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PFAFFIAN AND YOUNG TABLEAUX

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ABSTRACT. We consider a Pfaffian and its combinatorial model. We give a bijection between Pfaffian and the generating function of weights of generalized Young tableaux by this combinatorial model, and we find an explicit formula for the Pfaffian by this bijection.

1. Introduction

The Pfaffian is introduced by Pfaffian in 1815 to find a solution of a differential equation. Since then, Hurst and Green (1960) [6], Kasteleyn (1963) [7] and Montrol (1964) [8] obtained a simple solution of Ising problem for square network of two dimensions in using the Pfaffian. The importance of Pfaffian in physical problems turned out to be the well known relation between the Pfaffian and the determinant of antisymmetric matrix. Recently, Desainte-Catherine [3] used the Pfaffian as a combinatorial tool for the number of crossings of an involution without fixed points. Choi and Lee [2] expressed the generating function of cells of generalized Young tableaux by a Pfaffian. We consider a Pfaffian which is given by weights of nontouching paths, and we show that this Pfaffian corresponds to the generating function of weights of generalized Young tableaux. So we can give a good expression for this Pfaffian by using the formula $\prod_{1\le i\le j\le n}\frac{1-q^{2k+i+j}}{1-q^{i+j}}$ given by Desarmenien [4];

2. Definitions

Let T be a tableau $(t_{i,j})_{1\le i<j\le 2k}$. We call *Pfaffian of the tableau T* the following sum :

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$$Pf(T) = \sum_{\sigma \in S^*_{2k}} \epsilon_\sigma t_{\sigma(1)\sigma(2)} t_{\sigma(3)\sigma(4)} \ldots t_{\sigma(2k-1)\sigma(2k)},$$

where $S^*_{2k} = \{\sigma \in S_{2k} | \sigma(1) < \sigma(2), \sigma(3) < \sigma(4), ..., \sigma(2k-1) < \sigma(2k)\}$ and $\{\sigma(1) < \sigma(3) < ... < \sigma(2k-1)\}$, and ϵ_σ is the sign of the permutation σ.

NOTATION 1. We denote Pf(T), the Pfaffian of the tableau T, as the upper part of a upper triangular matrix :

$$Pf(T) = \begin{pmatrix} t_{1,2} & t_{1,3} & t_{1,4} & \ldots\ldots & t_{1,2k} \\ & t_{2,3} & t_{2,4} & \ldots\ldots & t_{2,2k} \\ & & & \ldots\ldots & . \\ & & & \ldots\ldots & . \\ & & & & t_{2k-1,2k} \end{pmatrix}$$

We can also define the Pfaffian of antisymmetric matrix :

DEFINITION 1. Let A be an antisymmetric matrix $\left(a_{i,j} \right)_{1 \leq i,j \leq 2k}$, with

$$a_{i,j} = \begin{cases} 0 & \text{if } i = j,\ i \in [1, 2k] \\ -a_{j,i} & \text{if } i \neq j\ (i,j) \in [1, 2k]^2. \end{cases}$$

Let $T = \left(a_{i,j} \right)_{1 \leq i < j \leq 2k}$. We get

$$Pf(A) = Pf(T).$$

THEOREM 1. (Cayley 1847) *Let $T = (t_{i,j})$ a matrix such that, for $1 \leq i < j \leq 2k$, $t_{j,i} = 0$, and let M the antisymmetric matrix $M = T - T^t$ (where T^t denote the transpose of T). For the Pfaffian of the matrix T $Pf(T)$, we have*

$$Det(M) = (Pf(T))^2.$$

A partition is any sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ of non-negative integers in decreasing order; $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$. The λ_i, $1 \leq i \leq m$, are called the parts of λ, and the number of parts is the length of λ, denoted by $l(\lambda)$; and the sum of the parts is the *weight* of λ, denoted by $|\lambda|$: $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_m$.

The Ferrers diagram of a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ is an array of m rows of cells, with λ_i cells, left justified, in the ith row. The rows are numbered from bottom to top.

