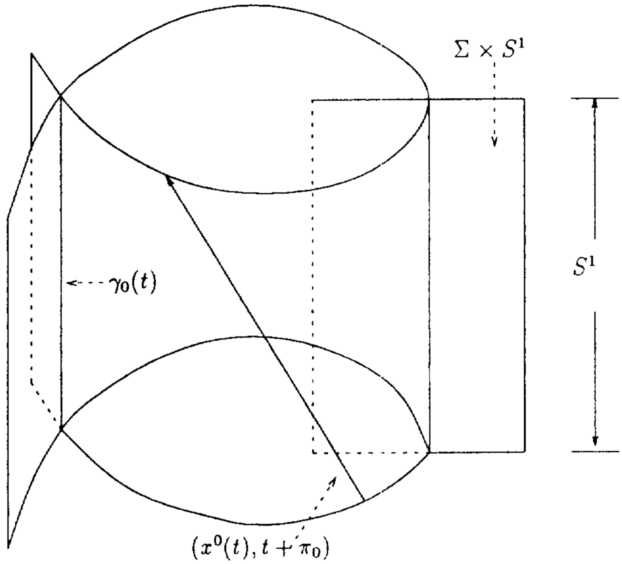

FIGURE 2: The full phase space of (2.1) with a periodic orbit $\gamma_0(t)$ and the section $\Sigma \times S^1$ are shown when $\epsilon = 0$.

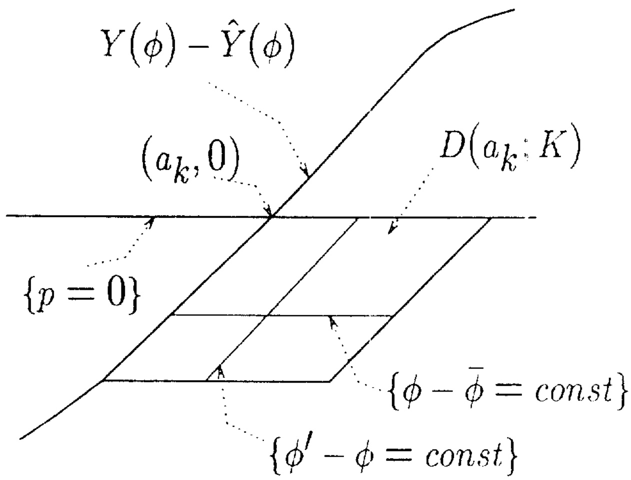

FIGURE 3: The connected open set $D(a_k; K)$ near the simple zero $(a_k, 0)$ of $Y(\phi) - \hat{Y}(\phi)$ in the coordinate system $(\phi - \bar{\phi}, \phi' - \phi)$ are shown.

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