DEFINITION 2. A *generalized Young tableau* of shape λ is a filling of the Ferrers diagram of λ with positive integers which are increasing from bottom to top and strictly increasing from left to right.

2.1. Pfaffian and nontouching paths Let $\mathbb{Z}$ be the set of all integers.

A path is a sequence $w = (s_0, s_1, ..., s_n)$ of points in $\mathbb{Z} \times \mathbb{Z}$. A unit path is called *west step* if $s_i = (x, y)$ and $s_{i+1} = (x-1, y)$, and *north step* if $s_i = (x, y)$ and $s_{i+1} = (x, y+1)$ [9]. Let Π be the region limited by two half lines OX and OZ having lines equations $y = 0$ and $y = x$ respectively (cf. Figure 1). Let $A_1, A_2, ..., A_m, ..., A_{m+n}$ be points in Π with the following coordinates :

$$A_i = \begin{cases} (m+1-i, 0) & \text{if } 1 \leq i \leq m \\ (i-m, i-m) & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

We assume that $m + n$ is even. In this paper, we deal with paths consisting of two elementary steps : west step and north step, in the region Π. We have $\binom{m+1-i}{j-m}$ different paths linking A_i to A_j, $1 \leq i \leq m$, $m+1 \leq j \leq m+n$. There is no path going from A_i to A_j when $m+1-i < j-m$, with $1 \leq i \leq m+n$ and $m+1 \leq j \leq m+n$. We can describe $(m+n)/2$ nontouching paths going from A_i to A_j, $1 \leq i < j \leq m+n$, in the region Pi.

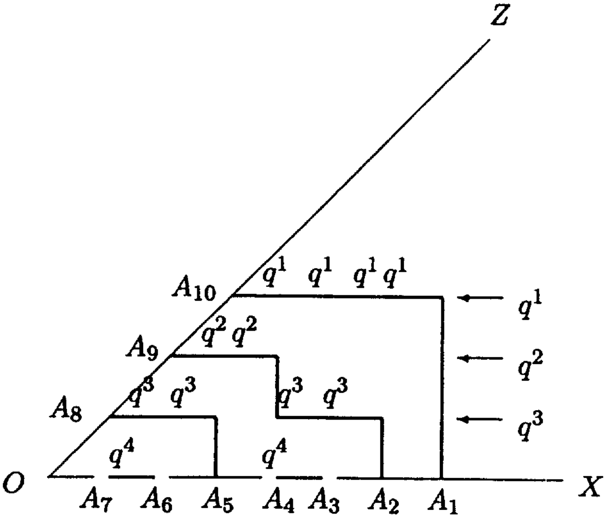
We give weights for elementary steps (West step or North step) of the path w of Π, and we define the weight $v(w)$ of the path w as the product of weights of elementary steps in w.

We define the weight on a path going from A_i to A_j, $1 \leq i, j \leq m+n$, in the following way:

- a West step of the path whose vertical coordinate is $n - i + 1$ is weighted by q^i,
- a North step is weighted by 1.

LEMMA 1. *There is a bijection between the set of generalized Young tableaux of height $2k$, with columns having only an even number of elements, and the k-tuples of nontouching paths in Π. This bijection preserves the weights of the generalized Young tableaux of height $2k$, with columns having only an even number of elementsand the k-tuples of nontouching paths*

PROOF. From a configuration of nontouching paths of Π, we construct a column of generalized Young tableaux, in taking all numbers i with decreasing order from the weight of West steps q^i of the path in Π, except the paths on the axis OX. The weight q^i of West step counts the weight of the corresponding generalized Young tableaux, which is the sum of all the entries (cf. Figure 1) □



$T =$

1	3	
1	3	
1	2	3
1	2	3

m = 7

n = 3

height of tableau : 4

The weights of two dominos on the axis OX are not related with the weight of tableau T.

$\alpha = (4, 2, 4)$
weight of tableau : 20
$(Q)^\alpha = q^4(q^2)^2(q^3)^4$

FIGURE 1: Configuration of nontouching paths and weight of generalized Young tableau.

Let $T = (t_{i,j})_{1 \le i,j \le m+n}$ be a matrix with $t_{i,j} = \sum_{w:A_i \to A_j} v(w)$, where $v(w)$ is the weight of a path w included in Π, having the points A_i, $1 \le i \le m$, on OX and the points A_j, $1 \le j \le n$, on OZ (cf. figure 1). There is no path going from A_i to A_j when $m + 1 - i < j - m$, with $1 \le i \le m$ and $m + 1 \le j \le m + n$, so in this case $t_{i,j}$ is equal to zero. If $i \ge j$, then $t_{i,j}$ is equal to zero too.

We can make also an antisymmetric matrix M $[m+n] \times [m+n]$ defined by $m_{i,j} = t_{i,j}$ for $1 \leq i < j \leq m+n$, and $m_{j,i} = -t_{i,j}$ for $1 \leq j \leq i \leq m+n$. According to the Theorem 1, $\mathrm{Pf}(T) = (det(M))^{\frac{1}{2}}$.

We denote $\begin{bmatrix} n \\ p \end{bmatrix}$ the Gaussian polynomial [1],[8]:

$$\frac{(1-q^{n-p+1})(1-q^{n-p+2})...(1-q^{n})}{(1-q)(1-q^2)...(1-q^p)}.$$

The tableau $T = (t_{i,j})_{1 \leq i,j \leq m+n}$ is described by the following definitions, which is given below in the tableau "Tableau T" :

(i) $t_{i,j} = 0$ if $1 \leq j \leq i \leq m+n$,

(ii) $t_{i,j} = q^{(j-i)(n+1)}$ if $1 \leq i < j \leq m$,

(iii) $t_{i,j} = q^{(2m-i-j+1)(m+n-j+1)} \begin{bmatrix} m-i+1 \\ 2m-i-j+1 \end{bmatrix}$ if $1 \leq i \leq 2m-j+1$ and $m+1 \leq j \leq m+n$,

(iv) $t_{i,j} = 0$, otherwise.

We denote $Pf_{m,n;q}$ the Pfaffian of the tableau T. For example, we can find that $Pf_{3,1;q} = q^2(1+q^2)$ (cf. Figure 2) and $Pf_{4,2;q} = q^3(1+q^2+q^3+q^4+q^6)$.

According to the Lemma 1, $Pf_{m,n;q}$ enumerate generalized Young tableaux with columns having only an even number of elements, and preserving their weights (cf. Figure 1).

Remark 1. If we give a weight of a West step by q and a weight of a North step by 1 $Pf_{m,n;q}$ can give the generating function of the weights of cells of generalized Young tableaux with columns, having only an even number of elements [2], [3], and [5].

Example 1. we have the following antisymmetric matrix $[m+n] \times [m+n]$ for m=3 and $n = 1$ (cf. Figure 1),

$$M = \begin{pmatrix} 0 & q^2 & q^4 & q^2\begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ -q^2 & 0 & q^2 & q\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ -q^4 & -q^2 & 0 & 1 \\ -q^2\begin{bmatrix} 3 \\ 2 \end{bmatrix} & -q\begin{bmatrix} 2 \\ 1 \end{bmatrix} & -1 & 0 \end{pmatrix}$$

$i \backslash j$	1	2	...	m	$m+1$	...	$m+n$
1	0	q^{n+1}	...	$q^{(m-1)(n+1)}$	$q^{(m-1)(n)} \begin{bmatrix} m \\ m-1 \end{bmatrix}$	...	$q^{(m-n)(1)} \begin{bmatrix} m \\ m-n \end{bmatrix}$
2	0	0	...	$q^{(m-2)(n+1)}$	$q^{(m-2)(n)} \begin{bmatrix} m-1 \\ m-2 \end{bmatrix}$	...	$q^{(m-n-1)(1)} \begin{bmatrix} m-1 \\ m-n-1 \end{bmatrix}$
.	0	0	...	.	.	...	.
.	0	0	...	.	.	...	.
$m-n$	0	0	...	.	.	...	$q^{(1)(1)} \begin{bmatrix} n+1 \\ 1 \end{bmatrix}$
$m-n+1$	0	0	0	.	.	...	1
.	0	0	0	0	.	...	0
.	0	0	0	0	0	...	0
.	0	0	0	0	0	...	.
.	0	0	0	0	0	...	.
.	0	0	0	0	0	...	.
$m+n-1$	0	0	0	0	0	0	0
$m+n$	0	0	0	0	0	0	0

Tableau T

We denote formal power series $b_{n,m-n;q} = \sum b_{n,m-n,i} q^i$ the generating function of the generalized Young tableaux with columns having only an even number of elements, where $b_{n,m-n,i}$ is the number of generalized Young tableaux with height bounded by $m-n$ and columns having only an even number of elements, having weights i, and the entries between 1 and n.

LEMMA 2. *The power series* $b_{n,m-n;q}$ *is equal to* $q^{-(n+1)(m-n/2)}Pf_{m,n;q}$.

PROOF. By Lemma 1, we have bijection between the configurations of nontouching paths of Π and generalized Young tableaux. There is $(m-n)/2$ dominos, which is a path of unit length on OX axis, in all of the configurations of nontouching paths without fixed point in Π, and these dominos play no role in the enumeration of the generalized Young tableaux. □

For example, the configuration of Figure 2 contains one domino, so we have : $Pf_{3,1;q} = (det(M_2))^{\frac{1}{2}} = q^2(1+q^2)$, and $b_{1,2;q} = (q^{-4}det(M))^{\frac{1}{2}} = 1+q^2$.

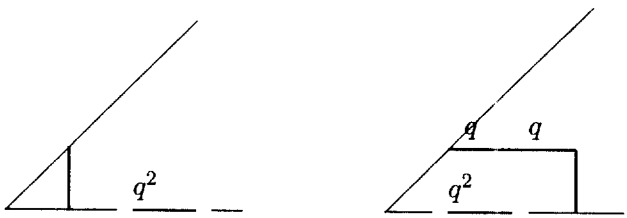


Figure 2 : Configuration of nontouching path for m = 3 and n = 1.

The determinant of antisymmetric matrix made by tableau T, can be expressed as the sum of determinants of the extracted minors of infinite matrix whose coefficients are the product of a monomial and a Gaussian polynomial : $D_n = (d_{n;i,j})_{0\le i,j}$, where

$$d_{n;i,j} = \begin{cases} q^{(n-j)(i-j)} \begin{bmatrix} i \\ j \end{bmatrix} & \text{if } 0 \le j \le i \\ 0 & 0 \le i < j. \end{cases}$$

The polynomial $d_{n;i,j}$, $(i,j \ge 0)$, is the sum of the weights of paths going from A_i to A_j. Although we don't know how to simplify the determinants of the extracted minors of matrix D_n, we can obtain simple formula for the determinant of the antisymmetric matrix in using the formula of Desarmenien. Desarmenien [4] have given a formula for the generating function of weights of generalized Young tableaux with height bounded by $2k$ and columns having only an even number of elements, and the entries between 1 and n as follows:

$$\prod_{1 \leq i \leq j \leq n} \frac{1 - q^{2k+i+j}}{1 - q^{i+j}}.$$

According to the Lemma 1 and Lemma 2, we obtain the following theorem.

THEOREM 2. *The Pfaffian of the tableau T, $Pf_{m,n;q}$, with $m+n$ even, is given by*

$$Pf_{m,n;q} = q^{(n+1)(m-n/2)} \prod_{1 \leq i \leq j \leq n} \frac{1 - q^{m-n+i+j}}{1 - q^{i+j}}.$$

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